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The weighted Exponentiated inverted exponential distribution

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Abstract
This paper introduces the Weighted Exponentiated Inverted Exponential distribution as a modification of the Exponentiated Inverted Exponential distribution. Its various basic statistical properties were explicitly derived and the method of maximum likelihood estimation was used in estimating the model parameters. The model was applied to a real life data sets and its performance was assessed with respect to existing distribution using the log-likelihood and Akaike Information Criteria as basis for judgment.

Keywords: Exponential distribution, generalization, inversion, statistical properties, weighted distributions, azzalini

1. Introduction

The weighted distributions are applied in many fields of real life such as medicine, ecology, reliability and so on. This distribution was first introduced by (Fisher 1934) [5] to model ascertainment bias, weighted distributions were later formalized in a unifying theory by (Rao 1965) [8]. Weighted distributions are used to adjust the probabilities of the events as observed and recorded. It also provides appropriate approach to dealing with model specification and data interpretation problem. The weighted distribution is referred to as length biased distribution when the weight function is the length of the units.

There are several weighted distributions in literature. For instance, (Mahdavi, 2015) [6] proposed new classes of weighted distributions by absorbing exponential distribution in Azzalini’s method also (Nasiru, 2015) [7] proposed a new generalization of the Weibull distribution based on a modified weighted version of (Azzalini’s, 1985).

(Fatima & Ahmad, 2017) [4] proposed and studied new family of distribution called Exponentiated Inverted Exponential distribution (EIED) by introducing a shape parameter to Inverted Exponential distribution (IED). The Probability density function (pdf) of the Exponentiated Inverted distribution is given by

$$f(x) = \frac{\alpha\lambda}{x^2}\left(e^{-\frac{\lambda}{x}}\right)^{\alpha} \qquad ; x > 0, \alpha, \lambda > 0 \qquad \dots (1)$$

The corresponding cumulative density function (cdf) is given by

$$F(x) = \left(e^{-\frac{\lambda}{x}}\right)^{\alpha} \qquad ; x > 0, \alpha, \lambda > 0 \qquad \dots (2)$$

Where α Shape and λ Scale

In this paper however, a weighted version of the Exponentiated Inverted distribution is introduced and studied. In section 2, the densities of the Weighted Exponentiated Inverted Exponential (WEIE) distribution are derived with its various statistical properties, in section 3, an application to a real life data set is provided.

2. The weighted Exponentiated inverted exponential distribution

Let X denote a non-negative continues random variable, considering the weight function

$w(x)=x^{-\beta}$ and pdf of the two-parameter Exponentiated Inverted Exponential distribution as given in Eqn.(1), therefore, the pdf of the Weighted Exponentiated Inverted Exponential distribution is derived as

$$f_w(x)=\frac{w(x)f(x)}{w_D} \quad \dots (3)$$

Where

$$w_D=\int_0^\infty w(x)f(x)dx \quad \dots (4)$$

$$w_D=\alpha\lambda\int_0^\infty x^{-\beta}x^{-2}e^{-\frac{\alpha\lambda}{x}}dx \quad \dots (5)$$

$$w_D=\alpha\lambda\int_0^\infty x^{-(2+\beta)}e^{-\frac{\alpha\lambda}{x}}dx \quad \dots (6)$$

Let $t=\frac{\alpha\lambda}{x}$ then $x=\frac{\alpha\lambda}{t}$ and $dx=-\frac{\alpha\lambda}{t^2}dt$

$$w_D=\frac{\Gamma(1+\beta)}{(\alpha\lambda)^\beta} \quad \dots (7)$$

Insert equation (7) into (3) and $f(x)$ as defined in (1)

$$f_w(x)=\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)}x^{-(2+\beta)}e^{-\frac{\alpha\lambda}{x}} \qquad ;x>0,\alpha,\lambda,\beta>0 \quad \dots (8)$$

Equation (8) is the pdf of the Weighted Exponentiated Inverted Exponential distribution.
Its associated CDF is obtained as follows:

$$F_w(x)=\frac{\Gamma\left(1+\beta,\frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)} \quad \dots (9)$$

Where α and β are shape parameters and λ is the scale parameter

Theorem: Let X denote a continuous random variable, then the pdf of the weighted Exponentiated Inverted Exponential distribution as derived in Equation (8) is a valid pdf.

