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Improving the efficiency of estimators of population mean in systematic sampling using inverse exponentiation

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Abstract
It is established that the ratio and product-type estimators discriminate between negatively and positively correlated populations for optimal efficiency performance. This paper introduces a new approach to estimation theory that produces a more precise and efficient ratio and product estimators that are substantially adaptable to both negatively and positively correlated populations. Analytical and numerical results showed that at optimal conditions the proposed ratio and product estimators have equal optimal efficiency with the classical regression estimator of population mean but are always more efficient than all existing related estimators under review.

Keywords: estimation technique, correlations, log-ratio estimator, log-product estimator, natural logarithm, optimal efficiency, systematic sampling

1. Introduction

The ratio and product estimators most practically have the limitation of having efficiency not exceeding that of the regression estimator. Consequently, most survey statisticians have carried out researches towards the modification of existing ratio and product estimators to provide better alternatives and improve their precision. Among the authors include; [Bahl and Tuteja (1991) [1], Kadilar and Cingi (2003) [19], Singh (2003), Singh and Vishwakarma (2007) [31], Sharma and Tailor (2010) [26], Onyeka (2012) [25], Tailor (2012) [37], Khare and Sinha (2012) [22], Singh and Audu (2013) [28] and Clement (2015, 2017a) [8, 4]].

The incorporation of auxiliary information is very important for the construction of efficient estimators for the estimation of population parameters and increasing the efficiency of the estimators in different sampling designs. Using the knowledge of the auxiliary variables, several authors have proposed different estimation techniques for the finite population mean of the study variable; [Cochran (1977) [13]; Singh et al. (2011) [32]; Khan and Arunachalam (2014) [21]; Lone and Tailor (2015) [23]; Khan (2015) [20]; Clement and Enang (2015, 2017) [7, 8]; Clement (2016, 2018) [3, 6]; Enang and Clement (2020) [14] Clement (2020a, 2020b, 2021) [9, 10, 12]] have worked on the estimation of population parameters using auxiliary information.

The method of systematic sampling was first studied by Madow and Madow (1944) [24] and is widely used in survey of finite populations. Systematic sampling is a method of selecting sample members from a larger population according to a random starting point and a fixed, periodic interval. Typically, every "nth" member is selected from the total population for inclusion in the sample population. Systematic sampling is still thought of as being random, as long as the periodic interval is determined beforehand and the starting point is random.

Systematic sampling has got the nice feature of selecting the whole sample with just one random start. Apart from its simplicity, which is of considerable importance, this procedure in many situations provides estimators that are more efficient than simple random sampling and/or stratified random sampling for certain types of populations. Consequently, many Survey Statisticians have worked on the estimation of population mean in systematic sampling, these include: [Gautschi (1957) [15]; Hajeck (1959) [16]; Swain (1764); Shukla (1971) [34]; Cochran (1977) [13] Singh and Solanki (2012) [30]; Singh and Jatwa (2012) [29]; Singh *et al.* (2012) [29, 30];

Verma and Singh (2014) [38, 39]; and Verma *et al*. (2014) [38, 39]; Clement (2017b) [5] Clement and Inyang (2020) [10]] among others. It is well established in sampling surveys that ratio estimators are effective when correlations between the study and auxiliary variables is highly positive while the product estimators are effective when the correlation between the study and the auxiliary variables is highly negative. In the present paper, a new estimation technique that makes both the ratio and product-type estimators substantially adaptable to high positive and high negative correlations between the study and auxiliary variables is developed.

