International Journal of Statistics and Applied Mathematics 2021; 6(1): 35-38

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2021; 6(1): 35-38

www.mathsjournal.com
Received: 07-11-2020
Accepted: 20-12-2020

V Durai Murugan
Assistant Professor, Department of Mathematics, St Joseph College of Arts and Science, Veikalipatty, Tenkasi, Tamil Nadu, India

R Seethalakshmi
Assistant Professor, Department of Mathematics, Sri Parasakthi College for Women, Courtallam, Tenkasi, Tamil Nadu, India

Corresponding Author:
V Durai Murugan
Assistant Professor, Department of Mathematics, St Joseph College of Arts and Science, Veikalipatty, Tenkasi, Tamil Nadu, India

A study on complementedness in the subgroup lattices of 2x2 matrices over Z_3, Z_5 and Z_7

V Durai Murugan and R Seethalakshmi

Abstract
In this paper, we verify the complementedness in the subgroup lattices of the group of 2x2 matrices over Z_3, Z_5 and Z_7.

Keywords: Matrix group, subgroups, Lagrange's theorem, lattice

Introduction

Let $\mathcal{G} = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$: a, b, c, d $\in Z_p$, ad – bc $\neq 0\}$. Then $\mathcal{G}$ is a group under matrix multiplication modulo p. Let G = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}$: ad-bc = 1$\}$. Then G is a subgroup of $\mathcal{G}$. We have, $o(\mathcal{G}) = p(p^2-1)(p-1)$ [6] and $o(G) = p(p^2-1)$ [6].

In this paper we are going to the study about the complementedness in the subgroup lattice of the group of 2 x 2 matrices over Z_3, Z_5 and Z_7

Preliminaries

In this section, we give the definitions needed for the development of the paper.

Definition 2.1

A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair (P, $\leq$) is called a partially ordered set or poset. A poset (P, $\leq$) is totally ordered if every x, y $\in$ P are comparable, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.

Definition 2.2

Let (P, $\leq$) be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the supremum or join of S.A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the infimum or meet of S.

Definition 2.3

A poset (P, $\leq$) is called a lattice if every pair x, y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.

Definition 2.4

A poset is said to be complete lattice if all its subsets have both join and meet. In particular, every complete lattice is a bounded lattice.

Definition 2.5

Let L be a bounded lattice with greatest element 1 and least element 0. Two elements x and y of L are said to be complements of each other if $x \vee y = 1$ and $x \wedge y = 0$. If every element of L has a complement, then L is called a complemented lattice.

We give below the diagrams of L(G) when p = 2,3 and 5 [6].

