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Poisson generalized Rayleigh distribution with properties and application

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Abstract
In this study, we have established a new three-parameter Poisson generalized Rayleigh distribution using the Poisson-Generating family of distribution. Some important mathematical and statistical properties of the proposed distribution including probability density function, cumulative distribution function, reliability function, hazard rate function, quantile, median, the measure of skewness, and kurtosis are presented. The parameters of the new distribution are estimated using the maximum likelihood estimation (MLE) method, and constructed the asymptotic confidence intervals also the Fisher information matrix is derived analytically to obtain the variance-covariance matrix for MLEs. All the computations are performed in R software. The capability and applicability of the proposed distribution is exposed by using graphical methods and statistical tests considering a real dataset. We have empirically verified that the Poisson generalized Rayleigh distribution provided a better fit and more flexible in comparison with some selected lifetime distributions.

Keywords: Poisson distribution, generalized Rayleigh distribution, reliability, estimation

1. Introduction

In the survival/reliability study, lifetime distributions are used to describe the length of the life of a component or device or a system. The lifetime distributions are most frequently used in the field like biological and medical sciences, engineering and manufacturing, etc. The exponential distribution is the most frequently used distribution due to the existence of simple elegant closed-form solutions to many survival analysis problems. The failure rate of the exponential distribution is constant but in real practice, the failure rates are not always constant. Hence in some situations, it seems to be inadequacy and unrealistic. For this, some modifications are needed to make exponential distribution more flexible. In recent, a new class of models has been introduced based on the modification of exponential distribution.

Gupta and Kundu (1999) [6] introduced the generalized exponential (GE) distribution, this extended family can accommodate data with increasing and decreasing failure rate functions, Nadarajah and Kotz (2006) [24] have introduced a generalization referred to as the beta exponential distribution generated from the logit of a beta random variable. There are lots of lifetime models which are obtained by compounding with Zero truncated Poisson distribution some of them are as follows,

A two-parameter exponential Poisson (EP) distribution has presented Kus (2007) [13] by modifying the exponential distribution with zero truncated Poisson distribution with a decreasing failure rate. The CDF of PE distribution is,

$$G\left(x;\beta,\lambda\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-e^{\left\{-\lambda\left(1-e^{-\beta x}\right)\right\}}\right] \quad ; x>0,\left(\beta,\lambda\right)>0 \tag{1.1}$$

While Barreto-Souza and Cribari-Neto (2009) [2] have introduced generalized EP distribution having the decreasing or increasing or upside-down bathtub shaped failure rate. This is the generalization of the distribution proposed by Kus (2007) [13] adding a power parameter to this distribution. Following a similar approach, Percontini *et al*. (2013) [18] have proposed the five-parameter beta Weibull Poisson distribution, which is obtained by compounding the Weibull Poisson and beta distributions. Following the same trend, Cancho (2011) [3] has developed a new distribution family also based on the exponential distribution with an increasing failure rate function known as Poisson exponential (PE) distribution. The cumulative distribution function of PE distribution can be expressed as

$$W\left(z;\lambda,\theta\right)=1-\frac{1-\mathrm{e}^{\left\{-\theta\left(1-e^{-\lambda z}\right)\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; z>0,\left(\lambda,\theta\right)>0 \tag{1.2}$$

A two-parameter Poisson-exponential with increasing failure rate has been defined by (Louzada-Neto *et al*., 2011) [15] by using the same approach as used by (Cancho, 2011) [3] under the Bayesian approach. Alkarni and Oraby (2012) [1] have introduced a new lifetime family of distribution with a decreasing failure rate which is obtained by compounding truncated Poisson distribution and a lifetime model. The cumulative distribution function of the Poisson generating family is given by,

$$F_{Poisson}\left(x;\lambda,\underline{\omega}\right)=1-\frac{1-e^{\left\{-\lambda G\left(y,\underline{\omega}\right)\right\}}}{\left(1-e^{-\lambda}\right)} \quad ; \ \lambda>0 \tag{1.3}$$

