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Optimal stopping problem of American put options for the diffusion risk model

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Abstract
This paper investigates the optimal stopping problem of American put options for the diffusion risk model. First of all, by using the relevant conclusions of the first hitting time of diffusion risk model, the optimal stopping problem formulation can be obtained. Then, the optimal value function and the optimal trading time of the American put option are obtained. Finally, numerical examples are given to illustrate the applications of the optimal stopping problem for American put options.

Keywords: optimal stopping time, gain function, diffusion risk model, put option

1. Introduction

Option is a kind of financial derivative that first appeared in the United States in the mid-1970s and has been developing rapidly. Options can be divided into European options and American options according to the transaction time. However, American options can be traded at any time during the period of validity, they are more widely used in international finance. In recent years, the pricing problem and trading time of American options has become one of the most popular optimal-stopping problems in the field of financial mathematics. And domestic and foreign scholars have widely used the optimal stopping time methods to study American options. For example, Bensoussan (1984) [1] used the optimal stopping time problem to characterize the execution time of American-style undetermined equity pricing under complete market conditions. Merton and Pliska (1995) [2] used the optimal stopping time theory to study the problem of securities investment with fixed transaction costs. Ekstrom (2004) [3] showed that the convexity of the optimal stopping boundary for the American put option in the standard Black-Scholes model. Chiarolla and Angelis (2015) [4] studied the optimal stopping problem of pricing American put option on a zero coupon bond in the Heath-Jarrow-Morton model for forward interest rates, and showed the first price of the American bond option equals the optimal payoff. Armerin (2019) [5] considered an optimal stopping problem of a killed exponentially growing process and obtained the valuation of American call option. Woo and Choe (2020) [6] solved the optimal stopping problem related to maximum and minimum and found the closed form pricing formula of the perpetual American look back spread option. Based on these, the optimal stopping time problem of American put options for the diffusion risk model is considered.

The diffusion risk model is given by

$$U_t = u + ct + \sigma B_t \tag{1.1}$$

Where

U_t Describes the put option price at time t, $U_0 = u > 0$ denotes the initial value. c, σ are positive constants, c is a drift parameter and σ represents the random volatility coefficient. $\{B_t, t \geq 0\}$ is a standard Brownian motion.

Define the value function V by

$$V(u) = ess\sup_{\tau} E_u[e^{-r\tau}G(U_\tau)] \tag{1.2}$$

Where

All τ are $\{\mathrm{F}_t, t \geq 0\}$-stopping times, which represents the option trading time and $\mathrm{F}_t = \sigma\{U_s, 0 < s \leq t\}$ are the natural σ-filtrations. Let $G(u) = (K-u)^+$, which is a gain function equal to the American put option. $r > 0$ Represents a constant rate and K is a constant that represents strike price. The esssup denotes the essential supremum (see Peskir and Shiryaev (2006) [7] for the definition).

In order to get the optimal value function of the American put option and the optimal trading time, the problem is transformed into solving an optimal stopping problem on $\{U_t, t \geq 0\}$. That is, finding an optimal stopping time τ^* satisfying

$$V(u) = E_u[e^{-r\tau^*}G(U_{\tau^*})], \tag{1.3}$$

Where

$V(u)$ is the optimal value function of America put option and τ^* is the optimal trade time.

The remainder of the paper is organized as follows. In section 2, the formulation of the optimal stopping times problem for the diffusion risk model are given. In section 3, the optimal stopping times are obtained. Furthermore, numerical examples are presented to illustrate the applications of the optimal stopping problem in section 4.

2. Formulation of the optimal stopping times

The model (1.1) is a time-homogeneous Markov process taking values in $\mathbb{R}$ with generator A (see Klebaner (1998) [8]) that satisfies

$$\mathrm{A}f(x) = \frac{\sigma^2}{2} f''(x) + cf'(x) \tag{2.1}$$

Where

f belongs to the domain $\mathrm{D}(\mathrm{A})$ of the generator A of $\{U_t, t \geq 0\}$ and f is twice continuously differentiable with respect to x. For a fixed $\alpha > 0$, let

$$\mathrm{A}f(x) = \alpha f(x)$$

That is

$$\frac{\sigma^2}{2} f''(x) + cf'(x) = \alpha f(x).$$

Above equation has two positive solutions, and every solution is a linear combination of the form

$$C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

Where C_1 and C_2 are arbitrary constants and

$$r_1 = \frac{-c - \sqrt{c^2 + 2\alpha\sigma^2}}{\sigma^2}, r_2 = \frac{-c + \sqrt{c^2 + 2\alpha\sigma^2}}{\sigma^2}.$$