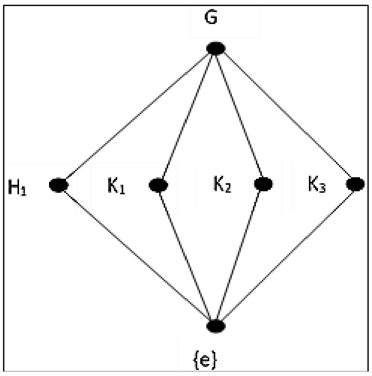

Fig 2.1: L(G) when p = 2

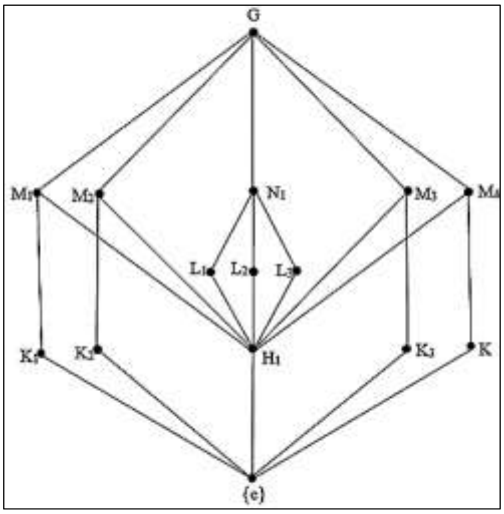

Fig 2.2: L(G) when p = 3

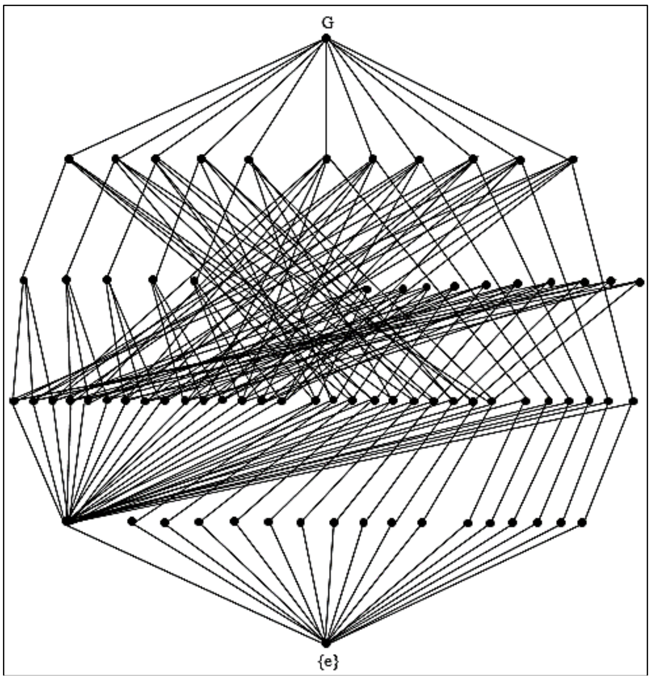

Fig 2.3: L(G) when p = 5

Row I: (Left to right): S_1 to S_5 and T_1 to T_6.
Row II: (Left to right): P_1 to P_5 and R_1 to R_{10}.
Row III: (Left to right): L_1 to L_{15}, N_1 to N_{10} and Q_1 to Q_6.
Row IV: (Left to right): H_1, K_1 to K_{10} and M_1 to M_6.

We tabulate the subgroups of G, when p = 7 in the order in which they lie in different maximal subgroups (co-atoms). This will make our work easy.

Table 2.1: Intervals [{e}, U_i] in L(G), i = 1,2,…..8

Order	Subgroups	Order	Subgroups
42	U_1	42	U_2
21	T_1	21	T_2
14	R_1	14	R_2
7	N_1	7	N_2
6	M_1, M_{12}, M_{14}, M_{16}, M_{17}, M_{21}, M_{23}	6	M_1, M_{11}, M_{13}, M_{15}, M_{18}, M_{22}, M_{24}
3	K_1, K_{12}, K_{14}, K_{16}, K_{17}, K_{21}, K_{23}	3	K_1, K_{11}, K_{13}, K_{15}, K_{18}, K_{22}, K_{24}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_3	42	U_4
21	T_3	21	T_4
14	R_3	14	R_4
7	N_3	7	N_4
6	M_4, M_6, M_9, M_{17}, M_{20}, M_{24}, M_{26}	6	M_4, M_5, M_{10}, M_{18}, M_{19}, M_{23}, M_{25}
3	K_4, K_6, K_9, K_{17}, K_{20}, K_{24}, K_{26}	3	K_4, K_5, K_{10}, K_{18}, K_{19}, K_{23}, K_{25}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_5	42	U_6
21	T_5	21	T_6
14	R_5	14	R_6
7	N_5	7	N_6
6	M_3, M_6, M_8, M_{16}, M_{19}, M_{22}, M_{28}	6	M_3, M_5, M_7, M_{15}, M_{20}, M_{21}, M_{27}
3	K_3, K_6, K_8, K_{16}, K_{19}, K_{22}, K_{28}	3	K_3, K_5, K_7, K_{15}, K_{20}, K_{21}, K_{27}
Order	**Subgroups**	**Order**	**Subgroups**
42	U_7	42	U_8
21	T_7	21	T_8
14	R_7	14	R_8
7	N_7	7	N_8
6	M_2, M_8, M_{10}, M_{12}, M_{13}, M_{26}, M_{27}	6	M_2, M_7, M_9, M_{11}, M_{14}, M_{25}, M_{28}
3	K_2, K_8, K_{10}, K_{12}, K_{13}, K_{26}, K_{27}	3	K_2, K_7, K_9, K_{11}, K_{14}, K_{25}, K_{28}