Proof: For the pdf to be valid, it suffices that; $\int_0^\infty f(x)dx=1$

$$\int_0^\infty f(x)dx=\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)}\int_0^\infty x^{-(2+\beta)}e^{-\frac{\alpha\lambda}{x}}dx \quad \dots (10)$$

Let $t=\frac{\alpha\lambda}{x}$ then $x=\frac{\alpha\lambda}{t}$ and $dx=-\frac{\alpha\lambda}{t^2}dt$

$$=\frac{(\alpha\lambda)^\beta}{\Gamma(1+\beta)}\int_0^\infty\left(\frac{t}{\alpha\lambda}\right)^\beta e^{-t}dt \quad \dots (11)$$

$$=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta)} \quad \dots (12)$$

Possible plots for the pdf and cdf of the Weighted Exponentiated Inverted Exponential distribution at varied parameter values are as shown in Figure 1 & 2.

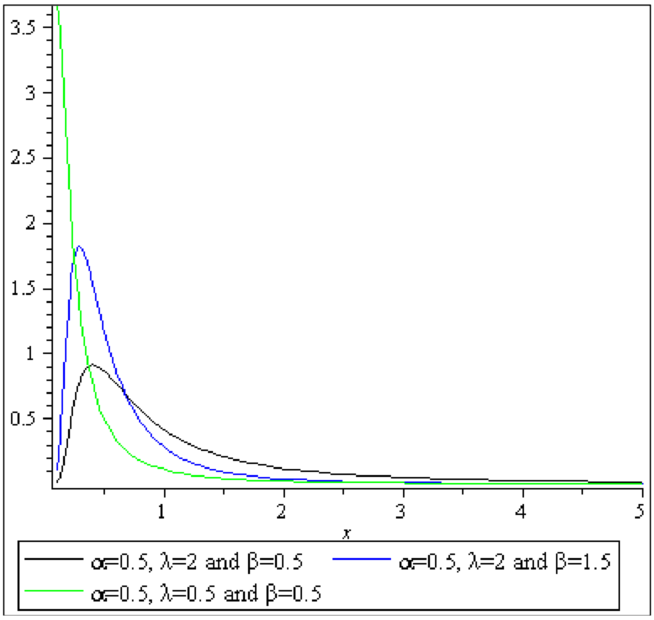

Fig 1: PDF plot of the Weighted Exponentiated Inverted Exponential Distribution

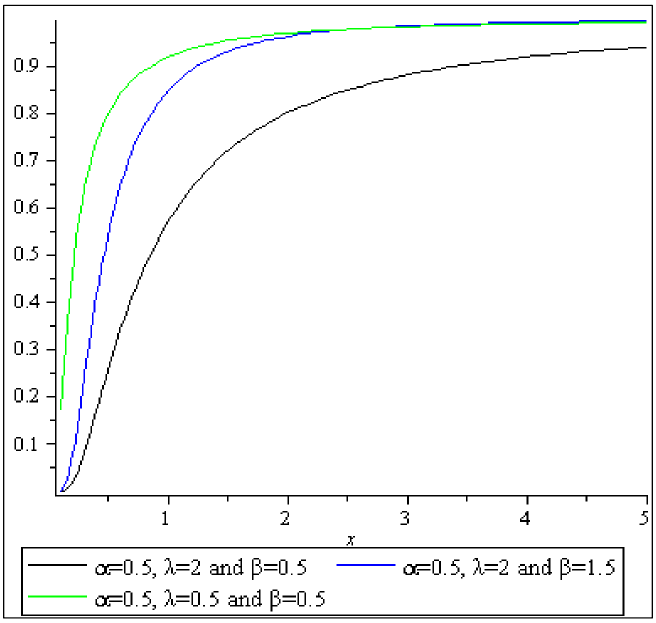

Fig 2: CDF plot of the Weighted Exponentiated Inverted Exponential Distribution

2.1 Reliability analysis

Survival function: The Survival function is given by:

$$S(x)=1-F(x)$$

$$S(x)=1-\frac{1}{\alpha}e^{-\frac{\beta}{x}}\left(\alpha+1-e^{-\frac{\alpha\beta}{x}}\right) \qquad x>0,\alpha>0,\beta>0 \tag{13}$$

