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Arthur Wanyonyi Wafula
University of Nairobi, School of Mathematics, PO Box 30197, Nairobi, Kenya

JM Khalaghai
University of Nairobi, School of Mathematics, PO Box 30197, Nairobi, Kenya

Corresponding Author:
Arthur Wanyonyi Wafula
University of Nairobi, School of Mathematics, PO Box 30197, Nairobi, Kenya

The numerical range of some operators on a finite dimensional Hilbert space

Arthur Wanyonyi Wafula and JM Khalaghai

Abstract
In this article we consider the geometry of the numerical range and show that if T is a hypernormal operator on a finite dimensional Hilbert space then its Numerical Range W (T) is Polygonal. AMS200 Mathematics classification 47B47, 47A30, 47B20.

Keywords: Numerical range, hypernormal, selfadjoint operators

Introduction

The well-known Toeplitz Horsdoff theorem touches on the geometry of the numerical range of an operator stating that it is a bounded convex subset of the complex numbers. However, if the operator T is a selfadjoint operator then the numerical range is an interval of the real numbers. An introduction to the classical numerical range is found in [1] and a simple proof that the numerical range of an operator on a two dimensional Hilbert space is elliptical is given. We proceed to show that if T is a Hypernormal operator on a finite dimensional Hilbert space then its numerical range is a polygon. We first consider an operator whose numerical range is a line segment in the complex plane. This type of operators includes all selfadjoint operators.

Theorem 1.1

An operator T has its numerical range on a line segment if and only if $T = \alpha I + \beta A$ where A is a selfadjoint operator.

Proof

Let W(T) be a line segment the $W(T) = \{\alpha + \beta\lambda;\ \lambda \in R\}$.
Hence $\beta^{-1}(\alpha + \beta\lambda) - \beta^{-1}\alpha$ is real. Now let $A = \beta^{-1}T - \beta^{-1}\alpha I$. Then for any unit vector x we have that $(Ax, x) = \beta^{-1}T(x, x) - \beta^{-1}\alpha I(x, x) = \beta^{-1}(\alpha + \beta\lambda) - \beta^{-1}\alpha$ is real. Thus, A is a self - adjoint operator and $T = \alpha I + \beta T$.
Conversely if $T = \alpha I + \beta A$ where A is selfadjoint then $\lambda \in W(T)$ implies there exists a unit vector x such that $\lambda = (Tx, x) = ((\alpha I + \beta T)x, x) = \alpha(x, x) + \beta(Ax, x) = \alpha + \beta\lambda$ where λ is real. Hence W(T) is a line segment.

Example 1.2

Let

$$T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and $\lambda \in W(T)\ then\ \lambda = (Tx, x)$ for some unit vector x. Thus

$$\lambda = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\binom{w}{z}, \binom{w}{z}\right) = \left(\binom{-z}{w}, \binom{w}{z}\right) = z\bar{w} + w\bar{z} = 2Im(z\bar{w})$$

Evidently the numerical range of T is a pure imaginary set and it must then be a line segment. If we parameterize w and z so that $w = r_1 e^{i\theta_1}$ and $z = r_2 e^{i\theta 2}$ then we have that $\lambda = -r_1 r_2 e^{\theta_2 - \theta_1} + r_1 r_2 e^{\theta_1 - \theta_2} = 2r_1(1 - r_1^2)^{\frac{1}{2}} sin(\theta_2 - \theta_1) = 2r(1 - r^2)^{\frac{1}{2}} sin\,\psi$.

The supremum of the absolute value of the last term occurs when $r = \frac{1}{\sqrt{2}}$ and $sin\,\psi = 1$. Hence $w(T) = 1$.
On the other hand we have that $\| Tx \|^2 = (Tx, Tx) = z\bar{z} + w\bar{w} = 1$ whenever x is a unit vector. Thus $\| T \|^2 = 1$ leading to $| T \| = w(T)$.

Theorem 1.3
If $T = \beta A$ where A is a selfadjoint operator and β is a non-zero complex number then $\| T \| = w(T)$.

