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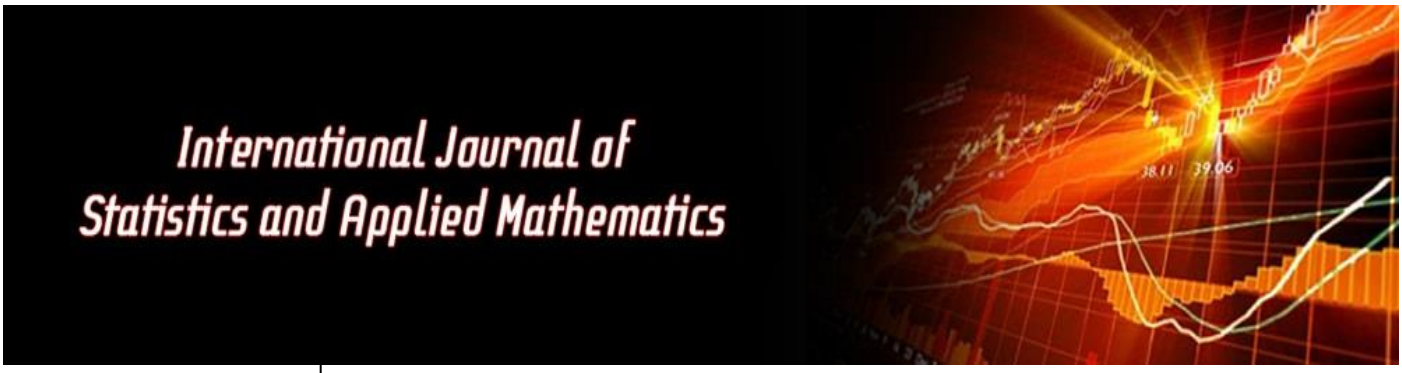

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Analysis of SIR and SEIR infectious disease models incorporating time delay and vertical transmission

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Abstract
This article proposed the SIR and SEIR models with delay and vertical transmission. We have incorporated the vertical transmission in a delayed SIR model assuming a fraction q of offspring from infectious class are infective at birth. Again, we have incorporated incubation period $h > 0$ as delay in the SEIR model with vertical transmission. The conditions of existence and boundedness of the steady states are executed. The basic reproduction number R_0 is derived, and disease-free equilibrium is shown globally stable at $R_0 < 1$. Global stability of the endemic equilibrium is shown under specified conditions. The numerical simulation also confirms our theoretical findings.

Keywords: Vertical transmission, latent period, incubation period, delay, stability

1. Introduction
Disease transmission is a dynamic process driven by the interaction between susceptible and infective. In nature disease transmitted through both horizontal and vertical modes. Horizontal transmission occurs through direct or indirect physical contact with the infectious host while vertical transmission can be accomplished in humans and animals through transplacental transfer of disease agents via mother to a new born and, in insects and plants, vertical transmission through eggs and seeds.
Some mathematicians studied an SIR infectious disease model considering the diseases transmitted in humans via direct contact [1], which has a latent period $h > 0$ to become infectious. However, in many human diseases such as rubella, herbs, infection is transmitted through vertical modes. In this article, we have studied and analysed an SIR model with latent period as delay for the diseases also transmitted vertically [2], assuming that a fraction of the offspring of the infected host is infected at birth and will stay latent before becoming infectious.
Some researchers [3] proposed an SEIR model for epidemic spreading in humans by vertical transmission, assuming that the infected host instantaneously shows the disease's symptoms. However, for many infectious diseases such as syphilis, hepatitis B, AIDS, the infected host stay in an incubation period before showing the symptom of the disease. Here, we have analysed an SEIR model with vertical transmission assuming a time lag $h > 0$ after that infected host will show symptoms of the infection.

2. Mathematical models
2.1 SIR model

$$\frac{dS}{dt} = b - \lambda e^{-bh} S(t) I(t-h) - qbI - bS$$

$$\frac{dI}{dt} = \lambda e^{-bh} S(t) I(t-h) - bI - \gamma I + qbI \qquad (1)$$

$$\frac{dR}{dt} = \gamma I - bR.$$

With $S(t) > 0, I(t) \geq 0, R(t) \geq 0, \forall t \geq t_0$ and $(S(t), I(t), R(t)) \in L^1(0, +\infty)$.

We denote the total population is constant and N (t) = S (t) + I (t) + R (t). Where S(t), I(t) and R(t) are the susceptible, infectious and recovered or removed population from the possibility of infection, b stands for the recruitment of susceptible and per capita death rate, λ is average no. of susceptible per infective per day, γ is removal rate of infective, q is fraction of offspring from infectious class who are infective at birth. We consider that the infected individual will not survive after time h units; hence we introduce a survival term e^{-bh} which is independent of the of the individual.

2.1.1 Boundedness of solutions

Lemma 1: The solutions of system (1), which start in $[0,1]^3 \subset \mathrm{IR}_+^{\,3}$ lies in the feasible region

$\Gamma = \{(S, I, R) \in \mathrm{R}_+^{\,3} : S + I + R \leq 1\}$ (2)

Moreover, they are uniformly bounded in this region.

Proof: We rewrite our system

$$\frac{dS}{dt} = -\lambda e^{-bh} S(t) I(t-h) - q b_1 I + b_1 \left(1 - \frac{S(t)}{K}\right) S(t),$$

$$\frac{dI}{dt} = \lambda e^{-bh} S(t) I(t-h) - b_2 I - \gamma I + q b_1 I, \quad (3)$$

$$\frac{dR}{dt} = \gamma I - b_3 R.$$

Here, K is the environment carrying capacity, and a positive steady state is feasible if $K > S^*$. Let $(S(t), I(t), R(t))$ be any solution of system (3) then $\frac{dS}{dt} \leq b_1 \left(1 - \frac{S(t)}{K}\right) S(t)$.