2. Basic Notations and Definition

Suppose a finite population consists of N units $U = (U_1, U_2, \dots, U_N)$ numbered from 1 to N in some order. A sample of size n units is taken at random from the first k units and every kth subsequent unit; then, $N = nk$ where n and k are positive integers; thus, there will be k samples (clusters) each of size n and observe the study variate y and auxiliary variate x for each and every unit selected in the sample. Let (x_{ij}, y_{ij}) for $i = 1,2,\dots,k$ and $j = 1,2,\dots,n$ denote the value of jth unit in the ith sample. Then, the systematic sample means are defined as follows: $\bar{y}_{sy} = \frac{1}{n}\sum_{j=1}^{n} y_{ij}$ and $\bar{x}_{sy} = \frac{1}{n}\sum_{j=1}^{n} x_{ij}$ are the unbiased estimators of the population means $\bar{Y} = \frac{1}{N}\sum_{j=1}^{N} y_{ij}$ and $\bar{X} = \frac{1}{N}\sum_{j=1}^{N} x_{ij}$ of y and x, respectively. Also, let $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(y_{ij} - \bar{Y})^2$ and $S_x^2 = \frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{X})^2$ be the population variances of the study variable and the auxiliary variable respectively, with the corresponding population covariance $S_{xy} = \frac{1}{N-1}\sum_{i=1}^{k}\sum_{j=1}^{n}(x_{ij} - \bar{X})(y_{ij} - \bar{Y})$. Also C_y^2 and C_x^2 are the known population coefficients of variation of the study variable and the auxiliary variable respectively.

Let $e_x = \left[(\bar{x}_{sy} - \bar{X})/\bar{X}\right]$ and $e_y = \left[(\bar{y}_{sy} - \bar{Y})/\bar{Y}\right]$

so that:

$$\left.\begin{aligned} \bar{x}_{sy} &= \bar{X}(1 + e_x) \\ \bar{y}_{sy} &= \bar{Y}(1 + e_y) \end{aligned}\right\} \quad (1)$$

where

$$\begin{aligned} &E(e_x) = E(e_y) = 0 \\ &E(e_x^2) = \gamma^*[1 + (n-1)P_x]C_x^2 \\ &E(e_y^2) = \gamma^*[1 + (n-1)P_y]C_y^2 \\ &E(e_x e_y) = \gamma^*[1 + (n-1)\rho_y]^{\frac{1}{2}}[1 + (n-1)\rho_x]^{\frac{1}{2}}\rho_{xy}C_x C_y \end{aligned} \quad (2)$$

$$\gamma^* = \left(\frac{N-1}{nN}\right); C_x^2 = \frac{S_x^2}{\bar{X}^2};\ C_y^2 = \frac{S_y^2}{\bar{Y}^2}; K = \rho_{xy}\frac{C_y}{C_x}; \rho_{xy} = \frac{S_{xy}}{S_x S_y}; R = \frac{\bar{Y}}{\bar{X}}$$

$$\varphi = [1 + (n-1)\rho_y];\ \tau = \frac{[1 + (n-1)\rho_y]^{\frac{1}{2}}}{[1 + (n-1)\rho_x]^{\frac{1}{2}}}$$

3. Review of Some Existing Estimators in Systematic Sampling

This section gives a review of some existing estimators in survey sampling literature under the systematic sampling scheme with their MSE expressions as summarized in Table 1.

Table 1: Some Existing Estimators in Systematic Sampling with their MSEs

S/N	Estimators	MSE
1	$\hat{\bar{Y}}_1^* = \bar{y}_{sy}$ *unbiased sample mean* [*Cochran* (1977)]	$\gamma^* S_y^2[1 + (n-1)\rho_x]$
2	$\hat{\bar{Y}}_2^* = \bar{y}_{sy}\left(\frac{\bar{X}}{\bar{x}_{sy}}\right)$ Classical Ratio [Swain (1964) [36]]	$\gamma^*\varphi[S_y^2 + \tau^2 R^2 S_x^2 - 2R\tau S_{xy}]$
3	$\hat{\bar{Y}}_2^* = \bar{y}_{sy}\left(\frac{\bar{x}_{sy}}{\bar{X}}\right)$ Classical Product [Shukla (1971) [34]]	$\gamma^*\varphi[S_y^2 + \tau^2 R^2 S_x^2 + 2R\tau S_{xy}]$
4	$\hat{\bar{Y}}_4^* = \bar{y}_{sy}\left(\frac{\bar{X}}{\bar{x}_{sy}}\right)^2$ Chain Ratio [Kadilar and Cingi(2003)]	$\gamma^*\varphi[S_y^2 + 4\tau^2 R^2 S_x^2 - 4R\tau S_{xy}]$
5	$\hat{\bar{Y}}_5^* = \bar{y}_{sy}\left(\frac{\bar{x}_{sy}}{\bar{X}}\right)^2$ *Chain Product* [*Kadilar and Cingi* (2003)]	$\gamma^*\varphi[S_y^2 + 4\tau^2 R^2 S_x^2 + 4R\tau S_{xy}]$
6	$\hat{\bar{Y}}_6^* = \bar{y}_{sy} + \hat{\beta}(\bar{X} - \bar{x}_{sy})$ Classical regression [Hansen *et al.* (1953)]	$\gamma^*\varphi S_y^2(1 - \rho_{xy}^2)$

