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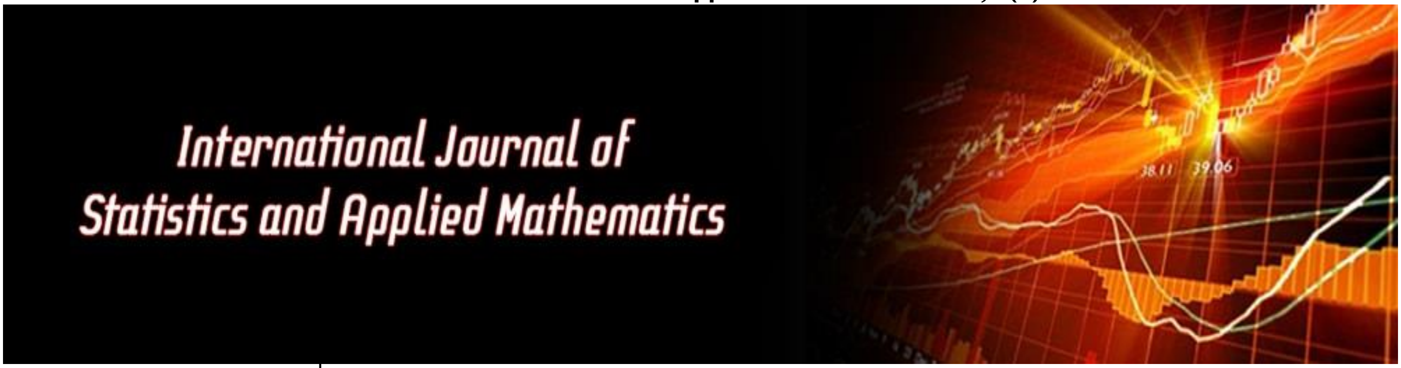

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P_1-Curvature tensor in the space time of general relativity

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Abstract
The P_1 - curvature tensor defined from W_3 - curvature tensor has been studied in the spacetime of general relativity. The Bianchi like differential identity is satisfied by P_1 - tensor if and only if the Ricci tensor is of Codazzi type. It is shown that Einstein like field equations can be expressed with the help of the contracted part of P_1 - tensor, which is conserved if the energy momentum tensor is Codazzi type. Considering P_1 -flat space time satisfying Einstein's field equations with cosmological term, the existence of Killing vector field ξ is shown if and only if the Lie derivative of the energy-momentum tensor vanishes with respect to ξ, as well as admitting a conformal Killing vector field is established if and only if the energy-momentum tensor has the symmetry inheritance property. Finally for a P_1 - flat perfect fluid spacetime satisfying Einstein's equations with cosmological term, some results are obtained.

Keywords: P_1 - curvature tensor, Einstein field equations, killing vector, perfect fluid space time *AMS classification: 53C50, 53C45*

Introduction
Consider an n-dimensional space V_n in which the curvature tensorW_3 has been defined (Pokhariyal, 1973) [7].

$$W_3(X,Y,Z,T)=R(X,Y,Z,T)+\frac{1}{n-1}\left[g(Y,Z)Ric(X,T)-g(Y,T)Ric(X,Z)\right] \tag{1.1}$$

It was noticed that

$$W_3(X,Y,Z,T)=-W_3(X,Y,T,Z);\ W_3(X,Y,Z,T)\neq -W_3(Y,X,Z,T) \text{ and } W_3(X,Y,Z,T)\neq W_3(Z,T,X,Y). \tag{1.2a}$$