And the hazard function is:

$$h(x)=\frac{\frac{(\alpha+1)}{\alpha x^2}\beta e^{-\frac{\beta}{x}}\left(1-e^{-\frac{\alpha\beta}{x}}\right)}{1-\frac{1}{\alpha}e^{-\frac{\beta}{x}}\left(\alpha+1-e^{-\frac{\alpha\beta}{x}}\right)} \tag{14}$$

Possible plots for the survival and hazard function at varied parameter values are displayed in Figure 3 & 4

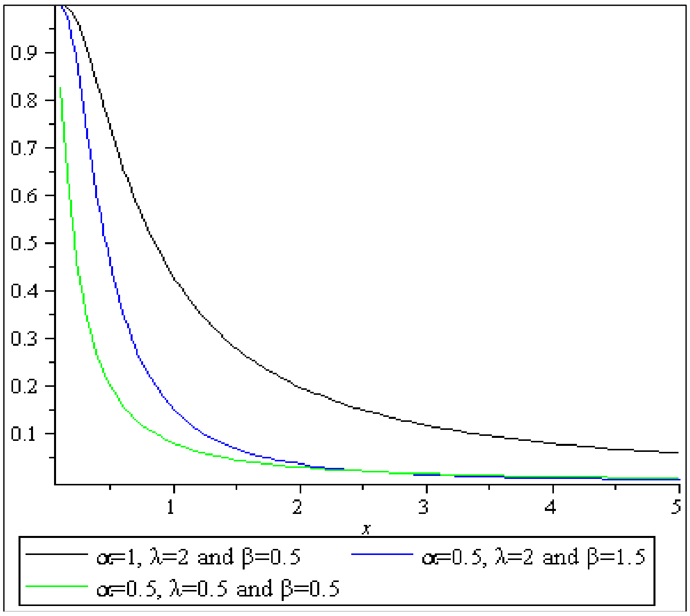

Fig 3: Survival plot of the Weighted Exponentiated Inverted Exponential Distribution

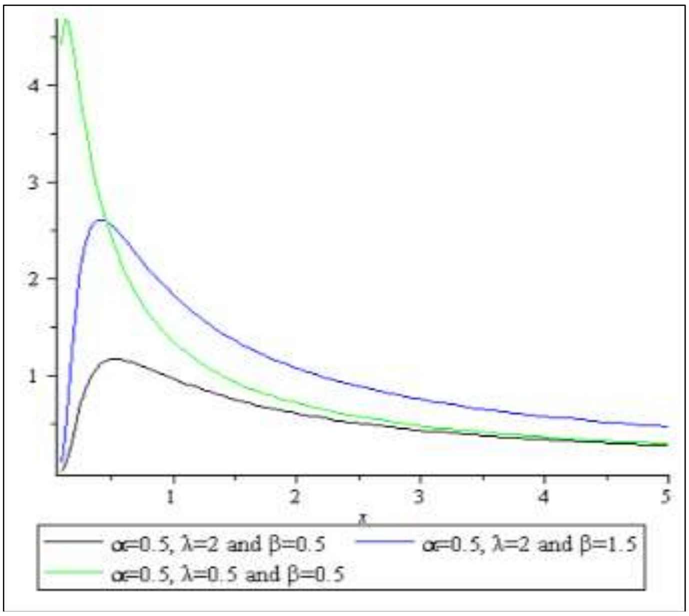

Fig 4: Hazard plot of the Weighted Exponentiated Inverted Exponential Distribution

From Figure 4, it can be seen that the hazard function of the Weighted Exponentiated Inverted Exponential Distribution exhibits unimodal (inverted bathtub) and decreasing shapes. This implies that the WEIE distribution can be used to describe or model real life phenomena with unimodal or decreasing failure rates.