4. The proposed Log-Ratio and Log-Product estimators

This section suggests new ratio and product-type estimators of population mean using the principle of inverse exponentiation.

4.1 The Proposed Log-Ratio estimator

Using known natural logarithm of the population and sample means of the auxiliary variable (X) , the proposed log-ratio estimator of the population mean $(\bar{Y})$, of the study variable (Y) in systematic sampling design is defined as:

$$\hat{\bar{Y}}_R^* = \bar{y}_{sy} \frac{ln(\bar{X})^{\alpha}}{ln(\bar{x}_{sy})^{\alpha}} \qquad ln(\bar{X}) \neq 0; \; ln(\bar{x}_{sy}) \neq 0 \tag{3}$$

where α is a suitably chosen scalar to be determined such that the MSE of the proposed estimator is minimized.

Let $\bar{x}_{sy} = \bar{X}(1+e_x)$

so that

$$\bar{x}_{sy}^{\alpha} = [\bar{X}(1+e_x)]^{\alpha}$$

and

$$ln(\bar{x}_{sy})^{\alpha} = ln(\bar{X})^{\alpha} + ln(1+e_x)^{\alpha} \tag{4}$$

Dividing both sides of (4) by $ln(\bar{X})^{\alpha}$ gives:

$$\frac{ln(\bar{x}_{sy})^{\alpha}}{ln(\bar{X})^{\alpha}} = 1 + \lambda\, ln(1+e_x)^{\alpha}$$

where $\lambda = (Ln(\bar{X})^{\alpha})^{-1}$

It is assumed that $|e_x| < 1$ so that expanding $(1+e_x)^{\alpha}$ as a series in power of e_x, multiplying out and retaining terms of the e's to the second degree, gives:

$$\frac{ln(\bar{x}_{sy})^{\alpha}}{ln(\bar{X})^{\alpha}} = 1 + \lambda\, ln\{1 + [\alpha e_x + \alpha(\alpha-1)e_x^2/2]\}$$

$$\frac{ln(\bar{x}_{sy})^{\alpha}}{ln(\bar{X})^{\alpha}} = 1 + \lambda\{\alpha e_x + \alpha(\alpha-1)e_x^2/2 - [\alpha e_x + \alpha(\alpha-1)e_x^2/2]^2/2! + \cdots\} \tag{5}$$

Taking the reciprocal of (5) gives:

$$\frac{ln(\bar{X})^{\alpha}}{ln(\bar{x}_{sy})^{\alpha}} = [1 + \lambda\{\alpha e_x + \alpha(\alpha-1)e_x^2/2 - [\alpha e_x + \alpha(\alpha-1)e_x^2/2]^2/2! + \cdots\}]^{-1} \tag{6}$$

Expanding (6) and retaining terms of the e's to the second degree gives:

$$\frac{ln(\bar{X})^{\alpha}}{ln(\bar{x}_{sy})^{\alpha}} = [1 + \lambda\alpha(e_x - e_x^2/2)]^{-1}$$

Now expanding $[1 + \lambda\alpha(e_x - e_x^2/2)]^{-1}$ as a series in power of $(\lambda\,\alpha[e_x - e_x^2/2])$, multiplying out and retaining terms of the e's to the second degree, gives

$$\frac{ln(\bar{X})^{\alpha}}{ln(\bar{x}_{sy})^{\alpha}} = [1 - \lambda\alpha e_x + \lambda\alpha e_x^2/2 + \lambda^2\alpha^2 e_x^2] \tag{7}$$