$$W_3(X,Y,Z,T)+W_3(Y,Z,X,T)+W_3(Z,X,Y,T)\neq 0;\ W_3(X,Y,Z,T)+W_3(X,Z,T,Y)+W_3(X,T,Y,Z)=0. \tag{1.2b}$$

This tensor has been studied in the space time of general relativity as well as various manifolds of differential geometry. For example, the contracted part of W_3 has been used to replace Ricci tensor in the Rainich (1952) [9] conditions for the existence of non-null electrovariance and the scalar invariant W_3 was found to be double of Riemann scalar curvature (Pokhariyal 1973) [7]. This tensor was studied in almost Tachibana and almost Kahler spaces as well as almost product and almost decomposable manifolds (Pokhariyal 1975) [8]. Moindi (2007) [5] has investigated its properties in Sasakian and *LP*– Sasakian manifolds. Pambo*et al*. (2020) [6] have studied Eta-Ricci Soliton on W_3 - semi symmetric *LP* – Sasakian manifold. In this paper, we study the tensorP_1 (X, Y, Z, T), defined by breaking W_3 - tensor in the skew symmetric parts in X, Y, in the space time of general relativity.

2. P_1 - Curvature tensor
The tensor $P_1(X, Y, Z, T)$, has been defined (Pokhariyal 1973) [7], as

$$\begin{aligned} P_1(X,Y,Z,T)&=\frac{1}{2}[W_3(X,Y,Z,T)-W_3(Y,X,Z,T)]\\ &=R(X,Y,Z,T)+\frac{1}{2(n-1)}\left[g(Y,Z)Ric(X,T)-g(Y,T)Ric(X,Z)-g(X,Z)Ric(Y,T)+g(X,T)Ric(Y,Z)\right] \end{aligned} \tag{2.1}$$

named as Pokhariyal Tensor$_1$, which possesses all symmetric and skew-symmetric as well as the cyclic properties satisfied by Riemann curvature tensor. Thus, we have

$$P_1(X,Y,Z,T) = -P_1(Y,X,Z,T);\ P_1(X,Y,Z,T) = -P_1(X,Y,T,Z) \text{ and } P_1(X,Y,Z,T) = P_1(Z,T,X,Y). \tag{2.2}$$

$$P_1(X,Y,Z,T) + P_1(X,Z,T,Y) + P_1(X,T,Y,Z) = 0 \text{ and } P_1(X,Y,Z,T) + P_1(Y,Z,X,T) + P_1(Z,X,Y,T) = 0 \tag{2.3}$$

In V_4 the tensor $P_1(X,Y,Z,T)$ becomes

$$P_1(X,Y,Z,T)= R(X,Y,Z,T) + (1/6)[g(Y,Z)Ric(X,T)- g(Y,T)Ric(X,Z)- g(X,Z)Ric(Y,T)+ g((X,T)Ric(Y,Z)] \tag{2.4}$$

The Bianchi differential identity is given by

$$(\nabla_U R)(X,Y,Z,T) + (\nabla_Z R)(X,Y,T,U) + (\nabla_T R)(X,Y,U,Z) = 0 \tag{2.5}$$

In order to check if $P_1(X, Y, Z, T)$ tensor satisfies Bianchi differential identity, we compute:

$$(\nabla_U P_1)(X,Y,Z,T) + (\nabla_Z P_1)(X,Y,T,U) + (\nabla_T P_1)(X,Y,U,Z) \tag{2.6}$$

$$= (\nabla_U R)(X,Y,Z,T) + \frac{1}{6}\left[g(Y,Z)(\nabla_U Ric)(X,T) - g(Y,T)(\nabla_U Ric)(X,Z) - g(X,Z)(\nabla_U Ric)(Y,T) + g(X,T)(\nabla_U Ric)(Y,Z)\right]$$
$$+ (\nabla_T R)(X,Y,U,Z) + \frac{1}{6}\left[g(Y,U)(\nabla_T Ric)(X,Z) - g(Y,Z)(\nabla_T Ric)(X,U) - g(X,U)(\nabla_T Ric)(Y,Z) + g(X,Z)(\nabla_T Ric)(Y,U)\right]$$
$$+ (\nabla_Z R)(X,Y,T,U) + \frac{1}{6}\left[g(Y,T)(\nabla_Z Ric)(X,U) - g(Y,U)(\nabla_Z Ric)(X,T) - g(X,T)(\nabla_Z Ric)(Y,U) + g(X,U)(\nabla_Z Ric)(Y,T)\right]$$