2.2 Distribution of order statistics

Let $x_1, x_2, \ldots, x_n$ denote random samples from a cdf and pdf distributed according to the Weighted Exponentiated Inverted Exponential Distribution, the pdf of the kth order statistics from the Weighted Exponentiated Inverted Exponential Distribution is derived from:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-j)!} f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k} \tag{15}$$

Therefore, the pdf of the kth order statistics for the Weighted Exponentiated Inverted Exponential Distribution is:

$$f_{k:n}(x) = \frac{n!}{(k-1)!(n-k)!}\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)} x^{-(2+\beta)} e^{-\frac{\alpha\lambda}{x}} \left[\frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}\right]^{k-1} \left[1-\frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}\right]^{n-k} \tag{16}$$

The distribution of minimum and maximum order statistics for the Weighted Exponentiated Inverted Exponential Distribution is:

$$f_{1:n}(x) = n\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)} x^{-(2+\beta)} e^{-\frac{\alpha\lambda}{x}} \left[1-\frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}\right]^{n-1} \tag{17}$$

And

$$f_{n:n}(x) = n\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)} x^{-(2+\beta)} e^{-\frac{\alpha\lambda}{x}} \left[\frac{\Gamma\left(1+\beta, \frac{\alpha\lambda}{x}\right)}{\Gamma(1+\beta)}\right]^{n-1} \tag{18}$$

2.3 Moments of weighted Exponentiated inverted exponential distribution

If X follows the WEIED with parameters α, λ and β then rth moment of X is given as

$$\mu_r^! = \frac{(\alpha\lambda)^r \Gamma(\beta - \mathrm{r} + 1)}{\Gamma(1+\beta)} \tag{19}$$

The mean, variance, coefficient of variation and coefficient of Skewness were obtained from the rth moment as follows:

$$\mu = E(\mathrm{X}) = \frac{(\alpha\lambda)\Gamma(\beta)}{\Gamma(1+\beta)} \tag{20}$$

$$\sigma^2 = E(\mathrm{X}) - \mu^2 = \frac{(\alpha\lambda)^2\left[\Gamma(1+\beta)\Gamma(\beta-1) - \Gamma^2(\beta)\right]}{\Gamma^2(1+\beta)} \tag{21}$$

$$CV = \frac{\sigma}{\mu} = \frac{\left[\Gamma(1+\beta)\Gamma(\beta-1) - \Gamma^2(\beta)\right]^{1/2}}{\Gamma(\beta)} \tag{22}$$

$$CS = \frac{\mu^3}{\sigma^3} = \frac{\Gamma(\beta-2)\Gamma^2(1+\beta) - 3\Gamma(\beta)\Gamma(\beta-1)\Gamma(\beta+1) + 2\Gamma^3(\beta)}{\Gamma^3(\beta)} \tag{23}$$

2.4 Moment generating function of WEIE distribution

Following (Cordeiro, 2011) [3] the expression for moment generating function is given as

$$M_x(t) = \sum_{r=0}^{n}\left[\frac{t^r}{r!}\frac{(\alpha\lambda)^r\Gamma(\beta-\text{r}+1)}{\Gamma(1+\beta)}\right] \qquad (24)$$

2.5 Parameter estimation of WEIE distribution

Let $X_1, X_2, \ldots X_n$ denote random samples drawn from the Weighted Exponentiated Inverted Exponential distribution with parameters as defined in equation (8). Using the method of Maximum Likelihood Estimation (MLE), the likelihood function L is given by:

$$L(x\,|\,\alpha,\lambda,\beta) = \prod_{i=1}^{n}\left\{\frac{(\alpha\lambda)^{(1+\beta)}}{\Gamma(1+\beta)}x^{-(2+\beta)}e^{-\frac{\alpha\lambda}{x}}\right\} \qquad (25)$$

Let log-likelihood function be denoted by l, therefore

$$l = n(1+\beta)\ln(\alpha) + \text{n}(1+\beta)\ln(\lambda) - (2+\beta)\sum_{i=1}^{n}\ln(\text{x}_i) - \sum_{i=1}^{n}\frac{\alpha\lambda}{x_i} - n\ln(\Gamma(1+\beta)) \qquad (26)$$

Differentiating l with respect to respectively gives

$$\frac{\partial l}{\partial \alpha} = \frac{n(1+\beta)}{\alpha} - \lambda\sum_{i=1}^{n}\frac{1}{x_i} \qquad (27)$$

$$\frac{\partial l}{\partial \lambda} = \frac{n(1+\beta)}{\lambda} - \alpha\sum_{i=1}^{n}\frac{1}{x_i} \qquad (28)$$

$$\frac{\partial l}{\partial \beta} = n\ln(\alpha) + \text{n}\ln(\lambda) - \sum_{i=1}^{n}\ln(\text{x}) - n\Psi(1+\beta) \qquad (29)$$

