International Journal of Statistics and Applied Mathematics 2021; 6(2): 01-05

International Journal of Statistics and Applied Mathematics

**ISSN:** 2456-1452
Maths 2021; 6(2): 01-05

www.mathsjournal.com
Received: 01-01-2021
Accepted: 05-02-2021

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# A study on $W_6$-curvature tensors and $W_8$-curvature tensors in Kenmotsu manifolds admitting semi-symmetric metric connection

**PW Njori, SK Moindi and GP Pokhariyal**

**Abstract**
This paper deals with the study of $W_6$ -Curvature tensors on Kenmotsu manifolds admitting semi-symmetric metric connection. It has been established that a flat $W_6$ -Curvature tensor in Kenmotsu manifold with respect to semi-symmetric connection does not imply the manifold is flat with respect to the connection. Further study has revealed the geometrical relationship between $W_6$-curvature tensor and $W_8$-curvature tensors in Kenmotsu manifold with respect to semi-symmetric metric connection and Levi-Civita connection respectively.

**Keywords:** $W_6$ −Curvature tensor, $W_6$ −Curvature tensor, $W_6$ −flat manifold, flat manifold, Kenmotsu manifold, semi-symmetric metric connection. AMS 200 Subject Classification: 53C15, 53C40

## 1. Introduction

In 1924, Friendmann and Schouten introduced the idea of semi-symmetric linear connection on differentiable manifold. In 1932, Hayden [2] introduced the idea of metric connection with torsion on a Riemannian manifold. A systematic study of the semi-symmetric metric connection on a Riemannian manifold was published by Yano [3] in 1970. After that the the properties of semi-symmetric metric connection have been studied by many authors like Amur and Pujar [4], Bagewadi [5], Shardfuddin and Hussain [6], De and Pathak [7], and others. In 1971, Kenmotsu studied a class of contact Riemannian manifolds satisfying some special conditions [11] We call it Kenmotsu manifold. Kenmotsu manifolds have been studied by many authors such as J. B. Jun. U. C.De and G. Pathak [12] and others.

## 2. Kenmotsu manifolds

A smooth $n$ −dimensional manifold $(M^n, g)$ is said to be almost contact metric manifold [11] if it admits a (1,1)-tensor field $\varphi$, a vector field $\xi$, a 1-form $\eta$ and a Riemannian metric g which satisfy

- $\varphi^2 X = -X + \eta(X)\xi,$
- $g(X,\xi) = \eta(X), \eta(\xi) = 1, \varphi\xi = 0, \eta(\varphi X) = 0$
- $g(\varphi X, \varphi Y) = g(X,Y) - \eta(X)\eta(Y)$

where $X$, and $Y$ are arbitrary vector fields on $M$.
An almost contact manifold $M^n(\varphi,\eta,\xi,g)$ is said to be a Kenmotsu manifold if the following conditions hold:

- $(\nabla_X \varphi)(Y) = g(\varphi X, Y)\xi - \eta(Y)\varphi X$
- $\nabla_X \xi = X - \eta(X)\xi$

where $\nabla$ is the Levi-Civita connection

In Kenmotsu manifold the following relations are true.

- $(\nabla_X \eta)Y = g(X,Y) - \eta(X)\eta(Y)$
- $R(X,Y)\xi = \eta(X)Y - \eta(Y)X$
- $R(\xi,X)Y = -R(X,\xi)Y = \eta(Y)X - g(X,Y)\xi$
- $\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X)$
- $S(X,\xi) = -(n-1)\eta(X), Q\xi = -(n-1)\xi$

for arbitrary vector fields $X, Y$ and $Z$on $M$ and $R$is Riemannian curvature tensor and $S$ the Ricci tensor of type (0,2) and $Q$ the Ricci operator such that

- $S(X,Y) = g(QX,Y) = -(n-1)g(X,Y)$

A linear connection $\tilde{\nabla}$ in a Riemannian manifold $M$ is said be a semisymmetric metric connection [1] if its tensor $T$ of the connection $\tilde{\nabla}$

- $T(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$

satisfies

- $T(X,Y) = \eta(Y)X - \eta(X)Y$

where $\eta$ is 1-form and $\xi$ is the vector field given by

- $g(X,\xi) = \eta(X)$

for all vector fields $X, Y \in \chi(M)$. Here $\chi(M)$ is the set of all differentiable vector fields on $M$.
A semi-symmetric connection $\tilde{\nabla}$ is called a semi-symmetric metric connection [4]
if it further satisfies

- $\tilde{\nabla} g = 0$

A relation between the semi-symmetric metric connection $\tilde{\nabla}$ and the Levi-Civita connection $\nabla$has been given by K. Yano [3] which is given by

- $\tilde{\nabla}_X Y = \nabla_X Y + \eta(Y) - g(X,Y)\xi,$

where $\eta(Y) = g(Y,\xi)$.

