International Journal of Statistics and Applied Mathematics 2021; 6(2): 72-80

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2021; 6(2): 72-80

www.mathsjournal.com
Received: 16-01-2021
Accepted: 27-02-2021

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Partially backlogging inventory model for perishable items with reliability and price varying demand

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DOI: https://doi.org/10.22271/maths.2021.v6.i2a.714

Abstract
Reliability is an important factor of any inventory system for attracting the customers towards its products. Mostly highly reliable products are preferred by the customers in the real/super market. Assuming, reliability any business organization can reduce their total cost or total profit. The proposed paper deals partially backlogging inventory model for perishable items with reliability and price varying demand. The shortages are partially backlogged. The unsatisfied demand is backlogged and governed by a function of waiting time. The objective of the study is to optimizing the total cost and selling price during a given period of time. The validity of the proposed model is shown by a numerical example along with the behavior of key parameters.

Keywords: Backlogging, inventory, reliability, perishability and selling price

Introduction
In all the realistic situations, the demand is not fixed, but it changes in dynamical nature. The demand is affected by the reliability and selling price of the products. The demand of the reliable products increases which also increases their selling price. Keeping these in view, the business organizations can reduce their total cost. In this area, Bag *et al.* [1] constructed a production inventory model for deteriorating items with fuzzy type random demand and reliability. Benkherouf and Balkhi [2] developed an inventory model for perishable items with time varying demand. Chang and Dye [3] analyzed an EOQ model for deteriorating items with time varying demand and shortages. Chen [4] studied an EOQ model for perishable items with time dependent demand and shortages under inflation and time discounting. Chung and Wee [5] presenting the scheduling and replenishment plan for an inventory model of deteriorating items with stock dependent selling rate. Das *et al.* [6] worked on a production inventory model for deteriorating items with partial trade credit policy and reliability. Datta and Pal [7] considered a deterministic inventory system for perishable items with stock dependent demand and shortages. Hou [8] analyzed an inventory model for deteriorating items with stock dependent demand and shortages under inflation and time discounting. Huang *et al.* [9] presented the coordination of inventory, pricing, and reliability for a supply chain inventory model of deteriorating items. Islam *et al.* [10] studied three stage supply chain inventory model with random capacities, disruptions and suppliers reliability. Jaggi *et al.* [11] analyzed the optimal replenishment policy for an EOQ model of perishable items with two level trade credit policy and credit period varying demand. Khanra *et al.* [12] focused an EOQ model for deteriorating items with time varying quadratic demand and permissible delay in payment. Liao *et al.* [13] constructed an inventory model for perishable items with permission delay in payment under inflation. Mahapatra *et al.* [14] worked on inventory model of deteriorating items with reliability and time varying demand along with shortages. Paul *et al.* [15] considered a production inventory model for perishable items with uncertainty and reliability varying demand. Paul *et al.* developed two inventory models [16, 17]. In the model [16] they were constructed a production inventory model with reliability dependent demand and disruptions. And in the model [17] they were discussed the real time disruption management for two stage production system with reliability. Pal *et al.* [18] focused on inventory model of deteriorating items with price and stock varying demand allowing permission delay in payment.

Pal *et al.* [19] analyzed a deterministic inventory model for perishable items with stock dependent demand. Sana and Chaudhuri [20] discussed a deterministic EOQ model for deteriorating items with price discount offer and permission delay in payment. Sarkar [21] constructed an inventory model consisting reliability in imperfect production process. Shah and Soni [22] studied multi object production system with fuzzy random demand and reliability. Tripathy *et al.* [23] analyzed an EOQ for deteriorating items with reliability.

Notations and Assumptions Used in the Study: The following notations and assumptions are used in the proposed model

1. Selling price per unit item is p (decision variable).
2. Reliability parameter is r.
3. Price and reliability varying demand is $R(t) = (\alpha - p\beta)r^a, \quad \alpha, \beta, a > 0$.
4. Constant deterioration rate is $\theta(t) = \theta$.
5. Ordering cost per order is c_1.
6. Lost sales cost per cycle is c_2.
7. Inventory carrying cost per unit per unit time is h.
8. Shortage cost per cycle is s.
9. Backlogging parameter is δ.
10. Time of zero inventory level is T_1 (decision variable).
11. Inventory cycle length is T (decision variable).
12. Total cost per cycle is $\Pi(T, T_1, p)$.
13. Lead time is zero.
14. Replenishment rate is finite.