By comparison theorem $\lim_{t \to \infty} sup\, S(t) \leq M$, where $M = max(S_0, K)$. Consider a function $y(t) = S(t) + I(t) + R(t)$.

$$\frac{dy}{dt} = b_1 \left(1 - \frac{S}{K}\right) S - b_2 I - b_3 R \leq (b_1 + 1) S - S - b_2 I - b_3 R \leq M(b_1 + 1) - b y(t)$$

$$\Rightarrow \frac{dy}{dt} + b y(t) \leq M(b_1 + 1).$$

Where $b = min\{b_2, b_3, 1\}$. After solving this, we get $0 \leq y(S, I, R) \leq \frac{M(b_1+1)}{b} + y\{S(0), I(0), R(0)\} e^{-bt}$. As $t \to \infty, 0 \leq y(S, I, R) \leq \frac{M}{b}(b_1 + 1)$. Therefore, all feasible solutions of extended system (3) enter in the region $\varphi = \left\{(S, I, R) \in [0,1]^3 : y \leq \frac{M(b_1+1)}{b} + \varepsilon, \varepsilon > 0\right\}$. Consequently, all feasible solutions of (1) will enter the region $\varphi_1 = \{(S, I, R) \in [0,1]^3 : y \leq 1 + \varepsilon, \varepsilon > 0\}$ and $\Gamma \subseteq \varphi_1$. Hence, feasible solutions of system (1) are bounded and consequently defined in [0, 1]^3. We also note that if $S_0 = 0$, then $(S(t), I(t), R(t)) \to (0, 0, 0)$ irrespective of initial value $I(t), R(t) \geq 0$. if $S(t) > 0$ then, $(S(t), R(t), I(t)) \to (1, 0, 0)$ for $I(t) = 0 = R(t)$. While solutions enter in the region $\emptyset_1$ and stay there otherwise. Subsequent analysis of the model will be based on the following. we consider any pair of variables from the three variable S, I, R and let $x(t) = \big(S(t), I(t)\big) \in \Omega, where\ \Omega = \big\{(S, I) \in R_+^{\,2} : S + I \leq 1\big\} x(t) = \big(I(t), R(t)\big) \in \Omega_1, where\ \Omega_1 = \big\{(I, R) \in R_+^{\,2} : I + R \leq 1\big\}$. With the understanding that $\Omega, \Omega_1 \subset \Gamma$. Therefore the endemic state in our analysis will be defined in terms (I^*, R^*) because based on the fact that $N \leq 1$, system (1) reduced to the following system (4), and this system now becomes the primary system for study.

2.1.2 Equilibrium classification

1. Disease-free equilibrium $E_S = (1, 0, 0)$.
2. Endemic Equilibrium $E_+ = (S^*, I^*, R^*)$.
3.

$$\left[S^* = \frac{b + \gamma - qb}{\lambda e^{-bh}}, I^* = \frac{b}{b + \gamma} - \frac{b}{\lambda e^{-bh}} + \frac{q b^2}{(b + \gamma) \lambda e^{-bh}}, R^* = \frac{\gamma}{b + \gamma} - \frac{\gamma}{\lambda e^{-bh}} + \frac{q \gamma b}{\lambda e^{-bh} (b + \gamma)}\right]$$

Provided $\lambda e^{-bh} \neq b + \gamma - qb$. True in the interior of E_+.

2.1.3 Basic reproduction number

The basic reproduction number R_0 is given by $R_0 = \frac{\lambda e^{-bh} S_0}{b + \gamma - qb} > 0$.

2.1.4 Global stability analysis

Theorem 1: The endemic equilibrium point E_+ of the system given by (1) is globally asymptotically stable with respect to the set Γ given in (2).

Proof: Since$\mathrm{S} + \mathrm{I} + \mathrm{R} \leq 1$, the system reduced to

$$\frac{dI}{dt} = \lambda e^{-bh} S(t) I(t-h) - bI - \gamma I + qbI, \quad (4)$$

$$\frac{dR}{dt} = \gamma I - bR$$

From Beretta and Takeuchi [4] we can assume that $S^* \lambda e^{-bh} = b + \gamma - qb$. Now $(I, R) \in (0,1]^2$ because $(I^*, R^*) \neq 0$, while $(S, I, R) \in [0,1]^3$.

$$\frac{dI}{dt} = \lambda e^{-bh} S(t) I(t-h) - S\lambda e^{-bh} I(t) = \lambda e^{-bh}(1 - I - R) I(t-h) - S\lambda e^{-bh} I(t)$$

$$\frac{dR}{dt} = \gamma I - bR$$

Where

$(I, R) \in \Omega_1 = \{(I, R) \in R_+ : I + R \leq 1\}$. Let $U_1 = I - I^*, U_2 = R - R^*$, we have

$$\frac{dU_1}{dt} = \lambda e^{-bh}\left[\{1-(U_2+R^*)-(I^*+U_1)\}\{I^*+U_1(t-h)\} - S^*(I^*+U_1)\right]$$

$$= \lambda e^{-bh}\left[\{1-I^*-R^*-U_1-U_2\}\{I^*+U_1(t-h)\} - S^*(I^*+U_1)\right] \quad (5)$$

$$= \lambda e^{-bh}\left[\{S^*-U_1-U_2\}I^* + \{S^*-U_1-U_2\}U_1(t-h)\right] - S^*I^*\lambda e^{-bh} - S^*U_1\lambda e^{-bh}$$

$$\frac{dU_2}{dt} = \gamma(I^*+U_1) - b(R^*+U_2) = \gamma U_1 - bU_2$$

Now we define a Lyapunov function for the system (5)

$$V(U(t)) = \frac{W_1}{2} U_1{}^2(t) + \frac{W_2}{2} U_2{}^2(t) + \frac{1}{2} W_1 \lambda (I^* + S^*) e^{-bh} \int_{t-h}^{t} U_1{}^2(v) dv$$

Where $W_i > 0$ are positive constant, and $V(.)$ is an extended real-valued function $V: \Gamma \to [0, \infty)$ satisfying $V(U(t)) \geq 0$ and

$\dot{V(U(t))} = 0$ at E_S [5] and $V(U(t)) \leq \frac{K}{2}\{U_1{}^2(t) + U_2{}^2(t)\} \geq 0$.