Where $\underline{\omega}$ the parameter is space and $G\left(y,\underline{\omega}\right)$ is the cumulative distribution function of any distribution. Using a parallel approach the Weibull power series class of distributions with Poisson has presented by (Morais & Barreto-Souza, 2011) [21]. Mahmoudi and Sepahdar (2013) [19] have defined a new four-parameter distribution with increasing, decreasing, bathtub-shaped, and unimodal failure rate called as the exponentiated Weibull–Poisson (EWP) distribution which has obtained by compounding exponentiated Weibull (EW) and Poisson distributions. Similarly, Lu and Shi (2012) have created the new compounding distribution named the Weibull–Poisson distribution having the shape of decreasing, increasing, upside-down bathtub-shaped, or unimodal failure rate function. Further Kaviayarasu and Fawaz (2017) [10] have made an extensive study on Weibull–Poisson distribution through a reliability sampling plan. Kyurkchiev *et al*. (2018) [14] has used the exponentiated exponential-Poisson as the software reliability model. Joshi & Kumar (2020) [9] has presented Poisson exponential power distribution and used different estimation methods to estimate the model parameter and hazard rate function can have increasing or j-shaped hazard rate. Chaudhary and Kumar (2020) [5] have presented the Poisson Inverse NHE distribution. Louzada *et al*. (2020) [16] has used different estimation methods to estimate the parameter of exponential-Poisson distribution using rainfall and aircraft data. The new Lindley-Rayleigh distribution has introduced by (Joshi & Kumar, 2020) [8] by compounding Rayleigh distribution with Lindley distribution.

In this study, we propose a new distribution based on the generalized Rayleigh distribution has introduced by Surles and Padgett (2001) [30] introduced two-parameter Burr Type X distribution and correctly named as the generalized Rayleigh distribution for more detail see also (Surles and Padgett, 2004) [31]. Note that the two-parameter generalized Rayleigh distribution is a particular member of the exponentiated Weibull distribution, originally proposed by Mudholkar and Srivastava (1993) [22]. Kundu and Raqab (2005) [12] have presented the different estimation methods to estimate the unknown parameters of the generalized Rayleigh distribution. Chaudhary and Kumar (2020) [4] have presented the Logistic – Rayleigh distribution with increasing or j-shaped hazard rate function.

The CDF and PDF of the generalized Rayleigh distribution are respectively as

$$F(x;\alpha,\beta)=\left\{1-\exp(-\beta x^2)\right\}^{\alpha} \quad ; \ \alpha,\beta>0, \ x>0 \tag{1.4}$$

and

$$f(x;\alpha,\beta)=2\alpha\beta^2 x e^{-\beta x^2}\left[1-e^{-\beta x^2}\right]^{\alpha-1} \ ; \ \alpha,\beta>0, \ x>0 \tag{1.5}$$

The main objective of this study is to develop a more flexible distribution by adding just one extra parameter to the generalized Rayleigh distribution to achieve a better fit to real data. The different sections of this study are arranged as follows; in Section 2 we present the new distribution Poisson generalized Rayleigh distribution (PGR) along with its mathematical and statistical properties. We widely discuss the maximum likelihood estimation method in Section 3. In Section 4 using a real dataset, we present the estimated values of the model parameters and their corresponding asymptotic confidence intervals and observed the Fisher information matrix. Besides, we have illustrated the different test criteria to assess the goodness of fit of the proposed model. Some concluding remarks are presented in Section 5.

2. The Poisson Generalized Rayleigh (PGR) Distribution

Alkarni and Oraby (2012) [1] have introduced a new lifetime class with a decreasing failure rate which is obtained by compounding zero truncated Poisson distribution and a lifetime distribution, where the compounding procedure follows the same way that was previously carried out by (Adamidis & Loukas, 1998). Let $G(x)$ and $g(x)$ be the baseline cumulative distribution function and probability density function respectively then the Poisson family may be defined as,

$$F\left(x;\lambda,\underline{\Psi}\right)=1-\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda G(x;\underline{\Psi})\right\}\right] \quad ; x>0,\, \lambda>0 \tag{2.1}$$

$$f\left(x;\lambda,\underline{\Psi}\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\lambda g(x;\lambda,\underline{\Psi})\exp\left\{-\lambda G(x;\underline{\Psi})\right\} \quad ; x>0,\, \lambda>0 \tag{2.2}$$

where $G(x;\underline{\Psi})$ and $g(x;\lambda,\underline{\Psi})$ are the CDF and PDF of any baseline distribution $\underline{\Psi}$ the parameter space of the baseline distribution. Taking (1.4) and (1.5) as baseline distribution then (2.1) and (2.2) is reduced to the Poisson generalized Rayleigh (PGR) distribution can be defined as, Let X be a nonnegative random variable representing the lifetime of an item or component or a system in some population. The random variable X is said to follow the PGR distribution with parameters $(\alpha,\beta,\lambda)>0$ if its cumulative distribution function is given by

$$F\left(x\right)=\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\left(1-e^{-\beta x^2}\right)^{\alpha}\right\}\right]; x>0, \tag{2.3}$$