Table 2.2: Intervals [{e}, V_i] in L(G), i = 1,2,…..14

Order	Subgroups	Order	Subgroups
48	V_1	48	V_2
16	S_{12}, S_{16}, S_{17}	16	S_{13}, S_{18}, S_{19}
12	Q_1, Q_4, Q_7, Q_8	12	Q_1, Q_3, Q_9, Q_{10}
8	P_{12}, P_{16}, P_{17}	8	P_{13}, P_{18}, P_{19}
6	M_1, M_4, M_7, M_8	6	M_1, M_3, M_9, M_{10}
4	$L_1, L_2, L_3, L_{10}, L_{11}, L_{12}, L_{14}, L_{16}, L_{17}$	4	$L_1, L_2, L_3, L_8, L_9, L_{13}, L_{15}, L_{18}, L_{19}$
3	K_1, K_4, K_7, K_8	3	K_1, K_3, K_9, K_{10}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_3	48	V_4
16	S_4, S_5, S_{15}	16	S_6, S_7, S_{14}
12	Q_2, Q_4, Q_{15}, Q_{16}	12	Q_2, Q_3, Q_{17}, Q_{18}
8	P_4, P_5, P_{15}	8	P_6, P_7, P_{14}
6	M_2, M_4, M_{15}, M_{16}	6	M_2, M_3, M_{17}, M_{18}
4	$L_1, L_4, L_5, L_{10}, L_{11}, L_{13}, L_{15}, L_{20}, L_{21}$	4	$L_1, L_6, L_7, L_8, L_9, L_{12}, L_{14}, L_{20}, L_{21}$
3	K_2, K_4, K_{15}, K_{16}	3	K_2, K_3, K_{17}, K_{18}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_5	48	V_6
16	S_5, S_{10}, S_{21}	16	S_4, S_{11}, S_{20}
12	Q_5, Q_9, Q_{12}, Q_{22}	12	$Q_6, Q_{10}, Q_{11}, Q_{21}$
8	P_5, P_{10}, P_{21}	8	P_4, P_{11}, P_{20}
6	M_5, M_9, M_{12}, M_{22}	6	$M_6, M_{10}, M_{11}, M_{21}$
4	$L_3, L_4, L_5, L_7, L_8, L_{10}, L_{15}, L_{16}, L_{21}$	4	$L_2, L_4, L_5, L_6, L_9, L_{11}, L_{15}, L_{17}, L_{20}$
3	K_5, K_9, K_{12}, K_{22}	3	$K_6, K_{10}, K_{11}, K_{21}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_7	48	V_8
16	S_6, S_9, S_{20}	16	S_7, S_8, S_{21}
12	Q_5, Q_8, Q_{14}, Q_{24}	12	Q_6, Q_7, Q_{13}, Q_{23}
8	P_6, P_9, P_{20}	8	P_7, P_8, P_{21}
6	M_5, M_8, M_{14}, M_{24}	6	M_6, M_7, M_{13}, M_{23}
4	$L_3, L_4, L_6, L_7, L_9, L_{11}, L_{14}, L_{19}, L_{20}$	4	$L_2, L_5, L_6, L_7, L_8, L_{10}, L_{14}, L_{18}, L_{21}$
3	K_5, K_8, K_{14}, K_{24}	3	K_6, K_7, K_{13}, K_{23}
Order	**Subgroups**	**Order**	**Subgroups**
48	V_9	48	V_{10}
16	S_3, S_{10}, S_{16}	16	S_2, S_{11}, S_{17}
12	$Q_{11}, Q_{17}, Q_{19}, Q_{27}$	12	$Q_{12}, Q_{18}, Q_{20}, Q_{28}$
8	P_3, P_{10}, P_{16}	8	P_2, P_{11}, P_{17}
6	$M_{11}, M_{17}, M_{19}, M_{27}$	6	$M_{12}, M_{18}, M_{20}, M_{28}$
4	$L_3, L_5, L_9, L_{10}, L_{12}, L_{16}, L_{17}, L_{19}, L_{21}$	4	$L_2, L_4, L_8, L_{11}, L_{12}, L_{16}, L_{17}, L_{18}, L_{20}$
3	$K_{11}, K_{17}, K_{19}, K_{27}$	3	$K_{12}, K_{18}, K_{20}, K_{28}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_{11}	48	V_{12}
16	S_2, S_8, S_{18}	16	S_3, S_9, S_{19}
12	$Q_{14}, Q_{15}, Q_{19}, Q_{26}$	12	$Q_{13}, Q_{16}, Q_{20}, Q_{25}$
8	P_2, P_8, P_{18}	8	P_3, P_9, P_{19}
6	$M_{14}, M_{15}, M_{19}, M_{26}$	6	$M_{13}, M_{16}, M_{20}, M_{25}$
4	$L_2, L_7, L_8, L_{11}, L_{13}, L_{17}, L_{18}, L_{19}, L_{21}$	4	$L_3, L_6, L_9, L_{10}, L_{13}, L_{16}, L_{18}, L_{19}, L_{20}$
3	$K_{14}, K_{15}, K_{19}, K_{26}$	3	$K_{13}, K_{16}, K_{20}, K_{25}$
Order	**Subgroups**	**Order**	**Subgroups**
48	V_{13}	48	V_{14}
16	S_1, S_{12}, S_{14}	16	S_1, S_{13}, S_{15}
12	$Q_{21}, Q_{22}, Q_{25}, Q_{26}$	12	$Q_{23}, Q_{24}, Q_{27}, Q_{28}$
8	P_1, P_{12}, P_{14}	8	P_1, P_{13}, P_{15}
6	$M_{21}, M_{22}, M_{25}, M_{26}$	6	$M_{23}, M_{24}, M_{27}, M_{28}$
4	$L_1, L_6, L_7, L_{12}, L_{13}, L_{14}, L_{15}, L_{16}, L_{17}$	4	$L_1, L_4, L_5, L_{12}, L_{13}, L_{14}, L_{15}, L_{18}, L_{19}$
3	$K_{21}, K_{22}, K_{25}, K_{26}$	3	$K_{23}, K_{24}, K_{27}, K_{28}$