Setting $\frac{\partial l}{\partial \alpha} = 0$, $\frac{\partial l}{\partial \lambda} = 0$, $\frac{\partial l}{\partial \beta} = 0$ and solving the resulting nonlinear equations gives the Maximum Likelihood Estimates of parameters α, λ and β

3. Application to data sets

For illustration purposes, real life application of the Weighted Exponentiated Inverted Exponential Distribution is provided. The performance of the WEIE distribution was compared with that of the Exponentiated Inverted Exponential distribution using log-likelihood and Akaike Information Criterion as selection criteria. The distribution that corresponds to the highest log-likelihood value and lowest AIC value is selected as the best for the data set used.

Data set I: The first data set has been previously used by Alqallaf *et al*. (2015) [1] the data set represents waiting time (in minutes) before service of 100 bank customers.

Table 1: Waiting time of 100 bank Customers

0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.

Table 2: Summary of waiting time (Minutes) before service of bank Customers

n	Mean	Med.	Var.	Skewness	Kurtosis
100	9.877	8.100	52.3741	1.4727	5.5403

Table 3: Analysis of the performance of the competing distributions

Models	Estimates	LL	AIC
WEIED	$\hat{\alpha} = 2.975$ $\hat{\lambda} = 2.871$ $\hat{\beta} = 0.597$	-330. 766	667.533
EIED	$\hat{\alpha} = 2.314$ $\hat{\lambda} = 2.311$	-336.558	677.117

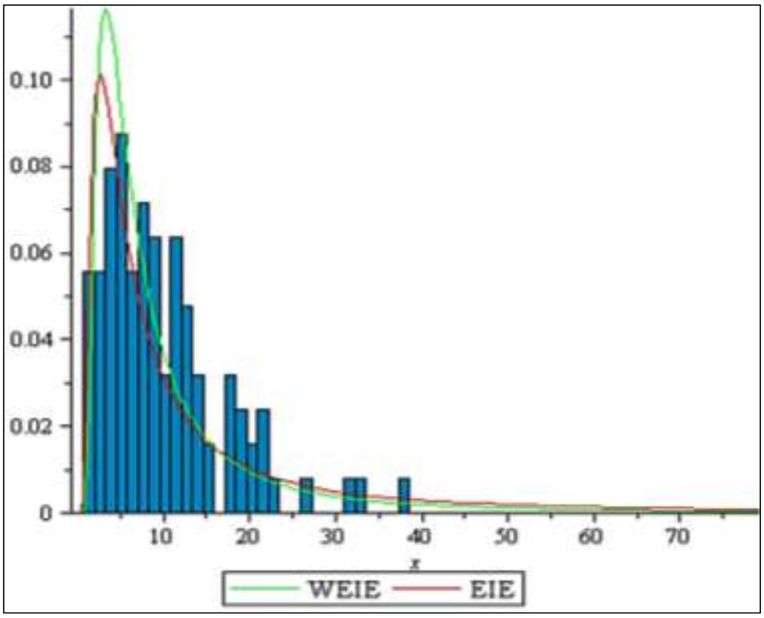

Fig 1: Histogram and theoretical densities

Remark: From Table 3, the WEIE distribution has the highest log-likelihood value and the lowest AIC value, therefore, it can be concluded that it fits the data set better than the Exponentiated Inverted Exponential distribution.

4. Conclusion

The Weighted Exponentiated Inverted Exponential distribution has been successfully derived. The model has unimodal (inverted bathtub) and decreasing shapes (depending on the value of the parameters). Explicit expressions for its basic statistical properties have been successfully derived. The model exhibits unimodal and decreasing failure rates, this implies that the model can be used to describe and model real life phenomena with unimodal or decreasing failure rates. For the real life application provided, the Weighted Exponentiated Inverted Exponential distribution performs better than the Exponentiated Inverted Exponential distribution; it is however a good and competitive model.

Conflict of interest

The authors declare that there is no conflict of interest.

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