And its corresponding probability density function is

$$f\left(x\right)=\frac{2\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}xe^{-\beta x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x^2}\right)^{\alpha}\right\}; x>0 \tag{2.4}$$

Reliability function

The reliability function of PGR is denoted by $R(t)$, which is the probability of an item not failing up to time *t*, is defined by $R(t)=1-F(t)$. The survival /reliability function of the Poisson generalized Rayleigh distribution is given by

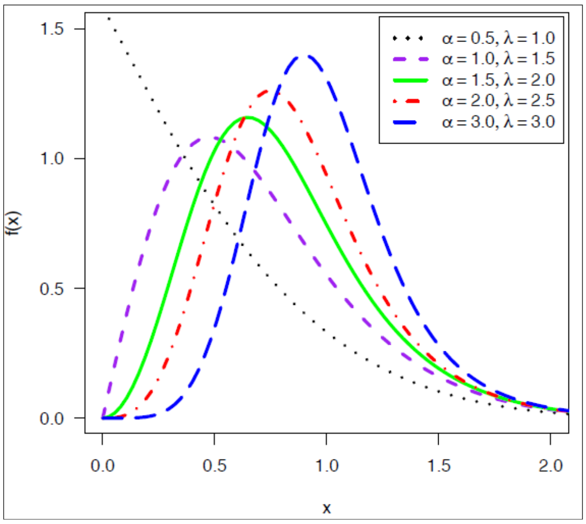
$$R\left(t;\alpha,\beta,\lambda\right)=1-\frac{1}{\left(1-e^{-\lambda}\right)}\left[1-\exp\left\{-\lambda\left(1-e^{-\beta x^2}\right)^{\alpha}\right\}\right]; t>0 \tag{2.5}$$

The hazard rate function (HRF)

The hazard rate function for the PGR distribution can be defined as; $h(x)=\frac{f(x)}{R(x)}$ here R(x) is a reliability function. Hence let, $X \sim PGR(\alpha,\beta,\lambda)$ then its hazard rate function is

$$h\left(x;\alpha,\beta,\lambda\right)=\frac{2\alpha\beta\lambda xe^{-\beta x^2}\left(1-e^{-\beta x^2}\right)^{\alpha-1}\exp\left\{-\lambda\left(1-e^{-\beta x^2}\right)^{\alpha}\right\}}{\exp\left\{-\lambda\left(1-e^{-\beta x^2}\right)^{\alpha}\right\}-e^{-\lambda}}; x>0 \tag{2.6}$$

In Figure 1 we have demonstrated the graph for PDF and hazard function for PGR distribution for different values of the parameters. From Figure 1 (left panel), the density function of the PGR distribution can exhibit different shapes according to the values of the parameters. Figure 1 (right panel) demonstrates the increasing, decreasing, the j-shaped, and constant shape of the hazard rate.



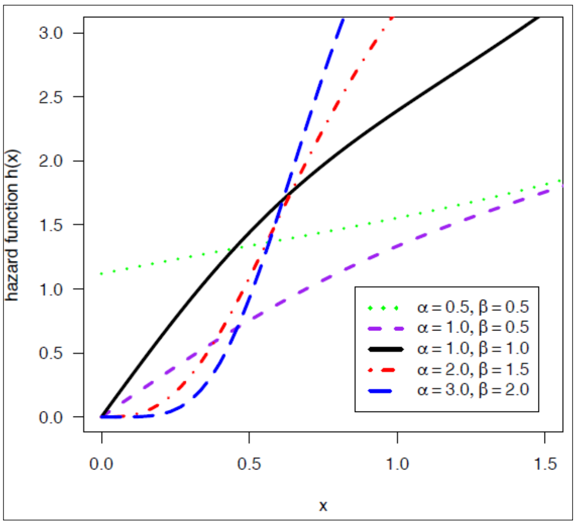


Fig 1: Graph of PDF (left panel) and hazard function (right panel) for different values of the parameters.