Mathematical Derivation of the Model: Graphically, it is assumed that the inventory system consists the maximum inventory level Q in the beginning of the cycle. In the interval $[0, T_1]$, the inventory level decreases due to demand and deterioration and becomes zero at time $t = T_1$. Just after the time $t = T_1$ shortages start and which are partially backlogged at a rate of $\frac{1}{1+\delta(T-t)}$, where δ is the backlogging parameter and t is the waiting time.
In the cycle, the inventory level at any time t is given by the following differential equations,

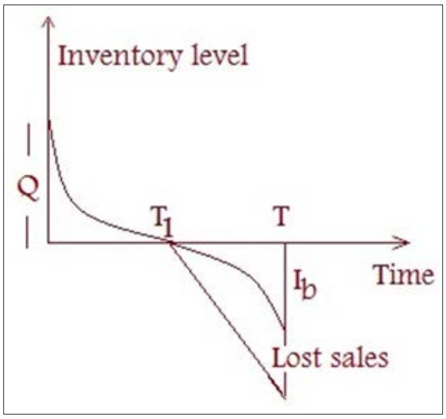


Fig 1: Inventory Model

$$\frac{dI}{dt} + \theta I = -(\alpha - p\beta)r^a, \qquad 0 \le t \le T_1 \tag{1}$$

With condition, $I(T_1) = 0$

$$\frac{dI}{dt} = -\frac{(\alpha - p\beta)r^a}{1+\delta(T-t)}, \qquad T_1 \le t \le T \tag{2}$$

With condition, $I(T_1) = 0$
The solutions of the equations (1) and (2) are given by the equations (3) and (4) respectively

$$I = (\alpha - p\beta)r^a\left[T_1 - t - \theta t T_1 + \theta t^2\right] \tag{3}$$

$$I=(\alpha-p\beta)r^{a}\left[T_{1}-t-\delta TT_{1}+\delta Tt+\frac{\delta}{2}T_{1}^{2}-\frac{\delta}{2}t^{2}\right] \tag{4}$$

The initial inventory level $Q=I(0)$ is obtained by putting $t=0$ in equation (3), we have

$$Q=(\alpha-p\beta)r^{a}T_{1} \tag{5}$$

The back ordered or shortage quantity I_b is obtained by putting $t=T$ in the equation (4), we have

$$I_{b}=-\frac{(\alpha-p\beta)r^{a}}{2}\left[2T_{1}-2T-2\delta TT_{1}+\delta T^{2}+\delta T_{1}^{2}\right] \tag{6}$$

Therefore, the maximum inventory level is $Q^{*}=Q+I_{b}$ Or

$$Q^{*}=\frac{(\alpha-p\beta)r^{a}}{2}\left[2T+2\delta TT_{1}-\delta T^{2}-\delta T_{1}^{2}\right] \tag{7}$$

Inventory ordering cost per cycle is, $C_{O}=c_{1}$ (8)

Inventory holding cost per cycle is, $C_{H}=h\int_{0}^{T_{1}}I(t)dt$ Or

$$C_{H}=\frac{(\alpha-p\beta)r^{a}}{6}\left[3T_{1}^{2}-\theta T_{1}^{3}\right] \tag{9}$$

Shortage cost per cycle is, $C_{S}=-s\int_{T_{1}}^{T}I(t)dt$ Or

$$C_{S}=\frac{s(\alpha-p\beta)r^{a}}{6}\left[3T_{1}^{2}+3T^{2}-6TT_{1}+6\delta T_{1}T^{2}-6\delta TT_{1}^{2}+2\delta T_{1}^{3}-2\delta T^{3}\right] \tag{10}$$