When p = 7, we display two typical intervals [{e}, U_1] and [{e}, V_1] of L(G) in the following diagrams [9].

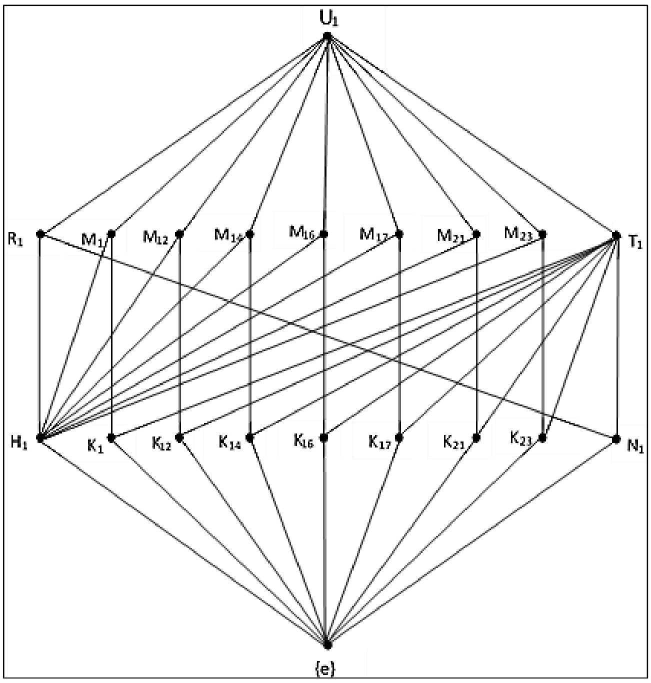

Fig 2.4: The Interval [{e}, U_1]