Proof
$\beta^{-1}T = A$ is a self adjoint operator. Hence $\| \beta^{-1}T \| = w(\beta^{-1}T)$. Consequently, $| \beta^{-1} | \| T \| = | \beta^{-1} | w(T)$ and so $\| T \| = w(T)$.

Theorem 1.4
If T is an operator whose numerical range is a line segment, then T is normal.

Proof
Let $T = \alpha I + \beta A$ where A is a selfadjoint then $TT^* = (\alpha I + \beta A)(\bar{\alpha} + \bar{\beta}A) = | \alpha |^2 + \alpha\bar{\beta}A + \beta\bar{\alpha}A + | \beta |^2$.
On the other hand $T^*T = (\bar{\alpha}I + \bar{\beta}A)(\alpha I + \beta A) = | \alpha |^2 + \bar{\alpha}\beta A + \bar{\beta}\alpha A + | \beta |^2$.
Hence $TT^* = T^*T$.

Theorem 1.5
If T is normal and λ is an eigenvalue for T then $\bar{\lambda}$ is an eigenvalue for T^*.
Let x be an eigen vector corresponding to λ then $Tx = \lambda x$. Hence $(T - \lambda I)x = 0$.
Now $\| (T^* - \bar{\lambda}I)x \| = \| (T - \lambda I)x \| = 0$. Hence $T^*x = \bar{\lambda}x$.

Theorem 1.6
If T is a normal operator then any eigenvectors corresponding to distinct eigenvalues of T are orthogonal.

Proof
Let x and y be eigenvectors of T i.e $Tx = \lambda x$ and $Ty = \beta y, \lambda \neq \beta$ then
$\lambda(x, y) = (\lambda x, y) = (Tx, y) = (x, T^*y) = (x, \bar{\beta}y) = \beta(x, y)$.
Hence $(\lambda - \beta)(x, y) = 0$. Consequently $(x, y) = 0$ since $\lambda - \beta \neq 0$.

Theorem 1.7
If T is a normal operator on a two dimensional Hilbert Space H then W (T) is a line segment.

Proof
Let u and v be normalized eigenvectors corresponding to distinct eigenvalues α, β then by the previous theorem they are independent and hence form a basis for H. For each unit vector x in H, we have that $x = pu + qv$. So that $(Tx, x) = (p\alpha u + q\beta v, pu + qv) = | p |^2 \alpha + | q |^2 \beta$ where $| p |^2 + | q |^2 = 1$. If we set $t = | p |^2$ then we obtain $(Tx, x) = t\alpha + \sqrt{(1 - t^2)}\beta$. Thus W (T) is a line segment from α to β.

Definition
An operator T is said to be Hyponormal if $\| T^*x \| \leq \| Tx \|$ for all x in H.

Theorem 1.8
If T is Hyponormal operator then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof
In [3].

Theorem 1.9
If T is a Hyponormal operator on a finite dimensional Hilbert space H then W (T) is a polygon with n sides.

Proof
Let $\{u_n\}$ be normalised eigenvectors corresponding to distinct eigenvectors α_n. Then each unit vector x in H has the unique representation $\sum_{k=1}^{k=n} p_k u_k$ where $\sum_{k=1}^{n} | p_k |^2 = 1$

$$(Tx, x) = \left(T\left(\sum_{k=1}^{n} p_k u_k\right), \sum_{k=1}^{n} p_k u_k\right)$$

$$= \left(\sum_{k=1}^{n} p_k T(u_k), \sum_{k=1}^{n} p_k u_k\right)$$

$$= \left(\sum_{k=1}^{n} p_k \alpha_k u_k, \sum_{k=1}^{n} p_k u_k \right)$$

$$= \sum_{k=1}^{n} | p_k |^2 \alpha_k$$

Corollary 1.10
If an operator T is normal on a finite dimensional Hilbert space, then W (T) is a polygon with n sides.

Example
Let

$$T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then for eigenvalues of T we must have $det(T - \lambda I) = 0$. Hence

$$\left\| \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix} \right\| = 0$$

This leads to the equation$-\lambda^3 + 1 = 0$. Hence the eigenvalues are the cube roots of unity. From which we obtain the fact that the numerical range is the equilateral triangle that joins the vertices $1, w, w^2$.

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