By virtue of (2.5), this reduces to

$$(\nabla_U P_1)(X,Y,Z,T) + (\nabla_Z P_1)(X,Y,T,U) + (\nabla_T P_1)(X,Y,U,Z) \tag{2.7}$$

$$= \frac{1}{6}[g(Y,Z)\{(\nabla_U Ric)(X,T) - (\nabla_T Ric)(X,U)\} + g(Y,T)\{(\nabla_Z Ric)(X,U) - (\nabla_U Ric)(X,Z)\}$$
$$+ g(X,Z)\{(\nabla_T Ric)(Y,U) - (\nabla_U Ric)(Y,T)\} + g(X,T)\{(\nabla_U Ric)(X,U) - (\nabla_U Ric)(Y,U)\}$$
$$+ g(X,U)\{(\nabla_Z Ric)(X,T) - (\nabla_T Ric)(Y,Z)\} + g(Y,U)\{(\nabla_T Ric)(X,Z) - (\nabla_Z Ric)(X,T)\}].$$

If Ricci tensor is of Codazzi type (Derdzinski and Shen, 1983), then

$$(\nabla_Z Ric)(X,T) = (\nabla_T Ric)(X,Z) = (\nabla_U Ric)(X,Z). \tag{2.8}$$

Using (2.8) in equation (2.7), we get

$$(\nabla_U P_1)(X,Y,Z,T) + (\nabla_Z P_1)(X,Y,T,U) + (\nabla_T P_1)(X,Y,U,Z) = 0 \tag{2.9}$$

This is Bianchi like differential (second) identity for ***P₁*** curvature tensor.
Conversely, if ***P₁***-curvature tensor satisfies Bianchi second identity then (2.7) reduces to

$$g(Y,Z)\{(\nabla_U Ric)(X,T) - (\nabla_T Ric)(X,U)\} + g(Y,T)\{(\nabla_Z Ric)(X,U) - (\nabla_U Ric)(X,Z)\}$$
$$+ g(X,U)\{(\nabla_Z Ric)(Y,T) - (\nabla_T Ric)(Y,Z)\} + g(Y,U)\{(\nabla_T Ric)(X,Z) - (\nabla_Z Ric)(X,T)\} = 0 \tag{2.10}$$

For (2.10) to hold, on simplification, we get

$$(\nabla_Z Ric)(X,T) = (\nabla_T Ric)(X,Z) \text{ and } (\nabla_Z Ric)(X,U) = (\nabla_U Ric)(X,Z). \tag{2.11}$$

This shows that Ricci tensor is Codazzi type. Thus, we have

Theorem (2.1): For a V_4, P_1-curvature tensor satisfies Bianchi type differential identity if and only if the Ricci tensor is of Codazzi type.

We consider the P_1 curvature tensor in the index notation as:

$$P_{1\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{2(n-1)}\left[g_{\beta\gamma}R_{\alpha\delta} - g_{\beta\delta}R_{\alpha\gamma} - g_{\alpha\gamma}R_{\beta\delta} + g_{\alpha\delta}R_{\beta\gamma}\right]. \tag{2.12}$$

This can be written as

$$P^{\alpha}_{1\ \beta\gamma\delta} = R^{\alpha}_{\beta\gamma\delta} + \frac{1}{2(n-1)}\left[g_{\beta\gamma}R^{\alpha}_{\delta} - g_{\beta\delta}R^{\alpha}_{\gamma} - g^{\alpha}_{\gamma}R_{\beta\delta} + g^{\alpha}_{\delta}R_{\beta\gamma}\right]. \tag{2.13}$$