Lost sales or opportunity cost per cycle is,

$$C_{L}=c_{2}\int_{T_{1}}^{T}(\alpha-p\beta)r^{a}\left[1-\frac{1}{1+\delta(T-t)}\right] \text{Or } C_{L}=\frac{c_{2}\delta(\alpha-p\beta)r^{a}}{2}\left[T_{1}^{2}+T^{2}-2TT_{1}\right] \tag{11}$$

Purchasing cost per cycle is, $C_{P}=cQ^{*}$ Or $C_{P}=\frac{c(\alpha-p\beta)r^{a}}{2}\left[2T+2\delta TT_{1}-\delta T^{2}-\delta T_{1}^{2}\right]$ (12)

Total inventory cost per cycle is,

$$\Pi(T,T_{1},p)=\frac{1}{T}\left[C_{O}+C_{H}+C_{S}+C_{L}+C_{P}\right]$$

$$\Pi(T,T_{1},p)=\frac{1}{T}\Big[c_{1}+c(\alpha-p\beta)r^{a}T+\frac{(\alpha-p\beta)r^{a}}{2}(h+s+c_{2}\delta-c\delta)T_{1}^{2}+\frac{(\alpha-p\beta)r^{a}}{2}(s+c_{2}\delta-c\delta)T^{2}$$

$$-(\alpha-p\beta)r^{a}(s+c_{2}\delta-c\delta)TT_{1}-\frac{(\alpha-p\beta)r^{a}}{6}(h\theta-2s\delta)T_{1}^{3}-\frac{s\delta(\alpha-p\beta)r^{a}}{3}T^{3}$$

$$+s\delta(\alpha-p\beta)r^{a}T_{1}T^{2}-s\delta(\alpha-p\beta)r^{a}TT_{1}^{2}\Big] \tag{13}$$

After partially differentiating equation (13), we obtain the following

$$\frac{\partial\Pi(T,T_1,p)}{\partial T_1}=\frac{(\alpha-p\beta)r^{a}}{T}\left[(h+s+c_2\delta-c\delta)T_1-(s+c_2\delta-c\delta)T-\frac{(h\theta-2s\delta)}{2}T_1^{2}+s\delta T^{2}-2s\delta TT_1\right] \tag{14}$$

$$\frac{\partial\Pi(T,T_1,p)}{\partial T}=\frac{(\alpha-p\beta)r^{a}}{T}\left[c+(s+c_2\delta-c\delta)T-(s+c_2\delta-c\delta)T_1-s\delta T^{2}+2s\delta TT_1-s\delta T_1^{2}\right]$$

$$-\frac{1}{T^{2}}\Big[c_1+c(\alpha-p\beta)r^{a}T+\frac{(\alpha-p\beta)r^{a}}{2}(h+s+c_2\delta-c\delta)T_1^{2}+\frac{(\alpha-p\beta)r^{a}}{2}(h+s+c_2\delta-c\delta)T^{2}$$

$$-(\alpha-p\beta)r^{a}(s+c_2\delta-c\delta)TT-\frac{(\alpha-p\beta)r^{a}}{6}(h\theta-2s\delta)T_1^{3}-\frac{s\delta(\alpha-p\beta)r^{a}}{3}T^{3}$$

$$+s\delta(\alpha-p\beta)r^{a}T_1T^{2}-s\delta(\alpha-p\beta)r^{a}TT_1^{2}\Big]$$

$$\frac{\partial\Pi(T,T_1,p)}{\partial p}=-\frac{\beta r^{a}}{T}\Big[cT+\frac{1}{2}(h+s+c_2\delta-c\delta)T_1^{2}+\frac{1}{2}(s+c_2\delta-c\delta)T^{2}-(s+c_2\delta-c\delta)TT_1$$

$$-\frac{1}{6}(h\theta-2s\delta)T_1^{3}-\frac{s\delta}{3}T^{3}+s\delta T_1T^{2}-s\delta T_1T^{2}-s\delta TT_1^{2}\Big] \tag{16}$$

The total cost will be necessarily minimum, if

$$\frac{\partial\Pi(T,T_1,p)}{\partial T_1}=0,\qquad \frac{\partial\Pi(T,T_1,p)}{\partial T}=0,\qquad \frac{\partial\Pi(T,T_1,p)}{\partial p}=0 \tag{17}$$

After solving the three equations in equation (17), we obtain the optimum values of the decision variables for which the total cost is optimum.