Where

$K = \max\left\{\frac{W_1}{2}\left[1 + \lambda(I^* + S^*)e^{-bh}\right], \frac{W_2}{2}\right\}$. Now

$$\frac{dV(U(t))}{dt} = W_1 U_1(t)\dot{U}_1(t) + W_2 U_2(t)\dot{U}_2(t) + \frac{1}{2} W_1 \lambda e^{-bh}(I^* + S^*)\left[U_1^2(t) - U_1^2(t-h)\right]$$

$$\ldots\ldots\ldots = W_1 U_1(t)\lambda e^{-bh}\left[(S^* - U_1 - U_2)U_1(t-h) - (I^* + S^*)U_1(t) - I^* U_2(t)\right] +$$

$$\ldots\ldots\ldots\ldots W_2 U_2(t)\left[\gamma U_1(t) - bU_2(t)\right] + \frac{W_1}{2}\lambda(I^* + S^*)e^{-bh}\left[U_1^2(t) - U_1^2(t-h)\right]$$

$$\ldots\ldots\ldots = -bW_2 U_2^2(t) + (W_2\gamma - \lambda W_1 I^* e^{-bh})U_1(t)U_2(t) - \frac{W_1}{2}\lambda(I^* + S^*)e^{-bh}U_1^2(t) +$$

$$\ldots\ldots\ldots\ldots \lambda W_1(S^* - U_1 - U_2)e^{-bh}U_1(t)U_1(t-h) - \frac{\lambda}{2} W_1(I^* + S^*)e^{-bh}U_1^2(t-h)$$

Choose $W_1, W_2 > 0$ such that $\lambda W_2 = W_1 \lambda e^{-bh} I^*$. Then

$$\dot{V(U(t))} = -bW_2 U_2^2(t) - \frac{\lambda}{2} W_1 e^{-bh}\{(I^* + S^*)U_1^2(t) - 2(S^* - U_1 - U_2)U_1(t-h)U_1(t) + (I^* + S^*)U_1^2(t-h)\}$$

$$= -bW_2 U_2^2(t) - \frac{\lambda}{2} W_1 e^{-bh}\left[\begin{pmatrix} U_1(t) \\ U_1(t-h) \end{pmatrix}^T \begin{Bmatrix} I^* + S^* & -\{S^* - (U_1 + U_2)\} \\ -\{S^* - (U_1 - U_2)\} & I^* + S^* \end{Bmatrix} \begin{pmatrix} U_1(t) \\ U_1(t-h) \end{pmatrix}\right]$$

$$= -bW_2U_2^2(t) - \frac{\lambda}{2}W_1e^{-bh}[V(t,h)A(t)V^T(t,h)] \quad (6)$$

Where,

$$A(t) = \begin{pmatrix} I^* + S^* & -\{S^* - (U_1 + U_2)\} \\ -\{S^* - (U_1 + U_2)\} & I^* + S^* \end{pmatrix} \text{ and } V(t,h) = \begin{bmatrix} U_1(t) \\ U_1(t-h) \end{bmatrix}$$

Then the determinant of $A(t)$ is positive if and only if $\Delta = [(I^* + S^*)^2 - \{S^* - (U_1(t) + U_2(t))\}^2] > 0$.With solutions satisfying the following inequality $-I^* < \frac{1}{2}(U_1(t) + U_2(t)) < S^*$. Now since Δ is more significant than zero then the matrix A is positive definite, now for any $\varepsilon > 0$, consider

$$\overline{\Omega}_{1,\varepsilon} = \left\{U \in \overline{\Omega_1} : U_1(t) + U_2(t) > -2I^* + \varepsilon\right\}$$

Then for any $U \in \Omega_1$, ∃ a minimum eigenvalue of A is strictly positive. Let us denote such an eigenvalue by φ_ε. We have

$$[V(t,h)A(t)V^T(t,h)] \geq \varphi_\varepsilon\left(U_1^2(t) + U_2^2(t-h)\right) \quad (7)$$

Substituting value from (7) in (6) we get

$$\dot{V(U(t))} \leq -bW_2U_2^2(t) - \frac{W_1}{2}\lambda e^{-bh}\varphi[U_1^2(t) + U_2^2(t-h)] \leq -\delta(U_1^2(t) + U_2^2(t)) < 0$$

Where
$\delta = min\left\{-W_2b, \frac{W_1}{2}\lambda e^{-bh}\varphi_\varepsilon\right\}$. By Barbalat's lemma [5] $(I - I^*)^2 + (R - R^*) \to 0$ as $t \to \infty$. Hence $\dot{V}(\leq 0)$ is semidefinite for any U_1 and U_2 and is equal to zero at E_+. Thus, since the only possible ω- limit sets of (I, R) on the boundary of $[0, 1]^2$ is $\{I^*, R^*\}$ and $\{0, 0\}$. The second set is to be ignored because $I^*, R^* \neq 0$ i.e., $(I^*, R^*) \in (0, 1]^2$. By Lasalle's invariance [6] principle, the only largest compact subset of Ω_1 is $M = \{I^*, R^*\}$. It, therefore, follows that $\exists \bar{I} > 0$such that $liminf\, I(t) \geq \bar{I}$ Hence E_+ is globally stable in $(0, 1]^2$, but all solutions of system (4) enter $[0, 1]^2$. Hence endemic equilibrium is unique and globally stable in $\dot{\varphi_1}$, the interior of φ_1.$(R_0 > 1)$.