For a fixed constant $a > 0$, define a stopping time

$$\tau_{(0,a)} = \begin{cases} \inf\{t \geq 0, U_t \in (0,a)\}, \\ \\ \infty, \qquad \text{if } U_t \notin (0,a) \text{ for all } t \geq 0. \end{cases}$$

Also define

$$V_a(u) = E_u[e^{-r\tau_{(0,a)}}(K - U_{\tau_{(0,a)}})^+], \tag{2.2}$$

Obviously, for $u \in (0, a]$, $\tau_{(0,a)} = 0$, then

$$V_a(u) = (K - u)^+$$

The Laplace-Stieltjes transforms of the hitting time for the diffusion risk model can be obtained (see He and Yang (2017) [9]) as follows.

Lemma 2.1. If $u > a$, then for $\alpha > 0$,

$$E_u[e^{-\alpha\tau_{(0,a)}}] = \frac{e^{r_1 u}}{e^{r_1 a}} = e^{\frac{-c-\sqrt{c^2+2\alpha\sigma^2}}{\sigma^2}(u-a)}$$

From Lemma 2.1, if let $\alpha = r$, then for $u > a$, the value function

$$\begin{aligned} V_a(u) &= E_u[e^{-r\tau_{(0,a)}}(K - U_{\tau_{(0,a)}})^+] \\ &= E_u[e^{-r\tau_{(0,a)}}(K - a)^+] \\ &= (K - a)^+ E_u[e^{-r\tau_{(0,a)}}] \\ &= (K - a)^+ e^{\frac{-c-\sqrt{c^2+2\alpha\sigma^2}}{\sigma^2}(u-a)}, \end{aligned}$$

and for $u \le a$, $V_a(u) = (K - u)^+$. Let $m = \dfrac{-c - \sqrt{c^2 + 2r\sigma^2}}{\sigma^2}$, thus

$$V_a(u) = \begin{cases} (K - a)^+ e^{m(u-a)}, & u > a, \\ (K - u)^+, & u \le a. \end{cases} \tag{2.3}$$

Next taking a derivative $V_a(u)$ with respect to a and let $\dfrac{\partial V_a(x)}{\partial a}\Big|_{a=a^*} = 0$. This implies that,

i) For $0 < a^* < K$

$$-e^{m(u-a^*)} - m(K - a^*)e^{m(u-a^*)} = 0$$

ii) Since $V_a(u) = 0$, for $a \in [K, \infty)$, and for all $a^* > K$

$$\frac{\partial V_a(x)}{\partial a}\Big|_{a=a^*} = 0$$

From (i), one can get

$$a^* = K + \frac{1}{m} \in (0, K)$$

Ignore (ii), since $V_{a^*}(u) = 0$, for all $a^* > K$, it is meaningless and a^* does not depend on x.

Go back to (2.2), let $a = a^*$, then

$$V_{a^*}(u)=\begin{cases}(K-a^*)^+e^{m(u-a^*)}, & u>a^*\in(0,K),\\ (K-u)^+, & u\le a^*\in(0,K),\end{cases}$$

$$=\begin{cases}-\dfrac{1}{m}\dfrac{e^{mu}}{e^{Km+1}}, & u>a^*\in(0,K),\\ K-u, & u\le a^*\in(0,K).\end{cases} \tag{2.4}$$

and the value function $V_{a^*}(u)$ satisfies

$$V_{a^*}(u)=\sup_a V_a(u)$$
$$=\sup_a E_u[e^{-r\tau_{(0,a)}}(K-U_{\tau_{(0,a)}})^+].$$