By contracting α and δ, we get on simplification

$$P_{1\beta\gamma} = \frac{3n-4}{2n-2} R_{\beta\gamma} + \frac{R}{2n-2} g_{\beta\gamma}. \quad (2.14)$$

For, $n=4$ in V_4 we have

$$P_{1\beta\gamma} = \frac{4}{3} R_{\beta\gamma} + \frac{R}{6} g_{\beta\gamma}. \quad (2.15)$$

Hence, $P_{1\beta\gamma}$ not vanish in an Einstein space.
The scalar invariant P_1 is obtained on simplification as (Pokhariyal 1973) [7]:

$$P_1 = 2R \quad (2.16)$$

2.1. Divergence of $P_{1\beta\gamma}$

The Bianchi differential identity for $P_{1\alpha\beta\mu\nu}$ from (2.9) on index notation is:

$$\nabla_\sigma P_{1\alpha\beta\mu\nu} + \nabla_\nu P_{1\alpha\beta\sigma\mu} + \nabla_\mu P_{1\alpha\beta\nu\sigma} = 0 \quad (2.17)$$

Multiplying through by $g^{\gamma\sigma} g^{\alpha\mu} g^{\beta\nu}$, we have by using the symmetric and skew symmetric properties

$$\nabla_\sigma g^{\gamma\sigma} P_1 - \nabla_\nu g^{\gamma\sigma} g^{\beta\nu} P_{1\alpha\beta} - \nabla_\mu g^{\gamma\sigma} g^{\alpha\mu} P_{1\sigma\alpha} = 0 \quad (2.18)$$

By further simplification we get

$$\nabla_\sigma g^{\gamma\sigma} P_1 - \nabla_\nu P_1^{\gamma\nu} - \nabla_\mu P_1^{\gamma\mu} = 0 \quad (2.19)$$

After renaming the indices and simplification, we have

$$\nabla_\sigma \left(P_1^{\gamma\sigma} - (1/2) P_1 g^{\gamma\sigma} \right) = 0 \quad (2.20)$$

We define

$$G_1^{\gamma\sigma} = P_1^{\gamma\sigma} - \frac{1}{2} P_1 g^{\gamma\sigma}$$

so that the divergence of $G^{\gamma\sigma}$ is zero. (2.21)

Einstein field equations, with cosmological term is given by

$$E_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2.22)$$

where $T_{\mu\nu}$ is stress-energy tensor and contains all forms of energy and momentum, κ is coupling constant with value $8\pi G/c^4$, Λ is the cosmological constant. These equations describe gravity as a consequence of space time being curved by means of mass and energy. The tensor $E_{\mu\nu}$ is determined by the curvature of space time at a specific point in space time, which is equated with the energy-momentum at that point. The Einstein field equations (2.22) can be expressed using the tensor $G_{1\mu\nu}$ defined by (2.21). Since $G_{1\mu\nu}$ is defined using $P_{1\mu\nu}$ and P_1 such that $P_{1\mu\nu}$ contains $R_{\mu\nu}$ and extra term having R and $g_{\mu\nu}$ (as compared to $R_{\mu\nu}$ used in the Einstein tensor). Further P_1 is twice the Riemann scalar curvature R and $P_{1\mu\nu}$ does not vanish in the Einstein space. Therefore $G_{1\mu\nu}$ is likely to provide different as well as additional physical and geometrical interpretations.

2.2. Conservation of $P_{1\mu\nu}$

Replacing Einstein tensor $E_{\mu\nu}$ by $G_{1\mu\nu}$, and R by P_1 the Einstein field equations in the presence of matter can be expressed as:

$$P_{1\mu\nu} - \frac{1}{2} P_1 g_{\mu\nu} = \kappa T_{\mu\nu}, \quad (2.23)$$

Multiplying this equation by $g^{\mu\nu}$, on simplification we get

$$P_1 = -\kappa T \quad (2.24)$$

Putting (2.24) in (2.23), we get

$$P_{1\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (2.25)$$

The energy-momentum tensor for the electromagnetic field is given by (Stephani, 1982):

$$T_{\mu\nu} = -F_{\mu\nu} F_\gamma^\nu + \frac{1}{4} g_{\mu\nu} F_{\delta\lambda} F^{\delta\lambda} \quad (2.26),$$

where $F_{\mu\nu}$ represents the skew-symmetric field tensor, satisfying Maxwell's equations.