Theorem 2: Whenever $\lambda e^{-bh} < b + \gamma - qb$, then disease-free equilibrium E_s is globally asymptotically stable.
Proof: We choose a variable (I, R) and consider the space $\Omega_1 = \{x = (I, R) \in R_+^2 : I + R \leq 1\}$. Dynamic equations for IR classes are given

$$\dot{I(t)} = -(b + \gamma - qb)I(t) + \lambda S(t)I(t-h)e^{-bh}, \dot{R(t)} = \gamma I(t) - bR(t).$$

Where

$0 \leq S \leq 1$ and$(I, R) \in \Omega_1$. Let us consider the Lyapunov function,

$$\dot{V(x(t))} = \dot{I(t)} + W\dot{R(t)} + \lambda e^{-bh}\int_{t-h}^{t} I(u)\,du, \text{W} > 0$$

Then, $V(x(t)) \geq min\{I, W\}(I(t) + R(t)) \geq 0$ for any $t \geq 0$.

$$\dot{V(x(t))} = \dot{I(t)} + W\dot{R(t)} + \lambda e^{-bh}\frac{d}{dt}\int_{t-h}^{t} I(u)\,du$$

$$= -(b + \gamma - qb)I(t) + \lambda S(t)I(t-h) + W\left(\gamma I(t) - bR(t)\right) + \lambda e^{-bh}[I(t) - I(t-h)]$$

$$\leq -(b + \gamma - qb)I(t) + \lambda e^{-bh}I(t) + W\left(\gamma I(t) - bR(t)\right) + \lambda e^{-bh}[I(t) - I(t-h)]$$

$$\leq -\{(b + \gamma - qb) - \lambda e^{-bh} - W\gamma\}I(t) - bWR(t) \leq -K[I(t) + R(t)] < 0.$$

Where
$K = min\{bW, (b + \gamma - qb) - \lambda e^{-bh} - W\gamma\}$. By choosing W Such that $b + \gamma - qb - W\gamma > \lambda e^{-\mu h}$. Hence the system (1) has only disease-free Equilibrium E_s, which is globally asymptotically stable.

2.2 SEIR model

$$\frac{dS}{dt} = b - \lambda e^{-bh} S(t) I(t-h) - pbE - qbI - bS,$$

$$\frac{dE}{dt} = \lambda e^{-bh} S(t) I(t-h) + pbE + qbI - (\varepsilon + b)E, \quad (8)$$

$$\frac{dI}{dt} = \varepsilon E - (\gamma + b)I,$$

$$\frac{dR}{dt} = \gamma I - bR.$$

With $S(t) > 0, I(t) \geq 0, E(t) \geq 0, R(t) \geq 0, \forall t \geq t_0$. We consider total population size $N(t) = S(t) + I(t) + E(t) + R(t) = 1$. Where, ε is the rate at which the exposed individual pass from latent to infectious class and p is fraction of off springs from exposed class who are infective at birth. Rest of the variables are same as described in section 2.1.

2.2.1 Boundedness of solution

Lemma 2: The feasible region for system (8) is $\Gamma = \{(S, E, I, R) \in IR_+^4 : S + E + I + R \leq 1\}$.
Moreover, the feasible solutions of the system (8), which start in $[0,1]^4 \subseteq IR_+^3$ are uniformly bounded.

Proof: Proof is similar to the lemma 1.

2.2.2 Equilibrium classification

We see that model has two equilibrium points

1. Disease-free Equilibrium $E_s = (1,0,0,0)$.
2. Endemic equilibrium point $E_+ = (S^*, E^*, I^*, R^*)$.

$$S^* = \frac{(\gamma + b)(\varepsilon + b) - pb(\gamma + b) - qb\varepsilon}{\lambda\varepsilon e^{-bh}}, E^* = \frac{b[\lambda\varepsilon e^{-bh} + pb(\gamma + b) + qb\varepsilon - (\gamma + b)(\varepsilon + b)]}{\varepsilon\lambda(\varepsilon + b)e^{-bh}},$$

$$I^* = \frac{b[\lambda\varepsilon e^{-bh} + pb(\gamma + b) + qb\varepsilon - (\gamma + b)(\varepsilon + b)]}{\lambda(\varepsilon + b)(\gamma + b)e^{-bh}}, R^* = \frac{\gamma[\lambda\varepsilon e^{-bh} + pb(\gamma + b) + qb\varepsilon - (\gamma + b)(\varepsilon + b)]}{\lambda(\varepsilon + b)(\gamma + b)e^{-bh}}$$

2.2.3 Basic reproduction number

The basic reproduction number R_0 for SEIR model is given by,

$$R_0 = \frac{\lambda e^{-bh}\varepsilon S_0}{(\varepsilon + b)(\gamma + b) - pb(\gamma + b) - qb\varepsilon} > 0.$$

2.2.5 Global stability analysis

Theorem 3: The endemic or positive equilibrium point E_+ of the system given by (8) is globally asymptotically stable for the set Γ given in (9) under the following condition

$$2b^2(\gamma + \xi) - (b^2 + \gamma^2)\lambda e^{-bh}(E^* + I^* + R^*) > 0.$$

Proof: Since $S + I + E + R \leq 1$, the system can be reduced to,

$$\frac{dE}{dt} = \lambda e^{-bh} S(t) I(t-h) + pbE + qbI - (\varepsilon + b)E, \frac{dI}{dt} = \varepsilon E - (\gamma + b)I, \frac{dR}{dt} = \gamma I - bR.$$

Now $(E, I, R) \in (0, I]^3$ because $(E^*, I^*, R^*) \neq 0$. While $(S, E, I, R) \in [0,1]^4$ and

$(E, I, R) \in \Omega_1 = \{(E, I, R) \in IR_+^3 : E + I + R \leq 1\}$. Let $U_1 = E - E^*$, $U_2 = I - I^*$, $U_3 = R - R^*$. We have

$$\frac{dU_1}{dt} = \lambda e^{-bh} S^* U_2(t-h) - \lambda e^{-bh}(U_1 + U_2 + U_3)U_2(t-h) - \lambda e^{-bh}(U_1 + U_2 + U_3)I^* + pbU_1 + qbU_2 - (\varepsilon + b)U_1,$$

$$\frac{dU_2}{dt} = \varepsilon U_1 - (\gamma + b)U_2, \quad (10)$$

$$\frac{dU_3}{dt} = \gamma U_2 - bU_3.$$

Now we define a Lyapunov function for the system,

$$V(U(t)) = \frac{W_1}{2}U_1^2(t) + \frac{W_2}{2}U_2^2(t) + \frac{W_3}{2}U_3^2(t) + \frac{W_2}{2}\lambda(E^* + I^* + S^*)e^{-bh}\int_{t-h}^{t} U_2^2(v)dv.$$

Where $W_i > 0$are positive constant and $V(.)$is extended real valued function $V: \Gamma \to [0, \infty)$ satisfying $V(U(t)) \geq 0$ and $V(\dot{U}(t)) = 0$ at E_s [5] and $V(U(t)) \leq \frac{K}{2}\{U_1^2(t) + U_2^2(t) + U_3^2(t)\} \geq 0$.