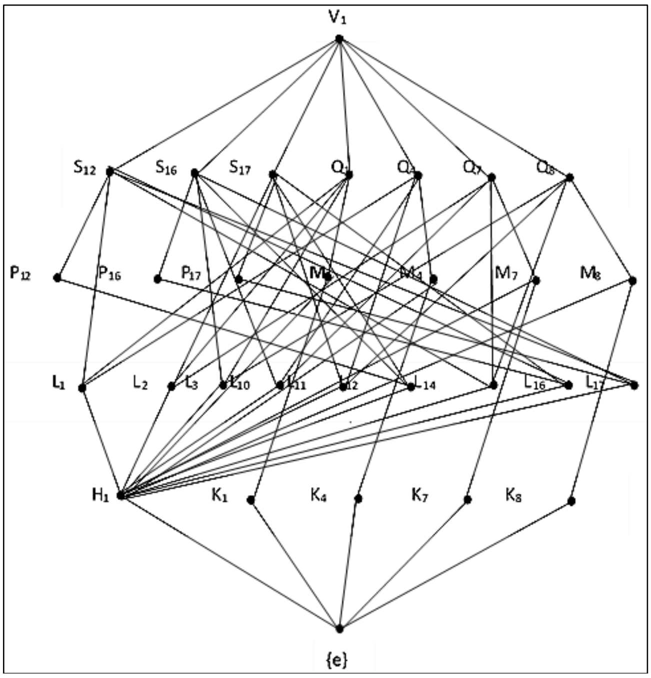

Fig 2.5: The Interval [{e}, V_1]

3. Complementedness in the lattice of subgroups of the group of 2 x 2 matrices over Z_3, Z_5 and Z_7

Lemma 3.1

For p = 3, 5 and 7, the two-element subgroup H_1 does not have a complement in L(G).

Proof:

An even order subgroup cannot be a complement of H_1.
So, if X were a complement of H_1, then in L(G).
$X \vee H_1 = G$ and $X \wedge H_1 = \{e\}$ where o(X) is odd.
If X is odd order which is a prime number, say k.
Now, $k-1 \equiv 0 \pmod 2$.
So, there exists a subgroup of order 2k.
$O(X \vee H_1) = 2k$.
Therefore, $X \vee H_1 \neq G$.
If o(X) = st, where s and t are odd primes and $s-1 \equiv 0 \pmod t$, (s>t)
Then $o(X \vee H_1) = 2\,st \neq (p-1)\,p\,(p+1) = o\,(G)$.
Therefore, $X \vee H_1 \neq G$.
So, H_1 has no complement in L (G).

Theorem: 3.2

L(G) is not complemented if p = 3, 5 and 7.

Proof:

Follows from the above lemma 3.1

Conclusion

In this paper, we proved that L(G) is not complemented when p = 3, 5 and 7.

References

1. Bourbaki N. Elements of Mathematics, Algebra I, Chapter 1-3 Springer Verlag Berlin Heidelberg, New York, London Paris Tokio 1974.
2. Fraleigh JB. A first course in Abstract Algebra, Addison-Wesley, London 1992.
3. Gardiner CF. A first course in group theory, Springer-Verlag, Berlin 1997.
4. Gratzer G, General Lattice theory: Birkhauser Verlag, Basel 1998.
5. Herstien IN. Topics in Algebra, John Wiley and sons, New York, 1975.
6. Vethamanickam A, Jebaraj Thiraviam. On Lattices of Subgroups, Int. Journal of Mathematical Archive 2015;6(9):1-11.
7. Vethamanickam A, Durai Murugan V. On the Lattice of Subgroups of 3x3 Matrices over Z_2 Int. Journal of Scientific Research and Reviews 2019;8(2):4107-4128.
8. Bashir Humera, Zahid Raza. On subgroups lattice of Quasidihedral group, International journal of Algebra 2012;6(25):1221-1225.
9. Seethalakshmi R, Durai Murugan V, Murugesan R. A Study on sun direct irreducibility of the subgroup lattices of the group of 2x2 matrices over Z_7 2020;9(4):1751-1760.