The quantile function of PGR distribution

According to Hyndman and Fan (1996), the value of the p^{th} quantile can be obtained by solving the following equation,

$$Q(p) = F^{-1}(p)$$

And we get the quantile function by inverting CDF of PGR as

$$Q(p) = \left[-\frac{1}{\beta} \ln \left[1 - \left[-\frac{1}{\lambda} \ln \left\{ 1 - p(1 - e^{-\lambda} \right\} \right] \right]^{\frac{1}{\alpha}} \right]^{1/2} \quad ; 0 < p < 1 \tag{2.7}$$

For the generation of the random numbers of the PGR distribution, we suppose simulating values of random variable X with the CDF (2.3). Let U denote a uniform random variable in (0, 1), then the simulated values of X can be obtained by

$$X = \left[-\frac{1}{\beta} \ln \left[1 - \left[-\frac{1}{\lambda} \ln \left\{ 1 - v(1 - e^{-\lambda} \right\} \right] \right]^{\frac{1}{\alpha}} \right]^{1/2} \quad ; 0 < v < 1 \tag{2.8}$$

Median of PGR distribution

The median of *X* from the PGR distribution is simply obtained by replacing p = 0.5 in equation (2.7) which gives

$$Median = \left[-\frac{1}{\beta} \ln \left[1 - \left[-\frac{1}{\lambda} \ln \left\{ 1 - 0.5(1 - e^{-\lambda} \right\} \right] \right]^{\frac{1}{\alpha}} \right]^{1/2}$$

Skewness and Kurtosis

The Bowley's measure of skewness based on quartiles was defined by (Kennedy & Keeping, 1962) [11] as,

$$Skewness(B) = \frac{Q(3/4) + Q(1/4) - 2Q(1/2)}{Q(3/4) - Q(1/4)}$$

and the coefficient of Moor's kurtosis measures based on octiles was defined by (Moors, 1988) [20] is given by

$$Kurtosis(M) = \frac{Q(0.875) - Q(0.625) + Q(0.375) - Q(0.125)}{Q(3/4) - Q(1/4)}$$

3. Methods of Estimation

The objective of estimation is to approximate the value of a model parameter based on sample information. The estimation theory deals with the basic problem of inferring some relevant features of a random experiment based on the observation of the experiment outcomes. There are so many methods for estimating unknown parameters of the model. We have considered the maximum likelihood estimation (MLE), method.

Maximum Likelihood Estimation (MLE) method

In this section, we have discussed the maximum likelihood estimators (MLE's) of the PGR (α, β, λ) distribution. Let $\underline{x} = (x_1, \ldots, x_n)$ be the observed values of size 'n' from PGR(α, β, λ), then the likelihood function for the parameter vector $\Upsilon = (\alpha, \beta, \lambda)^T$ can be written as,

$$L(\Upsilon) = \frac{2\alpha\beta\lambda}{\left(1 - e^{-\lambda}\right)} \prod_{i=1}^{n} x_i e^{-\beta x_i^2} \left(1 - e^{-\beta x_i^2}\right)^{\alpha - 1} \exp\left\{-\lambda\left(1 - e^{-\beta x_i^2}\right)^{\alpha}\right\}$$

It is easy to deals with the log-likelihood function as,

$$l = n\ln(2\alpha\beta\lambda) - n\ln(1 - e^{-\lambda}) + \sum_{i=1}^{n} \ln x_i - \beta \sum_{i=1}^{n} x_i^2 + (\alpha - 1)\sum_{i=1}^{n} \ln(1 - e^{-\beta x_i^2}) - \lambda \sum_{i=1}^{n} (1 - e^{-\beta x_i^2})^{\alpha} \tag{3.1}$$

The elements of the score function $Z(\Upsilon) = (Z_\alpha, Z_\beta, Z_\lambda)$ are obtained as

$$Z_\alpha = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln B(x_i) - \lambda \sum_{i=1}^{n} (B(x_i))^{\alpha} \ln B(x_i)$$

$$Z_\beta = \frac{n}{\beta} - \sum_{i=1}^{n} x_i^2 - (\alpha - 1)\sum_{i=1}^{n} \frac{x_i^2}{B(x_i)} - \alpha\lambda \sum_{i=1}^{n} x_i^2 e^{-\beta x_i^2} \left[B(x_i)\right]^{\alpha - 1} \tag{3.2}$$

$$Z_\lambda = \frac{n}{\lambda} - \frac{n}{(e^{\lambda} - 1)} - \sum_{i=1}^{n} \left[B(x_i)\right]^{\alpha}$$

where $B(x_i) = 1 - e^{-\beta x_i^2}$

Equating Z_α, Z_β *and* Z_λ to zero and solving these non-linear equations simultaneously gives the MLE $\hat{\Upsilon} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ of $\Upsilon = (\alpha, \beta, \lambda)^T$. These equations cannot be solved analytically and by using the computer software R, Mathematica, Matlab, or any other programs and Newton-Raphson's iteration method, one can solve these equations.