Where

$$K = max\left\{\tfrac{W_1}{2}[1 + \lambda(I^* + S^* + E^*)e^{-bh}], \tfrac{W_2}{2}, \tfrac{W_3}{2}\right\}.$$

$$V(\dot{U}(t)) = W_1U_1(t)\dot{U}_1(t) + W_2U_2(t)\dot{U}_2(t) + W_3U_3(t)\dot{U}_3(t) + [U_2^2(t) - U_2^2(t-h)] \times$$

$$\frac{W_2\lambda}{2}(E^* + I^* + S^*)e^{-bh}$$

$$= W_1U_1(t)[\lambda e^{-bh}\{S^* - (U_1 + U_2 + U_3)\}U_2(t-h) - \lambda e^{-bh}(U_1 + U_2 + U_3)I^*] + W_2U_2(t)[\varepsilon U_1(t) - (\gamma + b)U_2(t)]$$
$$+W_3U_3(t)[\gamma U_2(t) - bU_3(t)] + \frac{W_2\lambda}{2}(E^* + I^* + S^*)e^{-bh}[U_2^2(t) - U_2^2(t-h)]$$

$$= \left[\left\{\frac{W_2\lambda}{2}(E^* + I^* + S^*)e^{-bh} - W_2(\gamma + b)\right\}U_2^2(t) + \gamma W_3U_2(t)U_3(t) - \frac{bW_3}{2}U_3^2(t)\right]$$

$$-\left[\frac{\lambda e^{-bh}W_1}{2}I^*U_1(t) + \lambda e^{-bh}W_1I^*U_1(t)U_3(t) + \frac{bW_2}{2}U_3^2(t)\right]$$

$$-\frac{\lambda e^{-bh}}{2}[W_1I^*U_1^2(t) - 2W_1\{S^* - (U_1 + U_2 + U_3)\}U_1(t)U_2(t-h) + W_2(E^* + I^* + S^*)U_2^2(t-h)]$$

$$+\{\varepsilon W_2 - \lambda e^{-bh}W_1I^*\}U_1(t)U_2(t).$$

Taking $W_1, W_2 > 0$ such that $\varepsilon W_2 = \lambda e^{-bh}W_1I^*$;

$$V(U(t)) = -\frac{1}{2}\left[\begin{matrix}\{2W_2(\gamma + b) - \lambda W_2(E^* + I^* + S^*)e^{-bh}\}U_2^2(t) - 2\gamma W_3U_3(t)U_2(t) \\ +bW_3U_2^2(t)\end{matrix}\right]$$

$$-\frac{1}{2}[\lambda e^{-bh}W_1I^*U_1^2(t) + 2\lambda e^{-bh}W_1I^*U_1(t)U_3 + bW_3U_3^2(t)\,] - \frac{1}{2}\lambda e^{-bh}[U_1(t), U_2(t-h)]$$

$$\times\begin{bmatrix} W_1I^* & W_1\{S^* - (U_1 + U_2 + U_3)\} \\ W_1\{S^* - (U_1 + U_2 + U_3)\} & W_2(E^* + I^* + S^*)\end{bmatrix}\begin{bmatrix}U_1(t) \\ U_2(t-h)\end{bmatrix}$$

$$= -\frac{1}{2}[\{2W_2(\gamma + b) - \lambda W_2(E^* + I^* + S^*)e^{-bh}\}U_2^2(t) - 2\gamma W_3U_3(t)U_2(t) + bW_3U_2^2(t)]$$

$$-\frac{1}{2}[\lambda e^{-bh}W_1I^*U_1^2(t) + 2\lambda e^{-bh}W_1I^*U_1(t)U_3(t) + bW_3U_3^2(t)] - \frac{1}{2}\lambda e^{-bh}[V(t)A(t)V^T(t)]$$

Now if we choose $W_2 = 1$ and $W_1 = \frac{(E^*+I^*+S^*)}{I^*}$. Then above Lyapunov function is negative definite under the following conditions;

1. $b\{2(\gamma + b) - \lambda(E^* + I^* + S^*)e^{-bh}\} > \gamma^2W_3.$
2. $bW_3 > \lambda e^{-bh}(E^* + I^* + S^*)$
3. Determinant of A(t) is $+ve\ if\ (E^* + I^* + S^*)[I^{*2} - \{S^* - (U_1 + U_3 + U_3)\}^2] > 0.$

With solutions satisfying the following inequality $-I^* + S^* < (U_1 + U_2 + U_3) < I^* + S^*$,
And from condition (1) and (2), we get another condition

$$[2b^2(\gamma + \xi) - (b^2 + \gamma^2)\lambda e^{-bh}(E^* + I^* + R^*)] > 0$$

Hence $\dot{V}(\leq 0)$ is semidefinite for any U_1, U_2, and U_3, and this is equal to zero at E_+ Thus, the only possible ω- limit set of (E, I, R) on the boundary of $[0, 1]^3$ is $\{E^*, I^*, R^*\}$and $\{0, 0, 0\}$. The second set is to be ignored because $E^*, I^*, R^* \neq 0$, i.e., $\{I^*, E*, R^*\} \in (0, 1]^3$. Then by LaSalle's invariance principle [6], the only largest compact subset of Ω_1 is $M = \{E^*, I^*, R^*\}$. Hence, our endemic state is globally stable.