**3. $W_6$ −Curvature Tensor of a Kenmotsu manifold with respect to**
semi-symmetric metric connection
Pokhariyal and Mishra [8] have introduced new tensor fields $W$and studied their properties. Pokhariyal defined the $W_6$-curvature tensor field in Riemannian manifolds as [10].

- $W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Y)QZ - S(Y,Z)X]$

**Definition 3.1:** The $W_6$- curvature tensor in Kenmotsu manifold with respect to Levi-Civita connection $\nabla$ is given by

$$W_6(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(X,Y)QZ - S(Y,Z)X]$$

**Definition 3.2:** The $W_6$-curvature tensor in Kenmotsu manifold with respect to the semi-symmetric metric connection $\tilde{\nabla}$ is defined by

- $\tilde{W}_6(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}[g(X,Y)\tilde{Q}Z - \tilde{S}(Y,Z)X]$

**Definition 3.3:** In Kenmotsu manifolds, a relation between the curvature tensor $R$ and $\tilde{R}$ of type (1,3) of the connections $\nabla$ and $\tilde{\nabla}$ respectively is given by [9].

- $\tilde{R}(X,Y)Z = R(X,Y)Z + [g(X,Z)Y - g(Y,Z)X] + 2[g(\varphi X,\varphi Z)Y - g(\varphi Y,\varphi Z)X] + [g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi]$

From (3.3) it follows that

- $\tilde{S}(X,Y) = S(X,Y) - (n-1)g(X,Y) - 2(n-2)g(\varphi Y,\varphi Z)$

where $\tilde{S}$denotes the Ricci tensor with respect to semi-symmetric metric connection and $S$ the Ricci tensor with respect to Levi-Civita connection.

**Definition 3.4:** A semi-symmetric metric connection in a manifold is said to be flat if the Riemannian curvature tensor with respect to the connection vanishes

i.e. $\tilde{R}(X,Y)Z = 0$

**Definition 3.5:** A semi-symmetric metric connection in a manifold is said to be $W_6 -$ flat if the $W_6$-curvature tensor with respect to the connection vanishes

i.e. $\widetilde{W}_6(X,Y)Z = 0$

**Theorem 3.6:** A $W_6$ −flat Kenmotsu manifold with respect to semi-symmetric metric connection is not flat connection.

**Proof:** From (3.2) we have,

$$\widetilde{W}_6(X,Y)Z = \tilde{R}(X,Y)Z + \frac{1}{n-1}\left[g(X,Y)\tilde{Q}Z - \tilde{S}(Y,Z)X\right]$$

Expanding (3.2) with respect to U yields

- $\widetilde{W}_6{}'(X,Y,Z,U) = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}\left[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)\right]$

Suppose (3.5) is $W_6$ −flat, then (3.3) reduces to

$$0 = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}\left[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)\right]$$

- $\tilde{R}'(X,Y,Z,U) = \frac{1}{n-1}\left[\tilde{S}(Y,Z)g(X,U) - g(X,Y)\tilde{S}(Z,U)\right]$

Putting $Z = \xi$ in (3.4) gives

- $\tilde{R}'(X,Y,\xi,U) = \frac{1}{n-1}\left[\tilde{S}(Y,\xi)g(X,U) - g(X,Y)\tilde{S}(\xi,U)\right]$

From (3.4), we have

- $\tilde{S}(Y,\xi) = -(n-1)g(Y,\xi) - (n-1)g(Y,\xi)$

$= -2(n-1)\eta(Y)$

- $\tilde{S}(\xi,U) = -(n-1)g(\xi,U) - (n-1)g(\xi,U)$

$= -2(n-1)\eta(U)$

Therefore (3.7) becomes

- $\tilde{R}'(X,Y,\xi,U) = 2g(X,Y)\eta(U) - 2g(X,U)\eta(Y) \neq 0$

Hence, the proof of the theorem.

**4. Geometrical relationship between $W_6$ −curvature tensors and $W_8$-curvature tensors in Kenmotsu manifold admitting semi-symmetric metric connection**

**Definition 4.1:** The following theorem is a statement of the findings of the study investigating some of the geometrical relationship between curvature tensors.

**Theorem 4.2:** A $W_6$-curvature tensor on a semi-symmetric metric connection is geometrically equivalent to $W_8$-curvature tensor along the Levi-Civita connection in a Kenmotsu manifold.