Let us denote the parameter vector by $\Upsilon = (\alpha, \beta, \lambda)^T$ and the corresponding MLE of Υ as $\hat{\Upsilon} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ then the asymptotic normality results in, $(\hat{\Upsilon} - \Upsilon) \to N_3\left[0, (I(\Upsilon))^{-1}\right]$ where $I(\Upsilon)$ is the Fisher's information matrix given by,

$$I(\Upsilon) = -\begin{pmatrix} E(Z_{\alpha\alpha}) & E(Z_{\alpha\beta}) & E(Z_{\alpha\lambda}) \\ E(Z_{\alpha\beta}) & E(Z_{\beta\beta}) & E(Z_{\beta\lambda}) \\ E(Z_{\alpha\lambda}) & E(Z_{\beta\lambda}) & E(Z_{\lambda\lambda}) \end{pmatrix}$$

Further differentiating (3.2) we get,

$$Z_{\alpha\alpha} = -\frac{n}{\alpha^2} - \lambda \sum_{i=1}^{n} \left[B(x_i)\right]^{\alpha} \left[\ln B(x_i)\right]^2$$

$$Z_{\beta\beta} = -\frac{n}{\beta^2} - (\alpha - 1)\sum_{i=1}^{n} \frac{x_i^4 e^{\beta x_i^2}}{\left[\ln B(x_i)\right]^2} + \alpha\lambda \sum_{i=1}^{n} \frac{x_i^4 \left[B(x_i)\right]^{\alpha} \left(e^{\beta x_i^2} - \alpha\right)}{\left[\ln B(x_i)\right]^2}$$

$$Z_{\lambda\lambda} = -\frac{n}{\lambda^2} + \frac{ne^{\lambda}}{(e^{\lambda} - 1)^2}$$

$$Z_{\alpha\beta} = -\sum_{i=1}^{n} \frac{x_i^2}{B(x_i)} + \lambda \sum_{i=1}^{n} \frac{x_i^2 \left[\ln B(x_i)\right]^{\alpha} \left\{\alpha \ln B(x_i) + 1\right\}}{B(x_i)}$$

$$Z_{\alpha\lambda} = -\sum_{i=1}^{n} \left[B(x_i)\right]^{\alpha} \left[\ln B(x_i)\right]$$

$$Z_{\beta\lambda} = -\alpha \sum_{i=1}^{n} x_i^2 e^{\beta x_i^2} \left\{B(x_i)\right\}^{\alpha - 1}$$

The observed fisher information matrix $O(\hat{\Upsilon})$ as an estimate of the information matrix $I(\Upsilon)$ given by

$$O(\hat{\Upsilon}) = -\begin{pmatrix} Z_{\alpha\alpha} & Z_{\alpha\beta} & Z_{\alpha\lambda} \\ Z_{\alpha\beta} & Z_{\beta\beta} & Z_{\beta\lambda} \\ Z_{\alpha\lambda} & Z_{\beta\lambda} & Z_{\lambda\lambda} \end{pmatrix}_{\big|(\hat{\alpha}, \hat{\beta}, \hat{\lambda})} = -H(\Upsilon)_{\big|(\Upsilon = \hat{\Upsilon})}$$

Where H is the Hessian matrix.

The Newton-Raphson algorithm to maximize the likelihood produces the observed information matrix. Therefore, the variance-covariance matrix is given by,

$$\left[-H(\Upsilon)_{\big|(\Upsilon = \hat{\Upsilon})}\right]^{-1} = \begin{pmatrix} \operatorname{var}(\hat{\alpha}) & \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\beta}) & \operatorname{var}(\hat{\beta}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) \\ \operatorname{cov}(\hat{\alpha}, \hat{\lambda}) & \operatorname{cov}(\hat{\lambda}, \hat{\beta}) & \operatorname{var}(\hat{\lambda}) \end{pmatrix}$$

Hence from the asymptotic normality of MLEs, approximate 100(1-α) % confidence intervals for α, β, and θ can be constructed as,

$\hat{\alpha} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\alpha})}$, $\hat{\beta} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\beta})}$ and $\hat{\lambda} \pm Z_{\alpha/2}\sqrt{\operatorname{var}(\hat{\lambda})}$, where $Z_{\alpha/2}$ is the upper percentile of standard normal variate.