Example: Numerically, we consider the following data in appropriate units for the parameters of developed model

$\alpha=500,\ \ \beta=4,\ \ a=2,\ \ \theta=0.01,\ \ \delta=0.1,\ \ r=0.5,\ \ c_1=\$50,\ \ h=\$8,\ \ s=\$3,\ \ c_2=\$5,\ \ c=\$20$

Table 1: Variation in total cost w.r.to θ

θ	T	T_1	p	$\Pi(T,T_1,p)$
0.01	350.1461	289.0218	128.5224	8.2386
0.03	133.3068	96.6608	130.5135	0.3750
0.04	105.6570	73.0693	131.6331	0.5546
0.06	77.7353	49.6525	134.6439	0.6439
0.07	69.7211	43.0132	136.7302	1.1627
0.08	63.6499	38.0752	139.7058	1.3587

The table 1, shows that as we increase the deterioration parameter θ, the total cost and the selling price are increased. The reason is that the holding cost is increased. The cycle length T is decreased.

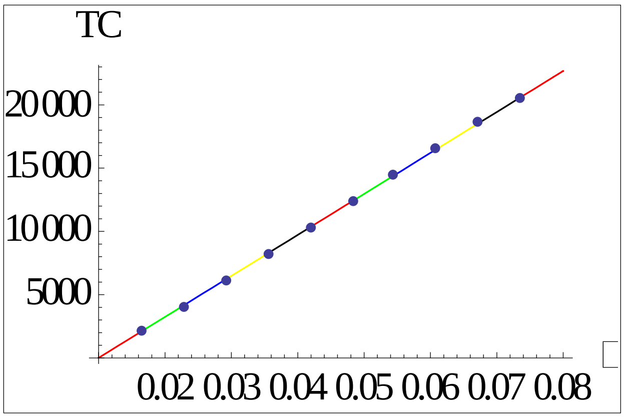


Fig 2: variation in total cost w.r.to θ

Table 2: variation in total cost w.r.to δ

δ	T	T_1	p	$\Pi(T,T_1,p)$
0.1	350.1461	289.0218	128.5224	8.2386
0.2	335.6951	293.0565	127.9668	7.5635
0.4	325.9253	296.3079	127.6309	7.1534
0.5	323.4917	297.1967	127.5513	7.0557
0.6	326.6734	301.9503	127.4439	50.5615
0.7	325.4909	302.8216	127.4057	52.0532

The table 2, shows that as we increase the backlogging parameter δ, the total cost is increased. The reason is that the purchasing or ordering cost is increased. The cycle length T is decreased.

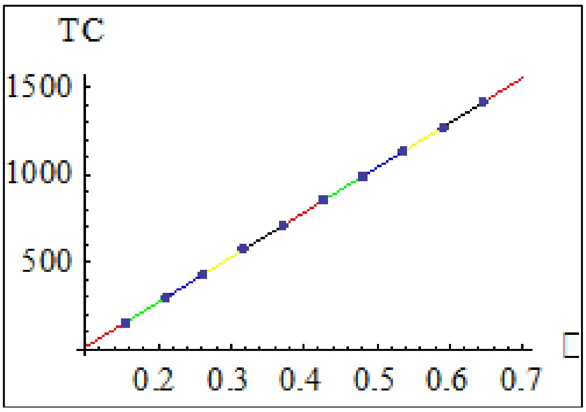
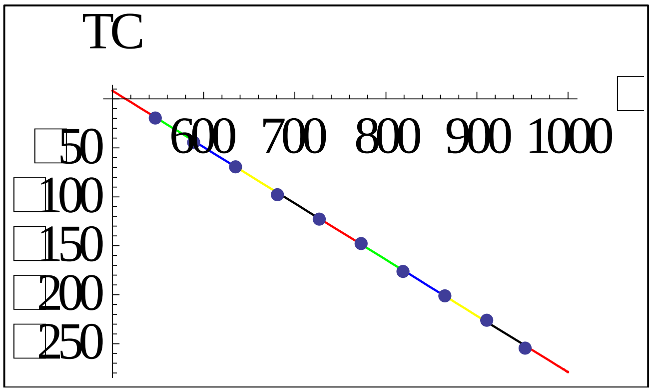