Theorem 4: Whenever $\lambda e^{-bh}\varepsilon S^* < \{(\varepsilon + b) - pb\}(\gamma + b) - qb$ then disease-free equilibrium point E_s is globally asymptotically stable.
Proof: We choose variable (E, I, R) and consider the space$\Omega_1 = \{x = (E, I, R) \in IR_+^3 : E + I + R \leq 1\}$.
Dynamical system is given by

$$\dot{E}(t) = \lambda e^{-bh}S(t)I(t-h) + pbE + qbI - (\varepsilon + b)E, \dot{I}(t) = \varepsilon E - (\gamma + b)I, \dot{R}(t) = \gamma I - bR.$$

Let us consider the Lyapunov function,

$$\dot{V}\big(\dot{x}(t)\big) = \dot{E}(t) + \dot{I}(t) + W\dot{R}(t) + \lambda e^{-bh}\int_{t-h}^{t} I(u)du\, \dot{V}\big(\dot{x}(t)\big)$$

$$= \dot{E}(t) + \dot{I}(t) + WR(t) + \lambda e^{-bh}[\dot{I}(t) - I(t-h)]$$

$$= \lambda e^{-bh}S(t)I(t-h) + pbE + qbI - (\varepsilon + b)E + \varepsilon E$$

$$-(\gamma + b)I + W\big(\gamma I(t) - bR(t)\big) + \lambda e^{-bh}[I(t) - I(t-h)]$$

$$\leq -[-\lambda e^{-bh} - qb - W\gamma + (\gamma + b)]\, I(t) - (b - bp)E(t) - bWR(t)$$

$$\leq -K[I(t) + E(t) + R(t)].$$

$Where\ K = min[\{-\lambda e^{-bh} - qb - W\gamma + (\gamma + b)\}, (pb - b), bW]$. By choosing W such that $\lambda e^{-bh} < \gamma + b - qb - W\gamma$. Hence, the system (8) has only one disease-free equilibrium, which is globally stable.

3. Results

We see that for the parameters, $b = 1, \lambda = 2, \gamma = .04, q = .5, h = .5$, the endemic equilibrium points of SIR model $S^* = .45$, $I^* = .516$, $R^* = .021$ exist and stable, as shown in fig. (a). Again, all the disease-free equilibrium point of SIR model exist and stable for the set of parameters s $b = .45, \lambda = 1, \gamma = .6, q = .5, h = .25$, as shown in fig. (b). the numerical simulation of SEIR model reveals that for the set of parameters $b = .05, \lambda = .3, \varepsilon = .2, .07, \gamma = .01, q = .5, p = .5, h = .025$, the endemic equilibrium point $S^* = .1420$, $I^* = .5720$, $E^* = .1716$, $R^* = .1144$ exist and stable as shown in fig. (c). Again, for the set of parameters $b = .05, \lambda = .15, \varepsilon = .2, .07, \gamma = .11, q = .5, p = .5, h = .025$, the disease-free equilibrium point exist and stable are satisfied as shown in fig. (d).

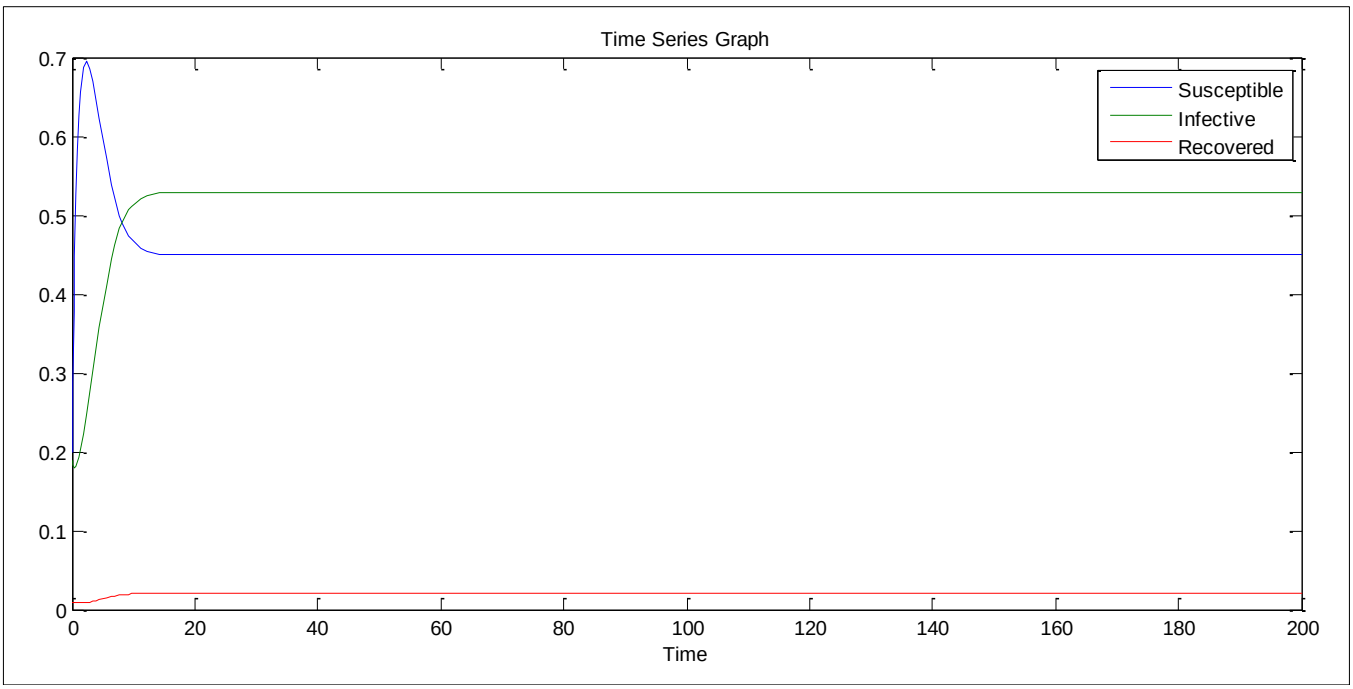

Fig (a): Stable population distribution for Susceptible, Infective and Recovered in endemic case

