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## Decomposition of normal projective curvature tensor fields in Finsler manifolds

**Manoj Singh Bisht and US Negi**

### Abstract

Takano (1966) have studied recurrent affine motion in a recurrent non-Riemannian space and later on he discussed a recurrent whose curvature tensor is decomposable. Also, Sinha and Singh (1970), has studied on decomposition of Recurrent Curvature Tensor Fields in Finsler Spaces; Pande and Khan (1973), has studied general Decomposition of Berwald's Curvature Tensor Fields in Recurrent Finsler Spaces. After that, Ram Hit (1975), have studied decomposition of Berwald's Curvature Tensor Fields; Pande and Shukla (1977) has studied a recurrent Finsler space whose curvature tensor is decomposable. In this paper we have defined and studied decomposition of normal projective curvature tensor fields in Finsler manifolds and some theorems established on its.

**Keywords:** Riemannian space, Kaehlerian manifold, Finsler manifold, H-projective recurrent, curvature tensors

### Introduction

The Euler's theorem on homogeneous functions, accordingly to the tensor  $g_{ij}(x, y)$  is positively homogeneous of degree zero in  $y^i$  and symmetric in  $i$  and  $j$ . The vector  $y_i$  satisfies the following relation:

$$g_{ij}(x, y) = \frac{1}{2} \hat{\partial}_i \hat{\partial}_j F^2(x, y). \tag{1.1}$$

$$y_i y^i = F^2, \tag{1.2}$$

Berwald covariant derivative of the function  $F$  and vector  $y^i$  vanish identically,

$$B_k F = 0 \text{ and } B_k y^i = 0. \tag{1.3}$$

The tensor  $H_{jkh}^i$  is called h-curvature tensor and defined by:

$$H_{jkh}^i = \partial_h G_{jk}^i + G_{jk}^r G_{rh}^i + G_{rk}^i G_j^r - h/k.$$

In view of Euler's theorem on homogeneous functions, we have the following relations:

$$\hat{\partial}_j H_{kh}^i = H_{jkh}^i, \tag{1.4a}$$

$$y^r \hat{\partial}_r H_{jkh}^i = y^r \hat{\partial}_j \hat{\partial}_r H_{kh}^i = 0, \tag{1.4b}$$

$$H_{jkh}^i y^j = H_{kh}^i, \tag{1.4c}$$

$$H_{ijkh} = g_{jr} H_{ikh}^r \tag{1.4d}$$

$$H_{kh}^i = \hat{\partial}_k H_h^i, \tag{1.4e}$$

$$H_{kh}^i y^k = H_h^i, \tag{1.4f}$$

$$H_{jkh} = g_{ik} H_{jh}^i, \tag{1.4g}$$

$$H_{jk} = H_{jkr}^r, \tag{1.4h}$$

$$H_{rkh}^r = H_{kh} - H_{hk}, \tag{1.4i}$$

$$H_k = H_{kr}^r, \tag{1.4l}$$

$$H = \frac{1}{n-1} H_r^r, \tag{1.4m}$$

$$H_{jk} y^k = H, \tag{1.4n}$$

**Decomposition of normal Projective Curvature Tensor in Finsler Manifolds.**

The connection coefficient  $\Pi_{jk}^i$  is positively homogeneous of degree zero in  $y^i$  and symmetric in their lower indices and defined the projective connection by [Yano (1957)],

$$\Pi_{jk}^i = G_{jk}^i - \frac{1}{n+1} y^i G_{jkr}^r.$$

The covariant derivative  $B_k T_j^i$  of an arbitrary tensor field  $T_j^i$  with respect to  $x^k$  in the sense of Berwald is given by:

$$B_k T_j^i = \partial_k T_j^i - (\partial_r T_j^i) \Pi_{ks}^r y^s + T_j^r \Pi_{kr}^i - T_r^i \Pi_{kj}^r. \tag{2.1a}$$

The communication formula for the above covariant derivative is given by

$$B_k B_h T_j^i - B_h B_k T_j^i = T_j^r N_{rkh}^i - T_r^i N_{jkh}^r - \hat{\partial}_r T_j^i N_{skh}^r y^s \tag{2.1b}$$

The relation between normal projective curvature tensor  $N_{jkh}^i$  and Berwald curvature tensor  $H_{jkh}^i$  obtained [Pandey (1981)], as follows:

$$N_{jkh}^i = H_{jkh}^i - \frac{1}{n+1} y^i \hat{\partial}_r H_{rkh}^r. \tag{2.2}$$

The normal projective curvature tensor  $N_{jkh}^i$  is homogeneous of degree zero in  $y^i$ . For the tensor  $N_{jkh}^i$ , we have the following identities:

$$N_{rkh}^r = H_{rkh}^r, \tag{2.3a}$$

$$N_{jkh}^i y^i = H_{kjh}^i, \tag{2.3b}$$

$$N_{jkh}^i = -N_{jhk}^i, \tag{2.3c}$$

$$N_{jkh}^i + N_{khj}^i + N_{kjh}^i = 0, \tag{2.3d}$$

$$N_{jk} = N_{jkr}^r. \tag{2.3e}$$

Let us consider the decomposition of the normal projective curvature tensor  $N_{jkh}^i$  of a Finsler space of the type (1, 3) as follows:

$$N_{jkh}^i = y^i Y_{jkh} \tag{2.4}$$

Where  $Y_{jkh}$  non-zero tensor field is called decomposition tensor field.