Fig 3: Variation in total cost w.r.to δ

Table 3: Variation in total cost w.r.to α

α	T	T_1	p	$\Pi(T,T_1,p)$
500	350.14610	289.02180	128.5224	8.2386
600	357.47150	294.15250	153.5598	84.6253
700	357.45151	294.15251	178.5548	84.5066
800	357.47152	294.15253	203.5396	84.1461
900	357.47154	294.15255	228.5346	84.0264
1000	357.47157	294.15257	253.5295	83.9066

From the table 3, we see that as we increase the parameter α, the total cost is decreased. The reason is that the ordering cost is increased. The cycle length T and selling price p are increased.



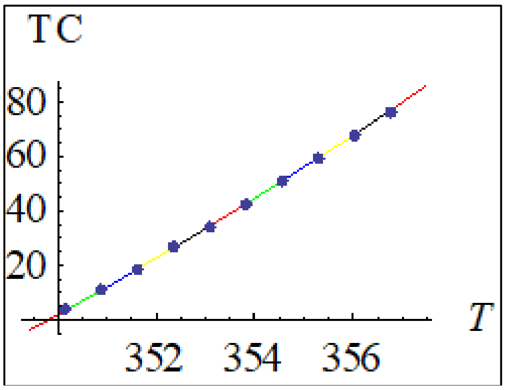
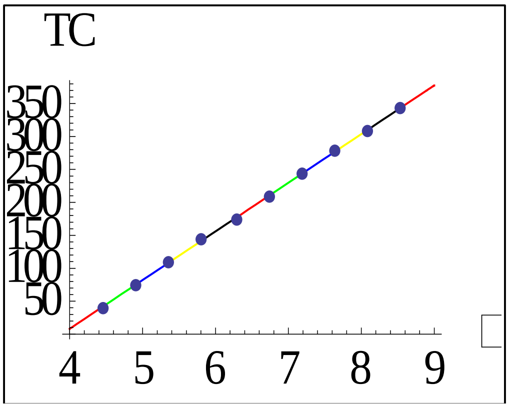


Fig 4: Variation in total cost w.r.to α **Fig 5:** variation in total cost w.r.to T

Table 4: variation in total cost w.r.to β

β	T	T_1	p	$\Pi(T, T_1, p)$
4	350.14610	289.021760	128.52240	8.23845
5	350.14611	289.021761	102.88012	8.41714
6	350.14613	289.021762	85.71035	8.33761
7	350.14615	289.021764	73.45193	8.34097
8	350.14616	289.021766	64.39085	8.83359
9	350.14618	289.021768	57.14023	8.83760

From the table 4, we observe that as we increase the parameter β, the total cost is increased. The cycle length T is slightly increased and selling price p is decreased.



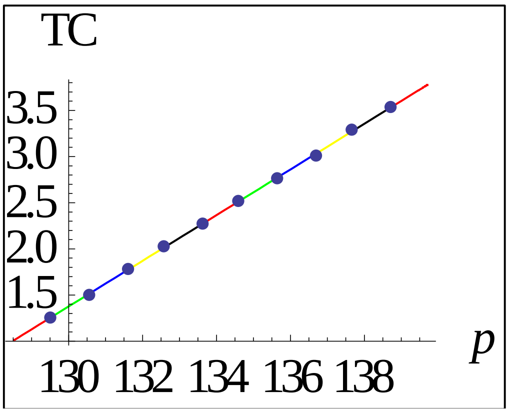


Fig 6: variation in total cost w.r.to β **Fig 7:** variation in total cost w.r.to p

Table 5: Variation in total cost w.r.to r

r	T	T_1	p	$\Pi(T, T_1, p)$
0.50	350.14610	289.021760	128.52238	8.23845
0.60	350.14611	289.021761	127.47839	8.33533
0.65	350.14613	289.021763	127.10904	8.34475
0.70	350.14615	289.021764	126.79638	8.34506
0.75	350.14617	289.021765	126.58617	8.34533
0.85	350.14618	289.021766	126.23330	8.34465

The table 5, shows that as we increase the reliability parameter r, the total cost is increased. The reason is that the ordering cost is increased. The cycle length T is slightly increased.