Recall the $V(u)$ in (1.2), obviously

$$V_{a^*}(u)\le V(u)$$

3. The optimal stopping problem

Theorem 3.1: Let $\{U_t, t\ge 0\}$ satisfy (1.1) and $V(u)$ be the value function of the optimal stopping problem defined in (1.2). For a family of functions $\{V_a(u), a\in(0,\infty)\}$ defined by

$$V_a(u)=E_u[e^{-r\tau_{(0,a)}}(K-U_{\tau_{(0,a)}})^+]$$

And if there exists a $a^*\in(0,\infty)$ (independent of u) such that $V_{a^*}(u)\ge V_a(u)$, for all a and u. Then

$$V(u)=V_{a^*}(u) \;=\begin{cases}(K-a^*)^+e^{m(u-a^*)}, & u>a^*\in(0,K),\\ (K-u)^+, & u\le a^*\in(0,K),\end{cases}$$

Where $a^*=K+\dfrac{1}{m}\in(0,K)$ and $\tau_{(0,a^*)}$ is an optimal stopping time.

Proof. It is known that there exists a $a^*\in(0,K)$ such that $V_{a^*}(u)\ge V_a(u)$, for all $a\in(0,\infty)$ and all $u\in(0,\infty)$. $V_{a^*}(u)$ is determined by (2.4), certainly $V_a(u)\le V(u)$ implies

$$\sup_a V_a(u)=V_{a^*}(u)\le V(u)$$

Next, proving

$$V(u)\le V_{a^*}(u).$$

The operator of the model (1.1) satisfies

$$\mathrm{A}f(x)=\frac{\sigma^2}{2}f''(x)+cf'(x)$$

Recall the value function $V_a(u)$ (and so $V_{a^*}(u)$ in (2.3)), by Dynkin formula, there is

$$\mathrm{A}V_{a^*}(u) = rV_{a^*}(u), \text{ for } u > a^*.$$

For $u \le a^*$, $V_{a^*}(u) = (K-u)^+$, so

$$\mathrm{A}V_{a^*}(u) \le rV_{a^*}(u), \text{ for } u \le a^*.$$

According to It'o formula, for $e^{-rt}V_{a^*}(U_t)$, $U_t \ne a$, one can get

$$\begin{aligned} e^{-rt}V_{a^*}(U_t) &= V_{a^*}(u) + \int_0^t e^{-rs}(\mathrm{A}V_{a^*} - rV_{a^*})(U_s)\times 1_{\{U_s \ne a^*\}}ds + \int_0^t e^{-rs}\frac{d}{du}V_{a^*}(U_s)\cdot\sigma U_s 1_{\{U_s\ne a^*\}}dB_s \\ &\le V_{a^*}(u) + \sigma\int_0^t e^{-rs}\frac{d}{du}V_{a^*}(U_s)U_s 1_{\{U_s \ne a^*\}}dB_s \\ &= V_{a^*}(u) + \mathrm{M}_t, \end{aligned} \tag{3.1}$$

Where

$$\mathrm{M}_t := \sigma\int_0^t e^{-rs}\frac{d}{du}V_{a^*}(U_s)U_s 1_{\{U_s\ne a^*\}}dB_s$$

Which is a continuous local martingale. From above all,

$$V_{a^*}(u) \ge (K-u)^+, u \le a^*$$

and for $x > a^*$

$$V_{a^*}(u) = \sup_a V_a(u) \ge V_u(u) = (K-u)^+$$

Hence it follows from (3.1) that

$$e^{-rt}(K-U_t)^+ \le e^{-rt}V_{a^*}(U_t) \le V_{a^*}(u) + \mathrm{M}_t$$

Since $\{\mathrm{M}_t, t\ge 0\}$ is continuous local martingale, and for all stopping time τ, let $\tau_N = inf\{t > 0 : |\mathrm{M}_t| < N\}$ such that $E\mathrm{M}_{\tau\wedge\tau_N} = 0$. Then

$$E_u[e^{-r\tau\wedge\tau_N}(K-U_{\tau\wedge\tau_N})^+] \le V_{a^*}(u)$$

Letting N™ ∞ in above equation, dominated convergence theorem yields

$$E_u[e^{-r\tau}(K-U_\tau)^+] \le V_{a^*}(u)$$

So

$$V(u) = ess\sup_\tau E_u[e^{-r\tau}(K-U_\tau)^+] \le V_{a^*}(u)$$

The proof is completed.