4. Application with a real dataset

In this section, we have illustrated the applicability of PGR distribution using a real dataset used by previous researchers. The data gives 100 observations on breaking the stress of carbon fibers (in Gba) (Nichols & Padgett, 2006) [25].

3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41, 3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97, 3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31, 3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17, 2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59, 3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17, 5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48, 0.85, 1.61, 2.79, 4.70, 2.03, 1.80,1.57, 1.08, 2.03, 1.61, 2.12, 1.89, 2.88, 2.82, 2.05, 3.65

The plots of profile log-likelihood function for the parameters α, β and λ have been displayed in Figure 2, and it is noticed that the ML estimates can be uniquely determined.

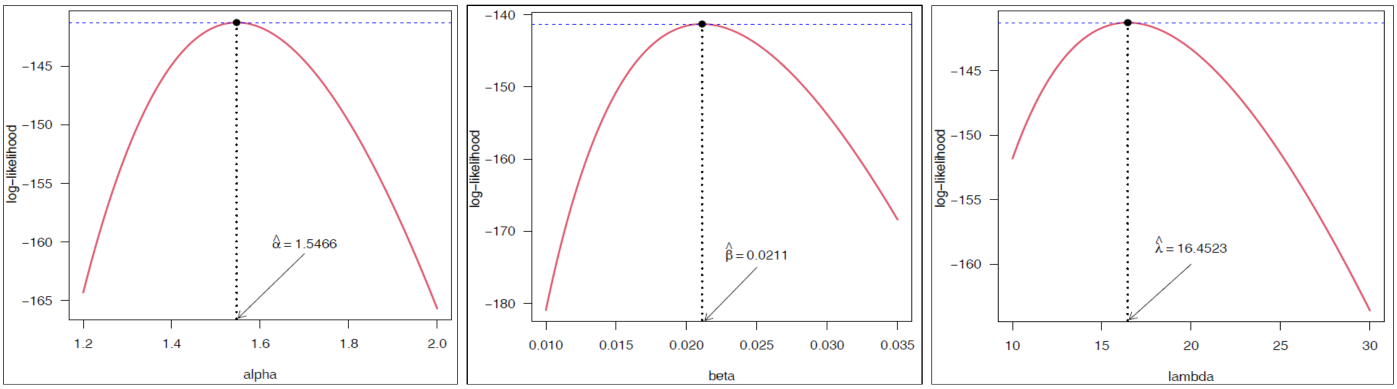


Fig 2: Graph of Profile log-likelihood function for the parameters α, β and λ.

The maximum likelihood estimates are calculated directly by using the optim() function in R software (R Core Team, 2020) [26] and (Rizzo, 2008) [28] by maximizing the likelihood function (3.1). We have obtained $\hat{\alpha}$ = 9.3340, $\hat{\beta}$ = 0.3010, $\hat{\lambda}$ = 104.4248 and corresponding Log-Likelihood value is -141.2833. In Table 1 we have demonstrated the MLE's with their standard errors (SE) and 95% confidence interval for α, β, and λ.

Table 1: MLE, SE and 95% confidence interval

Parameter	MLE	SE	95% ACI
alpha	1.5466	0.1492	(1.2542, 1.8389)
beta	0.0211	0.0053	(0.0106, 0.0315)
lambda	16.4523	2.9579	(10.6548, 22.2498)

The Hessian variance-covariance matrix is obtained as,

$$\left[-H(\Upsilon)\Big|_{(\Upsilon=\hat{\Upsilon})}\right]^{-1} = \begin{pmatrix} 0.02225 & 6.404\text{e-}04 & -0.0843 \\ 0.00064 & 2.843\text{e-}05 & -0.0103 \\ -0.0843 & -1.040\text{e-}02 & 8.7492 \end{pmatrix}$$

To judge the goodness-of-fit of the Poisson generalized Rayleigh distribution, we have considered some well-known distribution for comparison purpose which are listed below,

1. Exponentiated Exponential Poisson (EEP)

The probability density function of EEP (Ristić & Nadarajah, 2014) [27] can be expressed as

$$f(x) = \frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)} e^{-\beta x}\left(1-e^{-\beta x}\right)^{\alpha-1} \exp\left\{-\lambda\left(1-e^{-\beta x}\right)^{\alpha}\right\} \quad ; x>0, \alpha>0, \lambda>0$$