Further considering the decomposition of the tensor field (2.4) in the form

$$N_{jkh}^i = y^i y_j Y_{kh} \text{ and } N_{jkh}^i = y^i y_h Y_{jk} \tag{2.5}$$

Let us define

$$y^i \lambda_j = \sigma, \tag{2.6}$$

Such  $\lambda_j$  as recurrence vector and  $\sigma$  is decomposition scalar. Therefore, we have

**Theorem (2.1).** If the normal projective curvature tensor  $N_{jkh}^i$  of a Finsler manifolds is decomposable in the form (2.4), then the normal projective curvature tensor  $N_{jkh}^i$  and the  $h(v)$ -torsion tensor  $H_{kh}^i$  are decomposable and the tensor  $H_{rkh}^r$  is also decomposable in the form (2.4) and the decomposable tensor field  $Y_{jkh}$  satisfies.

**Proof.** In view of (2.4), the identities (2.3c) and (2.3d), can be written as

$$Y_{jkh} + Y_{jhk} = 0 \text{ and } Y_{jkh} + Y_{khj} + Y_{hjk} = 0 \quad (2.7)$$

In view of (2.3a), the contraction of the indices i and j in (2.4) gives

$$H_{rkh}^r = y^r Y_{rkh}. \quad (2.8)$$

Using (1.4i) in (2.8), we get

$$Y_{rkh} = \frac{1}{y^r} (H_{hk} - H_{kh}). \quad (2.9)$$

In view of (2.4), equation (2.9) can be written as

$$N_{jkh}^i = \frac{y^i}{y^j} (H_{hk} - H_{kh}). \quad (2.10)$$

Transecting (2.10) by  $y^j$  and using (2.3b), we get

$$H_{kh}^i = y^i (H_{hk} - H_{kh}). \quad (2.11)$$

Contraction of the indices i and j in (2.5), using (2.3a) and (1.2), we get

$$H_{rkh}^r = F^2 Y_{kh}. \quad (2.12)$$

Using (1.4i) in (2.12), we get

$$Y_{kh} = \frac{1}{F^2} (H_{hk} - H_{kh}). \quad (2.13)$$

In view of (2.5), equation (2.13) can be written as

$$N_{jkh}^i = y^i y_j \frac{1}{F^2} (H_{hk} - H_{kh}) \quad (2.14)$$

Transvecting (2.14) by  $y^j$  and using (2.3b), we get

$$H_{kh}^i = y^i (H_{hk} - H_{kh}). \quad (2.15)$$

Hence, we get

**Theorem (2.2).** If the normal projective curvature tensor  $N_{jkh}^i$  of a Finsler manifolds is decomposable in the form (2.5), then the normal projective curvature tensor  $N_{jkh}^i$  and the  $h(v)$ -torsion tensor  $H_{kh}^i$  are defined by (2.14) and (2.15) respectively, the tensor  $H_{rkh}^r$  is decomposable in the form (2.12) and the decomposable tensor field  $Y_{kh}$  satisfies (2.13). Now, we have obtained,

**Theorem (2.3).** In a normal Projective Curvature Tensor Finsler manifold, under the decomposition (2.4), the decomposition tensor field  $Y_{jkh}$  behaves like a recurrent tensor field.

**Proof.** A Finsler manifolds for which the normal projective tensor  $N_{jkh}^i$  satisfies the recurrence property with respect to Berwald's connection coefficient Projective Curvature Tensor Finsler manifold is characterized by [Pandey (1981)].

$$B_m P_{jkh}^i = \lambda_m P_{jkh}^i, P_{jkh}^i \neq 0, \quad (2.16)$$

Where  $\lambda_m$  is non-zero covariant vector field is recurrence vector field. Let us consider a Finsler manifolds whose normal projective curvature tensor  $N_{jkh}^i$  satisfies the condition (2.16). Transecting (2.16) by  $y^j$ , using (1.3) and (2.3b), we get

$$B_m H_{kh}^i = \lambda_m H_{kh}^i. \quad (2.17)$$

Contraction of the indices i and j in (2.16) and using (2.3a), we get

$$B_m H_{rkh}^r = \lambda_m H_{rkh}^r. \quad (2.18)$$

Transvecting (2.17) by  $y^k$ , using (1.3b) and (1.4f), we get

$$B_m H_h^i = \lambda_m H_h^i. \quad (2.19)$$

Contraction of the indices  $i$  and  $h$  in (2.17) using (1.4i), we get

$$B_m H_k = \lambda_m H_k \quad (2.20)$$

Transvecting (2.20) by  $y^k$ , using (1.3) and (1.4n), we get

$$B_m H = \lambda_m H. \quad (2.21)$$

We know that the projective curvature tensor  $P_{jkh}^i$  satisfying the following:

$$\lambda_m P_{jkh}^i + \lambda_k P_{jhm}^i + \lambda_n P_{jmk}^i = 0. \quad (2.22)$$