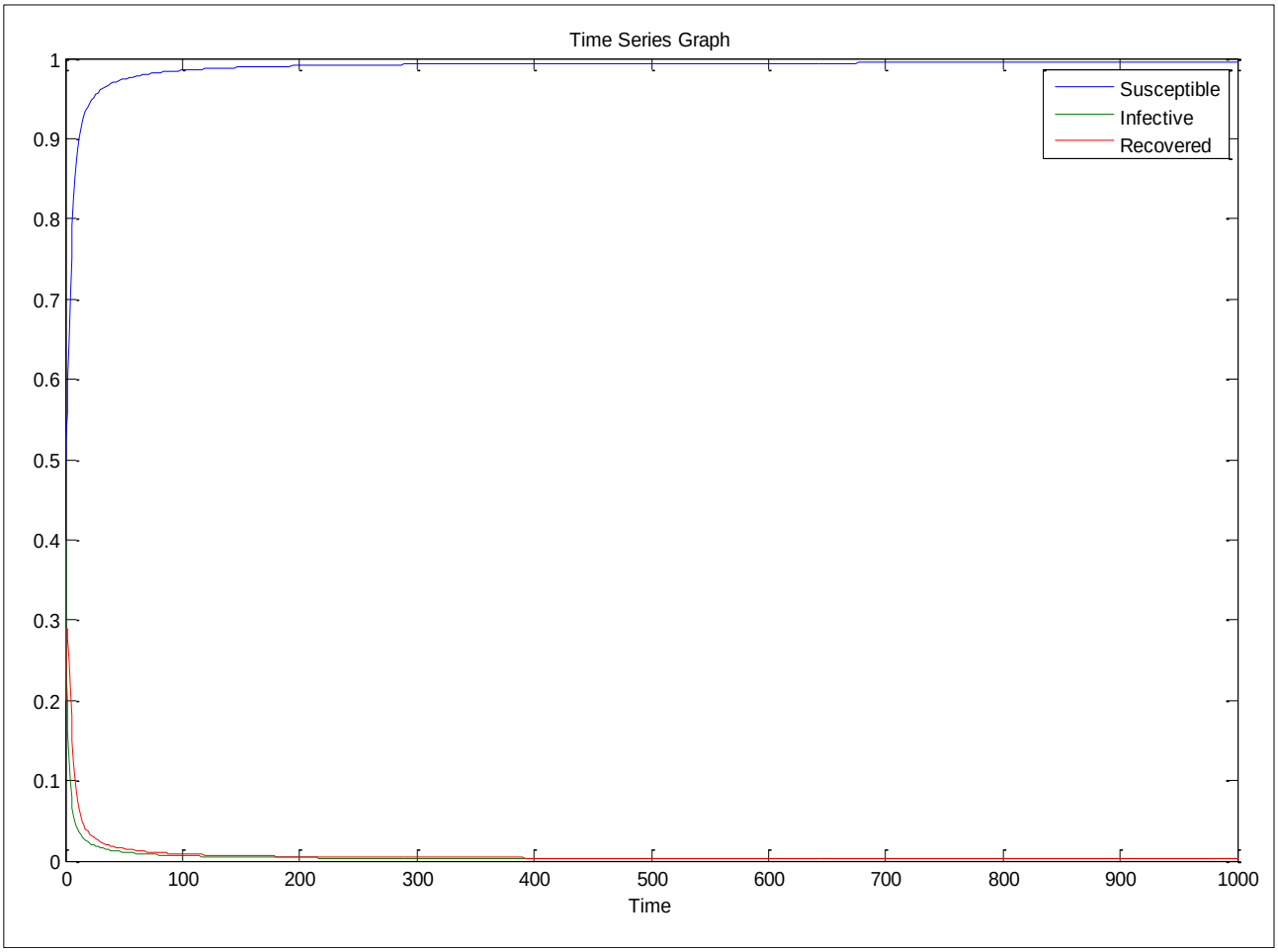

Fig (b): Stable population distribution for Susceptible, Infective, Recovered in disease-free case