2. Gamma distribution

The density of Gamma distribution with parameters α and θ is

$$f_{GA}(x) = \frac{1}{\theta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} \mathrm{e}^{-x/\theta} \quad ; x \geq 0, \alpha>0, \theta>0.$$

3. Poisson–exponential distribution (PE)

The probability density function of Poisson–exponential (PE) distribution was defined by (Louzada-Neto *et al*., 2011) [15] also it was used by (Rodrigues *et al*., 2018) [29] is

$$f(x) = \frac{\beta\lambda}{\left(1-e^{-\lambda}\right)} e^{-\beta x} \exp\left(-\lambda e^{-\beta x}\right); \quad \beta>0, \lambda>0, x>0$$

4. Generalized Exponential (GE) distribution

The probability density function of generalized exponential distribution (Gupta & Kundu, 1999) [6] can be written as

$$f_{GE}(x;\alpha,\lambda)=\alpha\,\lambda\,e^{-\lambda x}\left\{1-e^{-\lambda x}\right\}^{\alpha-1};(\alpha,\lambda)>0,x>0.$$

5. Gompertz distribution (GZ)

The probability density function of Gompertz distribution (Murthy *et al*., 2003) [23] with parameters α and θ is

$$f_{GZ}(x)=\theta\ e^{\alpha x}\ exp\left\{\frac{\theta}{\alpha}\left(1-e^{\alpha x}\right)\right\}\quad ;x\geq 0,\theta>0,-\infty<\alpha<\infty.$$

To assess the goodness of fit of a given distribution we generally use the PDF and CDF plot. To get the additional information we have to plot Q-Q and KS plots. In particular, the Q-Q plot may provide information about the lack-of-fit at the tails of the distribution, whereas the KS plot emphasizes the lack-of-fit. From Figure 4 it is verified that the PGR model fits the data very well.

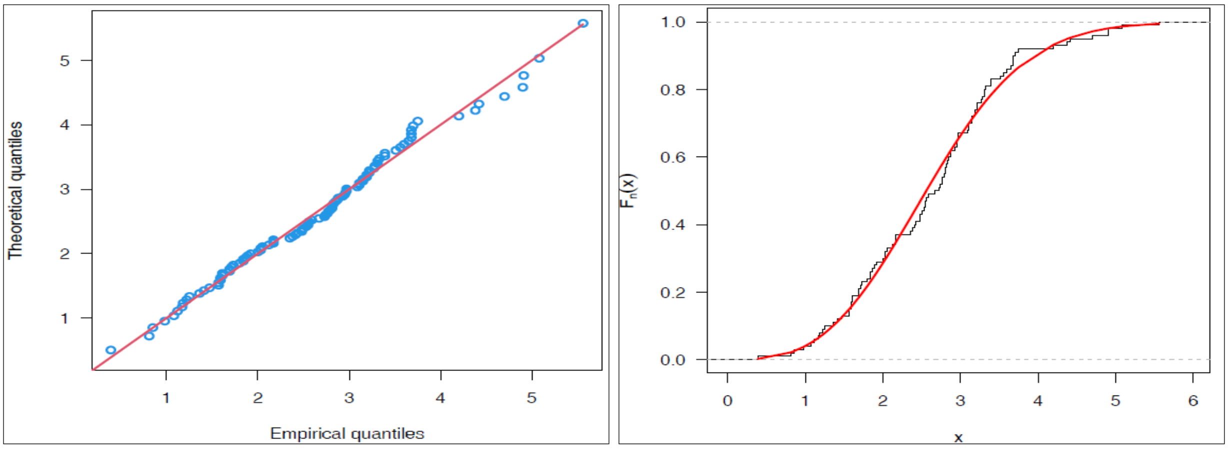


Fig 4: The P-P plot (left panel) and KS plot (right panel) of PGR distribution

For the assessment of potentiality of the Poisson generalized Rayleigh distribution, we have calculated the Akaike information criterion (AIC), Bayesian information criterion (BIC), Corrected Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) which are presented in Table 4.

