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A note on invariant subspaces of some operators

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Abstract
In this paper, we give a condition in which a paranormal operator T has invariant subspaces. We also show that a hyponormal operator T has an invariant subspace if it is complex symmetric.

Keywords: hyponormal, complex symmetric, paranormal

1. Introduction

Let H be Hilbert space and $B(H)$ denoted all bounded linear operators on H. An operator $T \in B(H)$ is complex symmetric operator if there exists a conjugation S such that $STS = T^*$. By a conjugation we mean an isometry or antilinear involution. An operator T is isometric if $T^*T = I$, normal if $T^*T = TT^*$, binormal $((TT^*)(TT^*) = (TT^*)(T^*T)$, that is, T^*T *and* TT^* commute, hyponormal if $T^*T \geq TT^*$,that is, $T^*T - TT^* \geq 0$, quasinormal if T commute with T^*T and subnormal if T is restriction of normal operator to an invariant subspace. these properties for the series of conclusion. Normality implies quasinormality, quasinormality implies surbnormality and subormality implies hyponormality. It has been shown that every hyponormal operator whose spectrum has Lebesque measure is normal [14]. A normal operator is both normal and complex symmetric. The study of complex symmetric operators was initiated by Garcia [6]. The study of binormal operators was initiated by Campbell [3] in 1972. Garcia and Wogen have shown that every binormal operator is complex symmetric [9].

2 Binormal operators

In this section we look into conditions under which a complex symmetric operator is binormal and vice versa.

Theorem 2.1: A complex symmetric operator T with conjugation S is binormal if and only if S commutes with TT^*T^*T, equivalently, T^*TTT^*.

Proof. Since T is complex symmetric ST=T^*S and TS=ST^*. Assume that S commutes with TT^*T^*T. Then we have

$T^*TTT^* = T^*TSST^*T$

$= T^*ST^*T^*ST^*$

$= STT^*T^*TS$

$= TT^*T^*TSS$

$= TT^*T^*T.$

Hence T is binormal. Conversely, suppose T is binormal. Then

$TT^*T^*TS = T^*TTT^*S$

$= T^*TSSTT^*S$

$= T^*ST^*T^*ST^*S$

$= STT^*T^*TSS$

$= STT^*T^*T$

Thus S commutes with TT^*T^*T. Hence the proof.

In the following we assume that T is binormal and find condition in which T is complex symmetric. We first define the following two operators.

Definition 2.1: If $T = U|T|$ is the polar decomposition of T,then the Duggal transform $\hat{T}$ of T is given by $\hat{T} = |T|U$.

Definition 2.2: If $T = U|T|$ is polar decomposition of T, then the aluthge transform $\tilde{T}\ of\ T$ is given by $\tilde{T} = \left|T|^{\frac{1}{2}}U\right| T|^{\frac{1}{2}}$.

Theorem 2.2: A binormal operator$T$ with polar decomposition $T = U|T|$ is complex symmetric if and only if$\hat{T} = T|U|$ is complex symmetric.

Proof. Suppose T is complex symmetric. Then, $\hat{T}$ is also complex symmetric according to [6, Theorem 1]. Let Tbe complex symmetric for a conjugation S. Since,T is binormal, the polar decomposition of $\hat{T}$ is $\hat{T} = \hat{U}\hat{T}$ [12, Theorem 3.24]. Since $\hat{T}$ is complex symmetric, $\hat{U} = U^*UU$ is unitary [8, corollary 1]. If U is proper partial isometry, then U^*U is proper projection and $U^*UU = U$.This shows that U is unitary and so $\hat{T}U^* = |T|$. We obtain $T = U|T| = U\hat{T}U^*$and therefore $T$and $\hat{T}$ are unitarily equivalent. Hence T is complex symmetric.

In the next result we show the link between $\tilde{T}$and T^2 but we first need the following lemma.

Lemma 2.3: An operator $T = U|T|$,with U unitary, is binormal if and only if $|T|and\ |\widehat{T}|$commute.

Proof. Let T be such that $[|\widehat{T}|, T] = 0$. According to [12, Theorem 3.2.9], $T$is binormal if and only if$\widehat{T}$is normal, equivalently $[|\widehat{T}|, |\hat{T}^*|] = 0$. Therefore $|\hat{T}^*| = |(\backslash T\backslash U)^*| = |U^*|T| = (|T|UU^*|T|)^{\frac{1}{2}} = |T|. Therefore, [|\widehat{T}|, |\hat{T}^*|] = [|\hat{T}|, |T|] = 0\ and\ so\ T$ is normal.

Suppose$T$ is binormal by [12], $\hat{T}$ is binormal, $[\left|\hat{T}\right|, |\hat{T}^*|\ ]=0$ Since $\left|\hat{T}^*\right| = |T|, 0 = [|\hat{T}|, |\hat{T}^*| = [\left|\hat{T}\right|, |T|]$, hence proof.

3 Main results

Theorem 3.1: A normal operator $T\ \in B(H)$ has nontrivial invariant subspaces.

Proof. Let $T\ \in B(H)$ be a normal operator. The von Neumann algebra $W^*(T)$ generated by T is abelian, and so it cannot be dense in $B(H)$. This means that its commutator $\{W^*(T)\}'$ is nontrivial, that is, it is not the scalar multiples of the identity operator. Any nontrivial von Neumann algebra has nontrivial projections. The range of any of these projections will be invariant for$T$.

Theorem 3.2: An operator$T\ \in\ B(H)$ which is both hyponormal and complex symmetric is normal.

Proof. Let T hyponormal operator,$||T^*x|| \geq ||T^*x||\ for$ all $x\ \in\ H$. Since T is complex symmetric, there is conjugation S so that$T = ST^*S, that\ is, ST = T^*S$. Thus

$\left||Tx|\right| = \left||ST^*Sx|\right| = \left||T^*Sx|\right| \leq \left||TSx|\right| = \left||ST^*x|\right| = \left||T^*x|\right|$.

Hence $\left||Tx|\right| = ||T^*x||$ hence T is normal.

The following result shows that a hyponormal operator$T\ \in\ B(H)$ has invariant subspaces if it is symmetric.

Theorem 3.3: Let $T\ \in B(H)$ be a hyponormal operator which is symmetric. Then T has an invariant subspace.

Proof. From Theorem 3.4, we observe that T is normal. Hence Thas nontrivial invariant subspace by Theorem 3.1.

Theorem 3.4: If T^2 is normal, then T is both binormal and comlex symmetric.

Proof. It T^2 is normal, then T is binormal by [5, Theorem 1] and complex symmetric by [9, corollary 3]

4. Paranormal operators

In [4], Campbell remarks that hyponormal operators is normal if and ony if its square is normal. In this section we relax that hypothesis from normality to paranormality

Lemma 4.1: (4, Theorem 4) A binormal operator $T\ \epsilon\ B(H)$ is hyponormal if and only if it is paranormal.

Lemma 4.2: A binormal, complex symmetric operator T is normal if and only if it is paranormal.

Proof. If T is paranormal and binormal, it is hyponormal by lemma 4.1. If T is hyponormal and complex symmetric, it is normal by theorem 3.2.

Theorem 4.3: A paranormal T is normal if and only if T^2 is normal.

Proof. Suppose T^2 is normal, by Theorem 3.4, T is binormal and comlex symmetric. Since T is paranormal, it is hyponormal by Lemma 4.1, Thus T is normal by lemma 4.2

Proposition 4.4: A paranormal operator T has nontrivial subspace if its square is normal.

Proof. By theorem 4.3 T is normal and hence has nontrivial invariant subspace by Theorem 3.1.

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