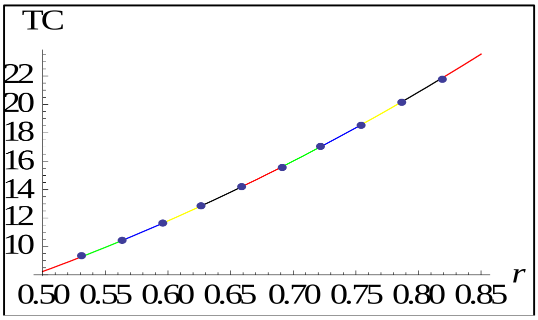


Fig 8: Variation in total cost w.r.to r

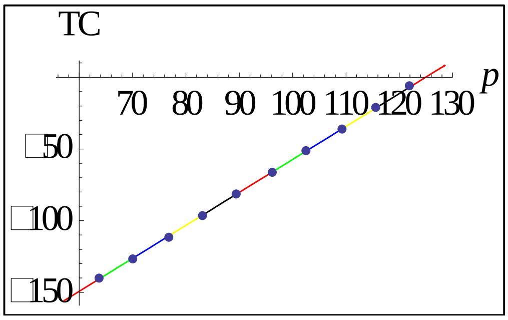


Fig 9: variation in total cost w.r.to p

Conclusion: In this work, partially backlogging inventory model for perishable items with reliability and price varying demand is studied. The relations among reliability, selling price and the total cost of the products show that the more reliable products have the much cost. Reliability is much affected by the backlogging and the cycle length which increases the demand. The more reliable products reduce the cost or profit of any firm. Therefore, the firm wants to sell or produce the reliable products for optimizing their cost or profit. In future this model can be generalized to incorporate some realistic features such as advertising, advance payment with installment facility, partial order cancellations, cash discount, delay in payment, replenishment policy etc. so that the system become more economical.