Differentiating (2.4) covariantly with respect to  $x^m$  in the sense of Berwald, using (2.16), (2.4) and (1.3), we get

$$B_m Y_{jkh} = \lambda_m Y_{jkh}, \quad (2.23)$$

Where  $\lambda_m$  is non-zero vector field. Again, Differentiating (2.21) covariantly with respect to  $x^l$  in the sense of Berwald and using (2.21), we get

$$B_l B_m Y_{jkh} = (B_l \lambda_m) Y_{jkh} + \lambda_m \lambda_l Y_{jkh}. \quad (2.24)$$

Interchanging the indices  $m$  and  $l$  in (2.24) and subtracting the equation obtained from (2.24), we get

$$B_l B_m Y_{jkh} - B_m B_l Y_{jkh} = (B_l \lambda_m - B_m \lambda_l) Y_{jkh}. \quad (2.25)$$

Using the commutation formula (2.1b) in (2.25), we get

$$(B_l \lambda_m - B_m \lambda_l) Y_{jkh} = -(Y_{rkh} N_{jml}^r + Y_{jrh} N_{kml}^r + Y_{jkr} N_{hml}^r + \dot{\partial}_r Y_{jkh} N_{sml}^r) y^s. \quad (2.26)$$

In view of (2.4) and the homogeneity property of  $Y_{jkh}$ , equ. (2.26) can be written as:

$$(B_l \lambda_m - B_m \lambda_l) Y_{jkh} = -(Y_{rkh} Y_{jml} + Y_{jrh} Y_{kml} + Y_{jkr} Y_{hml} - Y_{jkh} Y_{rml}) y^r. \quad (2.27)$$

Differentiating (2.27) covariantly with respect to  $x^n$  in the sense of Berwald, using (1.3b) and (2.27), we get

$$B_n (B_l \lambda_m - B_m \lambda_l) = \lambda_n (B_l \lambda_m - B_m \lambda_l). \quad (2.28)$$

Thus, we get obtained,

**Theorem (2.4).** In a Projective Curvature Tensor Finsler manifolds under the decomposition (2.4), the recurrence  $B_l \lambda_m - B_m \lambda_l$  behaves like a recurrent tensor field.

**Again,** We have considering the decomposition of the tensor field  $Y_{jkh}$  in the form

$$Y_{jkh} = \lambda_j Y_{jkh}. \quad (2.29)$$

Differentiating (2.29) covariantly with respect to  $x^m$  in the sense of Berwald, using (2.23) and (2.29), we get

$$\lambda_m \lambda_j Y_{jkh} = (B_m \lambda_j) Y_{jkh} + \lambda_j B_m Y_{jkh}. \quad (2.30)$$

Transvecting (2.30) by  $y^j$ , using (2.6) and (1.3a), we get

$$B_m Y_{kh} = \lambda_m Y_{kh}. \quad (2.31)$$

Thus, we get obtained,

**Theorem (2.5).** In a Projective Curvature Tensor Finsler manifolds, under the decomposition (2.4) and (2.29), the tensor field  $Y_{kh}$  behaves like a recurrent tensor field.

**Again we have** From (2.27) and (2.29), we get

$$\lambda_j \{ (B_l \lambda_m - B_m \lambda_l) Y_{kh} - Y_{ml} (\lambda_k Y_{rh} + \lambda_h Y_{kh}) y^r \} = 0. \quad (2.32)$$

Using (2.29) in (2.7), we get

$$-\lambda_j Y_{kh} = \lambda_k Y_{hj} + \lambda_h Y_{jk}. \quad (2.33)$$

From (2.32), (2.33), using (2.6), the fact that the vector field  $\lambda_m$  and the tensor field  $Y_{jkh}$  are non-zero, we get

$$B_l \lambda_m - B_m \lambda_l + \sigma Y_{ml} = 0. \quad (2.34)$$

Thus, we get obtained,

**Theorem (2.6).** In a Projective Curvature Tensor Finsler manifold, under the decomposition (2.4) and (2.34), the necessary and sufficient condition that  $B_l \lambda_m = B_m \lambda_l$  is that  $\sigma = 0$ .

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