4. Numerical examples

In this section, numerical examples are presented to illustrate the applications of the optimal stopping problem for the diffusion risk model. For example, the model is considered with the following parameter values:

$$c = 0.4,\ \sigma_1 = 3,\ \sigma_2 = 4,\ r = 0.1,\ K = 10$$

By $Theorem\ 3.1$, when the parameter values are $c = 0.4$, $\sigma_1 = 3$, $r = 0.1$, $K = 10$, the value function $V(u)$ is shown in $Figure\ 1$. from $Figure\ 1$, the intersection of the two curves is the level $a^* = 5$, which is optimal to exercise the option.

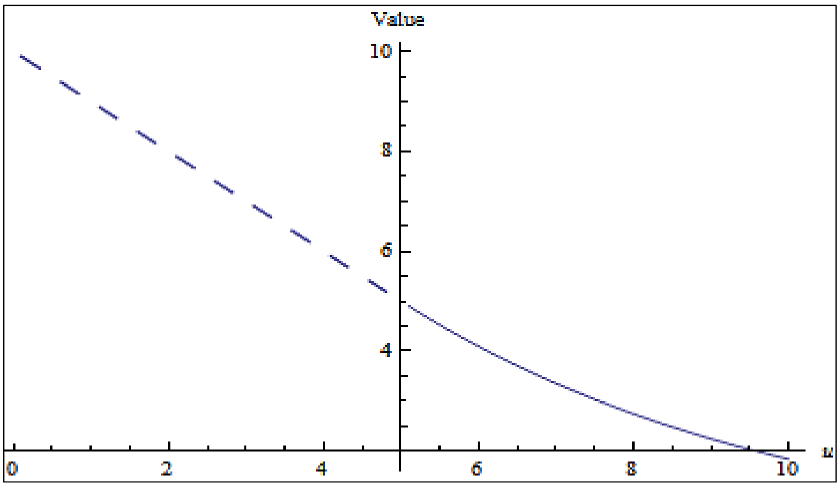

Fig 1: The optimal value function $V(u)$ for $c = 0.4$, $\sigma_1 = 3$, $r = 0.1$, $K = 10$.

By $Theorem\ 3.1$, when the parameter values are $c = 0.4$, $\sigma_2 = 4$, $r = 0.1$, $K = 10$, the value function $V(u)$ is shown in $Figure\ 2$. from $Figure\ 2$, the intersection of the two curves is the level $a^* = 2.86$, which is optimal to exercise the option.

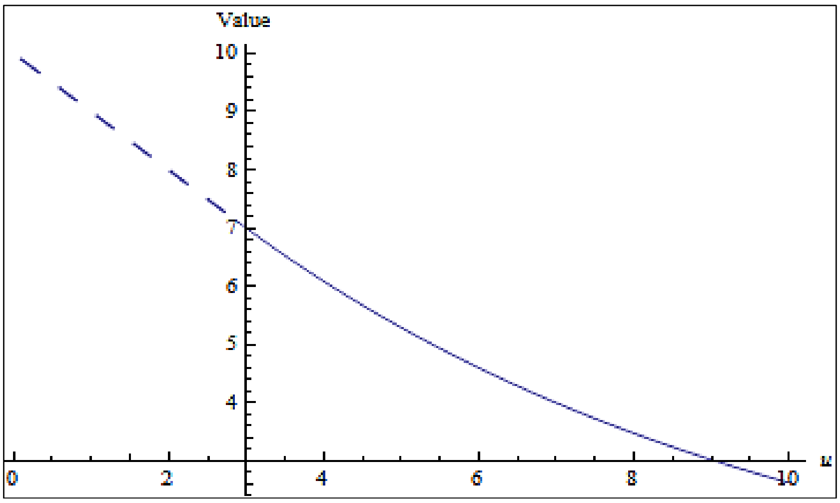

Fig 2: The optimal value function $V(u)$ for $c = 0.4$, $\sigma_2 = 4$, $r = 0.1$, $K = 10$

The dates from $Figure\ 1$ and $Figure\ 2$ shows that the optimal value function $V(u)$ are decreasing as the u increases. And when σ increasing, the optimal value function increases the optimal stopping level a^* decreases, so the optimal trade time and the optimal value function are affected by random volatility.

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