References

1. Bag S, Chakrabarty D, Roy AR. A production inventory model for deteriorating items with fuzzy type random demand and reliability. Computers and Industrial Engineering 2009;56(1):411-416.
2. Benkherouf L, Balkhi ZT. An inventory model for perishable items with time varying demand. Mathematical Methods of Operations Research 1997;45(2):221-233.
3. Chang HJ, Dye CY. An EOQ model for deteriorating items with time varying demand and Shortages. Journal of Operational Research Society 2001;50:1176-1182.
4. Chen JM. An EOQ model for perishable items with time varying demand and shortages under time discounting and inflation. International Journal of Production Economics 1998;55:21-30.
5. Chung CJ, Wee HM. A replenishment and scheduling plan for an inventory model of deteriorating items with stock dependent selling rate. International Journal of Advanced Management Technology 2008;35(7-8):665-679.
6. Das S, Khan Md. AA, Mahmoud EE, Aty AAH, Abualnja KM, Shaik AA. A production inventory model with partial trade credit policy and reliability. Alexandria Engineering Journal 2021;60:1325-1338.
7. Datta TK, Pal AK. A deterministic inventory model for deteriorating items with stock dependent demand and shortages. Opsearch 1990;27:213-224.
8. Hou KL. An inventory model for deteriorating items with stock dependent consumption rate and shortage under inflation and time discounting. Journal of Operational Research 2006;168:463-474.
9. Huang H, He Y, Li D. Coordination of inventory, pricing in a production supply chain inventory model for deteriorating items. International Journal of Production Research 2018;56(18):6201-6224.
10. Islam Md. T, Azeem A, Jabir M, Paul A, Paul SK. An inventory model for a three stage supply chain with random capacities considering disruptions and supplier reliability. Annals of Operations Research, 2021.
11. Jaggi CK, Kapur PK, Goyal SK, Goel SK. An optimal replenishment and credit policy for an EOQ model under two levels of trade credit policy with credit period induced demand. International Journal of System Assurance Engineering and Management 2012;3(4):352-359.
12. Khanra S, Ghosh SK, Chaudhuri KS. An EOQ model for deteriorating items with time dependent quadratic demand under permission delay in payment. Applied Mathematics and Computation 2011;218:1-9.
13. Liao HC, Tsai CH, Su CT. An inventory model for deteriorating items with permission delay in payment under inflation. International Journal of Production Economics 2000;63:207-214.
14. Mahapatra GS, Adak S, Mandal TK, Pal S. Inventory system for deteriorating items with reliability and time varying demand and shortages. International Journal of Operational Research 2017;29(3):344-359.
15. Paul SK, Azeem A, Sarkar R, Essam D. A production inventory model for deteriorating items with reliability and uncertainty. Optimization and Engineering 2014;15(3):697-720.
16. Paul SK, Sarkar R, Essam D. A production inventory model with disruption and considering reliability. In Proceeding of International Conference on Computers and Industrial Engineering Hong Kong 2013;1:291-304.
17. Paul SK, Sarkar R, Essam D. A two stage batch production inventory model for real time disruption and reliability. European Journal of Operational Research 2014;237(1):113-128.
18. Pal S, Mahapatra GS, Samantha GP. An inventory model for deteriorating items with stock varying demand under inflation and delay in payment. International Journal of System Assurance Engineering and Management 2014;5(4):591-601.
19. Pal S, Goswami A, Chaudhuri KS. An EPQ model for weibull deteriorating items with ramp type demand under inflation and fuzzy type finite horizon. International Journal of Production Economics 2014;156:159-166.
20. Sana SS, Chaudhuri KS. A deterministic EOQ model with price discount offers and delay in payments. European Journal of Operational Research 2008;184:509-533.
21. Sarkar B. An inventory model with imperfect production process and reliability. Applied Mathematics and Computation, 2012;218(9):4881-4891.

22. Shah NH, Soni H. A multi-object production inventory model with backorder and fuzzy random demand under reliability and flexibility. Journal of Mathematical Modeling and Algorithms 2011;10(4):341-356.
23. Tripathy PK, Wee WM, Majhi PR. An EOQ model for deteriorating items with reliability consideration. Journal of Operational Research Society 2003;54:549-554.

Appendix 1

For the optimality of total cost $\Pi(T, T_1, p)$, the second order derivatives are

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T_1^2} = \frac{(\alpha - p\beta)r^a}{T}\left[(h + s + c_2 s - c\delta) - (h\theta - 2s\delta)T_1 - 2s\delta T\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T_1 \partial T} = \frac{(\alpha - p\beta)r^a}{T}\left[-(s + c_2\delta - c\delta) - 2s\delta T - 2s\delta T_1\right] - \frac{(\alpha - p\beta)r^a}{T^2}\left[(h + s + c_2\delta - c\delta)T_1\right.$$

$$\left. -(s + c_2\delta - c\delta)T - \frac{(h\theta - 2s\delta)}{2}T_1^2 + s\delta T^2 - 2s\delta T_1\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T_1 \partial p} = -\frac{\beta r^a}{T}\left[(h + s + c_2\delta - c\delta)T_1 - (s + c_2\delta - c\delta)T - \frac{(h\theta - 2s\delta)}{2}T_1^2 + s\delta T^2 - 2s\delta T T_1\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T \partial T_1} = \frac{(\alpha - p\beta)r^a}{T}\left[-(s + c_2\delta - c\delta) + 2s\delta T - 2s\delta T_1\right] - \frac{(\alpha - p\beta)r^a}{T^2}\left[(h + s + c_2\delta - c\delta)T_1\right.$$