**Proof**

The $W_6$-curvature tensor In Kenmotsu manifold with respect to semi-symmetric metric connection is given by (3.2) and expands with respect to U as

- $\widetilde{W}_6{}'(X,Y,Z,U) = \tilde{R}'(X,Y,Z,U) + \frac{1}{n-1}[g(X,Y)\tilde{S}(Z,U) - \tilde{S}(Y,Z)g(X,U)]$

Putting $Z = \xi$ in (4.1) gives

- $\widetilde{W}_6{}'(X,Y,\xi,U) = \tilde{R}'(X,Y,\xi,U) + \frac{1}{n-1}[g(X,Y)\tilde{S}(\xi,U) - \tilde{S}(Y,\xi)g(X,U)]$

But from (3.3) we have

- $\tilde{R}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [g(X,\xi)g(Y,U) - g(Y,\xi)g(X,U)] + [g(Y,\xi)\eta(X)\eta(U) - g(X,\xi)\eta(Y)\eta(U)]$

Equation (4.3) reduces to

- $\tilde{R}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [\eta(X)g(Y,U) - \eta(Y)g(X,U)]$

Again, using (3.4) and (4.2) expression for $\tilde{S}$ can be easily made as

- $\tilde{S}(\xi,U) = -2(n-1)\eta(U)$ and
- $\tilde{S}(Y,\xi) = -2(n-1)\eta(Y)$

Hence, substituting $\tilde{S}$ and $\tilde{R}$ through (4.4), (4.5) and (4.6) into (4.2) gives

- $\widetilde{W}_6{}'(X,Y,\xi,U) = R'(X,Y,\xi,U) + [\eta(X)g(Y,U) - \eta(Y)g(X,U)] + 2[g(X,U)\eta(Y) - g(X,Y)\eta(U)]$

Interchanging$U$ and $\xi$ makes the tensors skew symmetric. Therefore, dividing by minus one yields;

- $\widetilde{W}_6{}'(X,Y,U,\xi) = R'(X,Y,U,\xi) - [\eta(X)g(Y,U) - \eta(Y)g(X,U)] - 2[g(X,U)\eta(Y) - g(X,Y)\eta(U)]$

Contracting (4.8) with respect to $\xi$ gives

- $\widetilde{W}_6(X,Y)U = R(X,Y)U - [g(Y,U)X - g(X,U)Y] - 2[g(X,U)Y - g(X,Y)U]$

Putting $Z = U$ in (4.9) gives

- $\widetilde{W}_6(X,Y)Z = R(X,Y)Z - [g(Y,Z)X - g(X,Z)Y] - 2[g(X,Z)Y - g(X,Y)Z]$

Simplifying (4.10) becomes

- $\widetilde{W}_6(X,Y)Z = 2[g(X,Z)Y - g(Y,Z)X] - 2[g(X,Z)Y - g(X,Y)Z]$

Therefore, (4.11) reduces to

- $\widetilde{W}_6(X,Y)Z = 2[g(X,Y)Z - g(Y,Z)X]$

**Definition 4.3:** Pokhariyal defined the $W_8$-curvature tensor field in Riemannian manifolds as [10].

- $W_8(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[S(X,Y)Z - S(Y,Z)X]$

This Curvature tensor in a Kenmotsu manifold with respect to Levi-Civita connection has the given geometrical properties expressed in the expansion of (4.13) below

- $W_8(X,Y)Z = R(X,Y)Z - [g(X,Y)Z - g(Y,Z)X]$

Expanding (4.14) with respect to $U$ becomes

- $W_8(X,Y,Z,U) = R(X,Y,Z,U) - [g(X,Y)g(Z,U) - g(Y,Z)g(X,U)]$

Putting $Z = \xi$ in (4.15) gives
$W_8(X,Y,\xi,U) = R(X,Y,\xi,U) - [g(X,Y)g(\xi,U) - g(Y,\xi)g(X,U)]$

- $W_8(X,Y,\xi,U) = R(X,Y,\xi,U) - [g(X,Y)\eta(U) - g(X,U)\eta(Y)]$

Interchanging$U$ and $\xi$ in (4.16) makes the tensors skew symmetric. Therefore, dividing by minus one yields;

- $W_8(X,Y,U,\xi) = R(X,Y,U,\xi) + [g(X,Y)\eta(U) - g(X,U)\eta(Y)]$

Contracting (4.17) with respect to $\xi$ gives

- $W_8(X,Y)U = R(X,Y)U + [g(X,Y)U - g(X,U)Y]$

Replacing $U$ with $Z$ in equation (4.18) to get

$W_8(X,Y)Z = R(X,Y)Z + [g(X,Y)Z - g(X,Z)Y]$

- $W_8(X,Y)Z = [g(X,Z)Y - g(Y,Z)X] + [g(X,Y)Z - g(X,Z)Y]$

Equation (4.19) simplifies to

- $W_8(X,Y)Z = [g(X,Y)Z - g(Y,Z)X]$

Comparing (4.20) and (4.12)

- $\widetilde{W}_6(X,Y)Z = 2W_8(X,Y)Z$

This completes the proof of the theory.

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