Table 2: Log-likelihood (LL), AIC, BIC, CAIC and HQIC

Model	-LL	AIC	BIC	CAIC	HQIC
PGR	141.2833	288.5666	296.3821	288.8166	291.7296
EEP	141.3240	288.6481	296.4636	288.8981	291.8111
Gamma	143.2336	290.4673	295.6776	290.5910	292.5760
PE	144.2051	292.4102	297.6205	292.5339	294.5189
GE	146.1823	296.3646	301.5749	296.4883	298.4733
GZ	149.1250	302.2500	307.4604	302.3737	304.3588

The Histogram and the density function of fitted distributions and Empirical distribution function with the estimated distribution function of PGR and some selected distributions are presented in Figure 5.

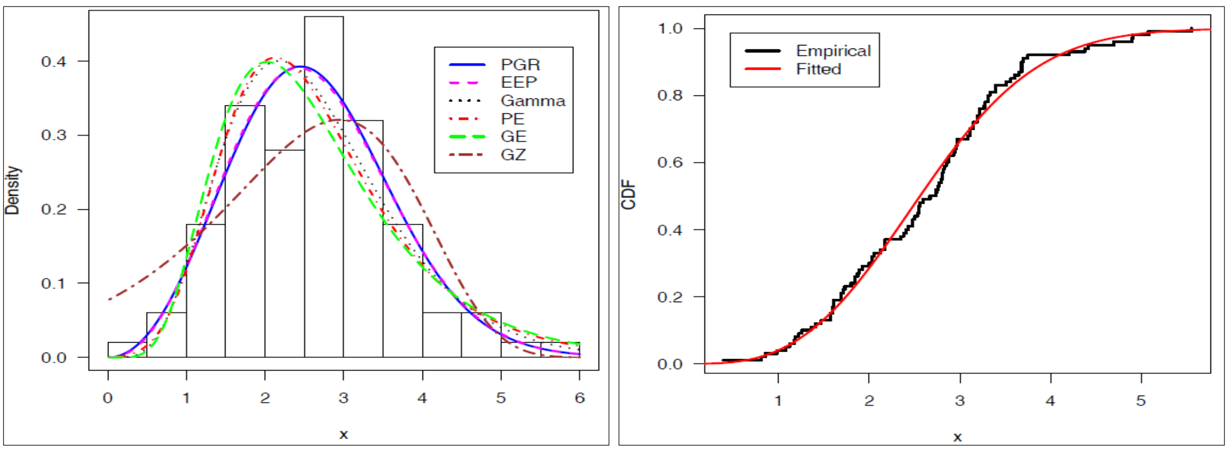


Fig 5: The Histogram and the density function of fitted distributions (left panel) and Empirical distribution function with estimated distribution function (right panel).

To compare the goodness-of-fit of the PGR distribution with other competing distributions we have presented the value of Kolmogorov-Simnorov (KS), the Anderson-Darling (AD) and the Cramer-Von Mises (CVM) statistics in Table 5. It is observed that the PGR distribution has the minimum value of the test statistic and higher *p*-value thus we conclude that the PGR distribution gets quite better fit and more consistent and reliable results from others taken for comparison.

Table 3: The goodness-of-fit statistics and their corresponding p-value

Model	*KS(p-value)*	*AD(p-value)*	*CVM(p-value)*
PGR	0.0656(0.7821)	0.0682(0.7636)	0.3931(0.8553)
EEP	0.0675(0.7527)	0.0723(0.7385)	0.4099(0.8385)
Gamma	0.0934(0.3479)	0.1500(0.3902)	0.7583(0.5117)
PE	0.0954(0.3229)	0.1724(0.3284)	0.9157(0.4044)
GE	0.1078(0.1959)	0.2293(0.2174)	1.2250(0.2581)
GZ	0.0962(0.3129)	0.2280(0.2193)	1.7537(0.1261)

5. Conclusion

In this study, we have studied the three-parameter Poisson generalized Rayleigh distribution. For our study, we provided the probability density function, the cumulative distribution function, and the shapes of the hazard rate function. The shape of the PDF of the PGR distribution is unimodal and positively skewed, while the hazard function of the PGR distribution is increasing, decreasing, and the j-shaped and constant shape of the hazard rate. The P-P and Q-Q plots showed that the purposed distribution is quite better for fitting the real dataset. Finally, using a real data set we have explored some well-known estimation methods namely the maximum likelihood estimation (MLE) method. Further, we also construct the asymptotic confidence interval for MLEs. The application illustrates that the proposed model provides a consistently better fit than other underling models. We expect that this model will contribute to the field of survival analysis.

6. References

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