From (2.23) it is clear that $T_{\mu}^{\nu} = T = 0$.

Einstein field equations (2.23) for purely electromagnetic distribution take the form

$$P_{1\mu\nu} = \kappa T_{\mu\nu} \tag{2.27}$$

From(2.23), we get

$$\nabla_{\gamma} P_{1\mu\nu} = \kappa \nabla_{\gamma} T_{\mu\nu} + \frac{1}{2} g_{\mu\nu} \nabla_{\gamma} P_1 \tag{2.28}$$

Thus, we can write

$$\nabla_{\gamma} P_{1\mu\nu} - \nabla_{\nu} P_{1\mu\gamma} = \kappa\left(\nabla_{\gamma} T_{\mu\nu} - \nabla_{\nu} T_{\mu\gamma}\right) + \frac{1}{2}\left(g_{\mu\nu} \nabla_{\gamma} P_1 - g_{\mu\gamma} \nabla_{\nu} P_1\right). \tag{2.29}$$

If $T_{\mu\nu}$ is Codazzi type, then (2.29) becomes

$$\nabla_{\gamma} P_{1\mu\nu} - \nabla_{\nu} P_{1\mu\gamma} = \frac{1}{2}\left(g_{\mu\nu} \nabla_{\gamma} P_1 - g_{\mu\gamma} \nabla_{\nu} P_1\right) \tag{2.30}$$

Multiplying (2.30) by $g^{\mu\nu}$, on simplification we get

$$\nabla_{\nu} P_{1\gamma}^{\nu} = -\frac{1}{2} \nabla_{\gamma} P_1 \tag{2.31}$$

Multiplying (2.31) by $g^{\mu\gamma}$, we get on simplification

$$\nabla_{\nu} P_1^{\mu\nu} = 0 \tag{2.32}$$

Thus, we have the following theorem

Theorem (2.2): For V_4 satisfying Einstein (like) field equations, the tensor $P_1^{\mu\nu}$ is conserved, if the energy-momentum tensor is Codazzi type.

2.3 P1- flat space time

Consider the equation (2.13) for P_1 - curvature tensor.

Definition (2.3)- The space time is said to be P_1 -flat, if the tensor $P_1{}^{\alpha}{}_{\beta\gamma\delta}$ vanishes in it.

$$R^{\alpha}_{\beta\gamma\delta} = \frac{1}{2(n-1)}\left[g_{\beta\delta} R^{\alpha}_{\gamma} - g_{\beta\gamma} R^{\alpha}_{\delta} - g^{\alpha}_{\delta} R_{\beta\gamma} + g^{\alpha}_{\gamma} R_{\beta\delta}\right]. \tag{2.33}$$

By contracting α and δ, on simplification, we get

$$R_{\beta\gamma} = -\frac{R}{8} g_{\beta\gamma}. \tag{2.34}$$

This shows that P_1 - flat space time is not an Einstein space.

The geometrical symmetries of space time are expressed through the equation (Ahsan and Ali, 2017) $£_{\xi}A - 2\Omega A = 0$, where $\boldsymbol{A}$ represents a geometrical/ physical quantity, Ω is a scalar and $£_{\xi}$ denotes Lie derivative with respect to a vector field ξ.