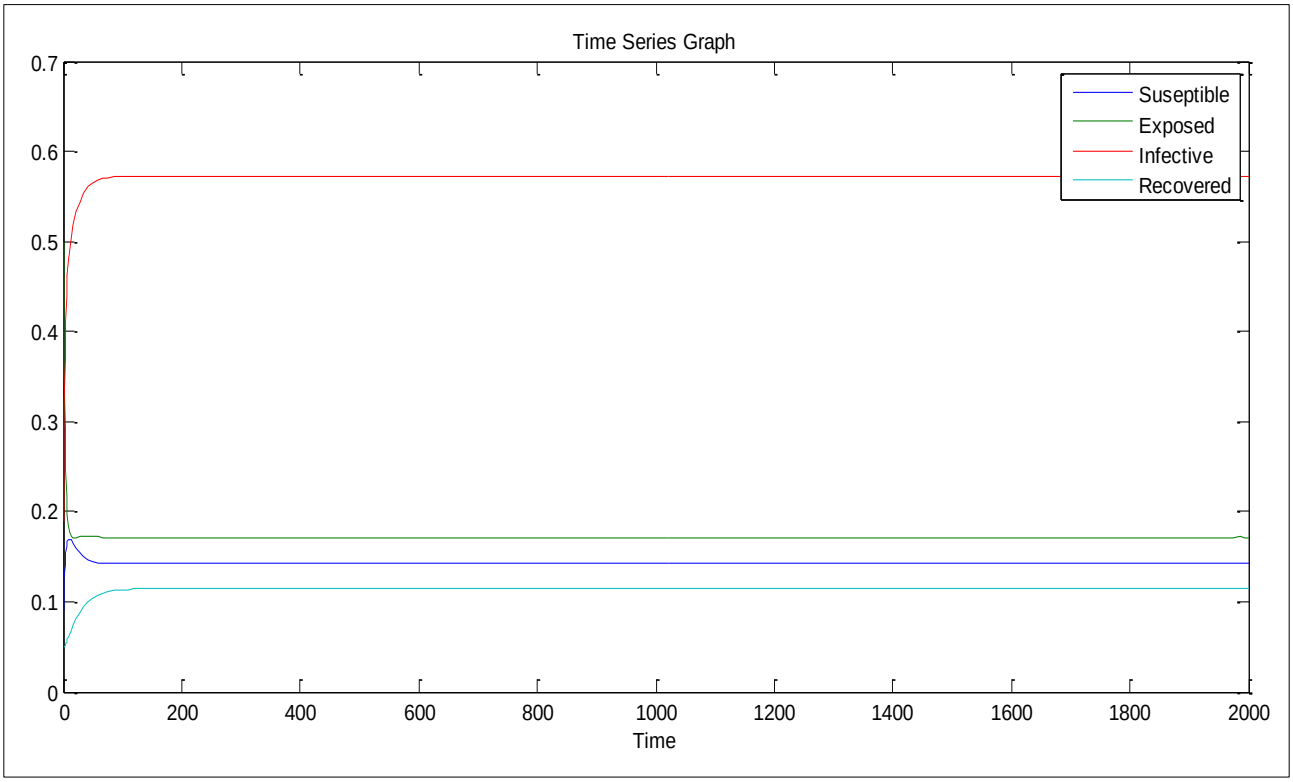

Fig (c): Stable population distribution for Susceptible, Exposed, Infective, Recovered in endemic case.

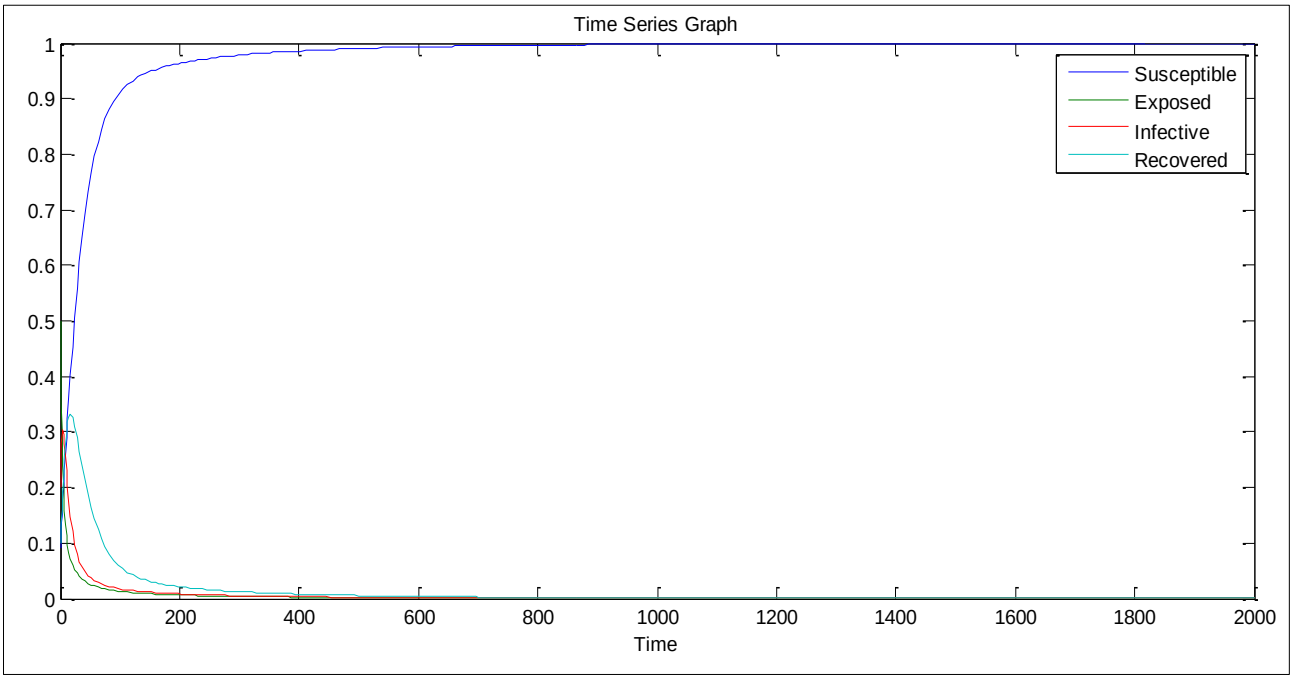

Fig (d): Stable population distribution for Susceptible, Exposed, Infective, Recovered in disease-free case

4. Conclusion

This paper investigated the SIR and SEIR epidemic model with delay and vertical transmission. In these models, the infection transfers through direct contact and vertical transmission. Delay in the SIR model reveals that the infected host stays in the latent period before becoming infectious, and delay in SEIR model reveals that the infected host stays in the incubation period before showing the symptom of the disease.

After analysis, we have concluded that the endemic and disease-free states are linearly asymptotically stable under some conditions involving disease-related parameters. The reproduction number shows that disease-free equilibrium is stable if $R_0 < 1$ and endemic equilibrium is stable if $R_0 > 1$. Reproduction for SIR model depends on$q$, $0 \leq q \leq 1$. Here, q is the fraction of offspring from the infectious class infected at birth. Now we can see that as q increases, R_0 also increases. Therefore, we can conclude that disease can be controlled if the number of infected newborns from the infective class is reduced. Again, the reproduction number for the SEIR model depends on q and$p, 0 \leq p \leq 1$. Where, p fraction of offspring from exposed class who are infected at birth. We can see from the reproduction number that as p increases, R_0 also increases. Therefore, if we want to control the disease, we have to reduce the number of newborn offspring from exposed class who are infective at birth.

5. References

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