$$\left. -(s + c_2\delta - c\delta)T - \frac{(h\theta - 2s\delta)}{2}T_1^2 + s\delta T^2 - 2s\delta T T_1\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T^2} = \frac{(\alpha - p\beta)r^a}{T}\left[(s + c_2\delta - c\delta) - 2s\delta T + 2s\delta T_1\right] - \frac{2(\alpha - p\beta)r^a}{T^2}\left[c + (s + c_2\delta - c\delta)T\right.$$

$$\left. -(s + c_2\delta - c\delta)T_1 - s\delta T^2 + 2s\delta T T_1 - 2s\delta T_1^2\right] + \frac{2}{T^3}\left[c_1 + c(\alpha - p\beta)r^a T\right.$$

$$+ \frac{(\alpha - p\beta)r^a}{2}(h + s + c_2\delta - c\delta)T_1^2 + \frac{(\alpha - p\beta)r^a}{2}(s + c_2\delta - c\delta)T^2 - (\alpha - p\beta)r^a(s + c_2\delta - c\delta)T T_1$$

$$\left. - \frac{(\alpha - p\beta)r^a}{6}(h\theta - 2s\delta)T_1^3 - \frac{(\alpha - p\beta)r^a}{3}s\delta T^3 + s\delta(\alpha - p\beta)r^a T_1 T^2 - s\delta(\alpha - p\beta)r^a T T_1^2\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial T \partial p} = -\frac{\beta r^a}{T}\left[c + (s + c_2\delta - c\delta)T - (s + c_2\delta - c\delta)T_1 - s\delta T^2 + 2s\delta T T_1 - s\delta T_1^2\right]$$

$$+ \frac{\beta r^a}{T^2}\left[cT + \frac{(h + s + c_2\delta - c\delta)}{2}T_1^2 + \frac{(s + c_2\delta - c\delta)}{2}T^2 - (s + c_2\delta - c\delta)T T_1\right.$$

$$\left. - \frac{(h\theta - 2s\delta)}{6}T_1^3 - \frac{s\delta}{3}T^3 + s\delta T_1 T^2 - s\delta T T_1^2\right]$$

$$\frac{\partial^2 \Pi(T, T_1, p)}{\partial p \partial T_1} = -\frac{\beta r^a}{T}\left[(h + s + c_2\delta - c\delta)T_1 - (s + c_2\delta - c\delta)T - \frac{(h\theta - 2s\delta)}{2}T_1^2 + s\delta T^2 - 2s\delta T T_1\right]$$

$$\frac{\partial^2 \Pi(T,T_1,p)}{\partial p \partial T} = -\frac{\beta r^a}{T}\left[c + (s + c_2\delta - c\delta)T - (s + c_2\delta - c\delta)T_1 - s\delta T^2 + 2s\delta T T_1 - s\delta T_1^2\right]$$

$$+\frac{\beta r^a}{T^2}\left[cT + (h + s + c_2\delta - c\delta)T_1^2 + (s + c_2\delta - c\delta)T^2 - (s + c_2\delta - c\delta)TT_1\right.$$

$$\left.-\frac{(h\theta - 2s\delta)}{6}T_1^3 - \frac{s\delta}{3}T^3 + s\delta T_1 T^2 - s\delta T T_1^2\right] \quad \frac{\partial^2 \Pi(T,T_1,p)}{\partial p^2} = 0$$

Appendix 2

Total cost $\Pi(T,\ T_1,\ p)$ will be sufficiently minimum, if all the principal minors of the Hessian matrix or H matrix are positive definite. The Hessian matrix is defined as follows

$$H = \begin{bmatrix} \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T_1^2} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T_1 \partial T} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T_1 \partial p} \\ \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T \partial T_1} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T^2} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial T \partial p} \\ \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial p \partial T_1} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial p \partial T} & \dfrac{\partial^2 \Pi(T,T_1,p)}{\partial p^2} \end{bmatrix}$$

Or

$$H = \begin{bmatrix} 0.5059 & 1.3857 & 0 \\ -0.3539 & 0.2959 & 2.8756 \\ 0 & 6.8619 & 0 \end{bmatrix}$$