Consider Einstein field equations with cosmological term:

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \kappa T_{\alpha\beta}. \tag{2.35}$$

Using (2.34) in (2.35), we get

$$(\Lambda - \frac{5}{8} R) g_{\alpha\beta} = \kappa T_{\alpha\beta}. \tag{2.36}$$

Since for a P_1 - flat space time $\boldsymbol{R}$ is constant, by taking the Lie derivative of both sides of (2.36) gives

$$(\Lambda - \frac{5}{8} R) £_{\xi} g_{\alpha\beta} = \kappa £_{\xi} T_{\alpha\beta}. \tag{2.37}$$

Thus we have the following theorem.

Theorem (2.3): For a P_1- flat space time satisfying Einstein field equations with a cosmological term, there exists a Killing vector fieldξ, if and only if the Lie derivative of the energy-momentum tensor vanishes with respect toξ.

Definition: A vector field ξ satisfying the equation,

$$£_{\xi} g_{\alpha\beta} = 2\Omega g_{\alpha\beta}, \tag{2.38}$$

is called a conformal vector field, where Ω is a scalar. The space time satisfying (2.38) is said to admit a conformal motion.

From (2.37) and (2.38), we have

$$(\Lambda - \frac{5}{8}R)2\Omega g_{\alpha\beta} = \kappa \pounds_{\xi} T_{\alpha\beta}. \tag{2.39}$$

Using (2.36) as a consequence of P_1 - flat space time, we get

$$\pounds_{\xi} T_{\alpha\beta} = 2\Omega T_{\alpha\beta}, \tag{2.40}$$

The energy-momentum tensor $T_{\alpha\beta}$ satisfying (2.40) is said to possess the symmetry inheritance property (Ahsan, 2005). Thus, we have the following theorem:

Theorem (2.4): A P_1 -flat space time satisfying the Einstein field equations with a cosmological term admits a conformal Killing vector field if and only if the energy-momentum tensor has the symmetry inheritance property.

2.4. Perfect Fluid space time

The energy-momentum tensor for a perfect fluid is given by:

$$T_{\alpha\beta} = (\mu + p) u_{\alpha} u_{\beta} + p g_{\alpha\beta}, \tag{2.41}$$

where p is the isotropic pressure, μ is the energy density, μ_α is the velocity of fluid, such that $g_{\alpha\beta}u^{\alpha}=u_{\beta}$ and $\mu_{\alpha}u^{\alpha} = -1.$ We now consider a perfect fluid space time with vanishing P_1 - curvature tensor.

From (2.36) and (2.41), we get

$$(\Lambda - \frac{5}{8}R - p\kappa) g_{\alpha\beta} = \kappa (\mu + p) u_{\alpha} u_{\beta}. \tag{2.42}$$

Multiplying (2.42) by $g_{\alpha\beta}$, on simplification we get

$$R = \frac{8}{5}\Lambda - 2\kappa\left(p - \frac{\mu}{5}\right). \tag{2.43}$$

Further, contracting (2.42) with $\mu_\alpha u^\beta$ gives

$$R = \frac{8}{5}(\Lambda + \mu\kappa). \tag{2.44}$$

Comparing (2.43) and (2.44) yields

$$\left(p + \frac{3}{5}\mu\right) = 0. \tag{2.45}$$

This means that either $p=0$, $\mu=0$(the empty space time) or the perfect fluid space time that satisfies the vacuum like equation of state (Kallingas*et al*, 1992) or alternatively pressure and density related by $p=(-3/5)\mu$.

Thus we have the following theorem.

Theorem (2.5): For a P_1 - flat perfect fluid space time satisfying Einstein field equations with cosmological term, the matter contents of the space time obey the vacuum like equation of the state or alternatively the isotropic pressure$p=(-3/5)\mu$ (energy density).

Discussion

The P_1 -curvature tensor satisfies symmetric, skew symmetric as well as cyclic properties that are satisfied by Riemann curvature tensor. Therefore this tensor can be used to study geometric properties of various manifolds as well as physical properties of different space times of general relativity. Expressing Einstein like field equations with the help of the contracted part of P_1 - curvature tensor may open new avenues for its applications.

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