Substituting (7) in (3) and using the results in (1) gives:

$$\hat{\bar{Y}}_R^* = \bar{y}_{sy}\left[1 - \lambda\alpha e_x + \lambda\alpha e_x^2/2 + \lambda^2\alpha^2 e_x^2\right]$$

$$\hat{\bar{Y}}_R^* = \bar{Y}(1+e_y)\left[1 - \lambda\alpha e_x + \lambda\alpha e_x^2/2 + \lambda^2\alpha^2 e_x^2\right]$$

so that

$$\hat{\bar{Y}}_R^* - \bar{Y} = \bar{Y}[e_y - \lambda\alpha e_x - \lambda\alpha e_x e_y + \lambda\alpha e_x^2/2\,(1 + 2\lambda\alpha)] \tag{8}$$

Taking expectation of both sides of (8) and using the results in (2), gives the bias of $\hat{\bar{Y}}_R^*$ to the first order of approximation (i.e. to terms of order $o(n^{-1})$) as:

$$Bias\left(\hat{\bar{Y}}_R^*\right) = \frac{\lambda\alpha}{2\bar{X}}\gamma^*[1 + (n-1)\rho_x]\left\{(1 + 2\lambda\alpha)RS_x^2 - \left(\frac{1+(n-1)\rho_y}{1+(n-1)\rho_x}\right)^{\frac{1}{2}}\right\} \tag{9}$$

Squaring both sides of (8) and retaining terms to the second degree, gives

$$\left(\hat{\bar{Y}}_R^* - \bar{Y}\right)^2 = \bar{Y}^2\left[e_y^2 - 2\lambda\alpha e_x e_y + \lambda^2\alpha^2 e_x^2\right] \quad (10)$$

Taking expectation of both sides of (10) and using the results in (2), gives the MSE of $\hat{\bar{Y}}_R^*$ to the first order of approximation as:

$$MSE\left(\hat{\bar{Y}}_R^*\right) = \gamma^*\left\{[1+(n-1)\rho_y]S_y^2 + \lambda^2\alpha^2[[1+(n-1)\rho_x]]R^2S_x^2 - 2\lambda\alpha RS_{xy}[1+(n-1)\rho_y]^{\frac{1}{2}}[1+(n-1)\rho_x]^{\frac{1}{2}}\right\} \quad (11)$$

The $MSE\left(\hat{\bar{Y}}_R^*\right)$ in (11) is minimized when

$$\alpha\lambda^2[[1+(n-1)\rho_x]]R^2S_x^2 \geq \lambda RS_{xy}[1+(n-1)\rho_y]^{\frac{1}{2}}[1+(n-1)\rho_x]^{\frac{1}{2}}$$

so that

$$\alpha_{R,opt} = \tau\frac{S_{xy}}{\lambda RS_x^2} \quad (12)$$

Substituting the value of $\alpha_{R,opt}$ in (12) for α in (11), gives the MSE of an asymptotically optimum log-ratio estimator (AOE) $\hat{\bar{Y}}_{R.opt}^*$ (or minimum MSE of $\hat{\bar{Y}}_R^*$) as:

$$MSE\left(\hat{\bar{Y}}_{R,opt}^*\right) = \gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \quad (13)$$

4.2 The Proposed Log-Product estimators

Following the procedure of section 4.1, the proposed log-product estimator is defined as:

$$\hat{\bar{Y}}_P^* = \bar{y}_{sy}\frac{ln(\bar{x}_{sy})^\alpha}{ln(\bar{X})^\alpha} \qquad ln(\bar{X}) \neq 0;\ \ ln(\bar{x}_{sy}) \neq 0 \quad (14)$$

Substituting (5) in (14) and using the results in (1) gives:

$$\hat{\bar{Y}}_P^* = \bar{Y}(1+e_y)[1+\lambda(\alpha e_x - \alpha e_x^2/2)]$$

so that

$$\hat{\bar{Y}}_P^* - \bar{Y} = \bar{Y}[e_y + \lambda\alpha e_x + \lambda\alpha e_x e_y - \lambda\alpha e_x^2/2] \quad (15)$$

Taking expectation of both sides of (15) and using the results in (2), gives the bias of $\hat{\bar{Y}}_P^*$ to the first order of approximation as:

$$Bias\left(\hat{\bar{Y}}_P^*\right) = \frac{\lambda\alpha}{\bar{X}}\gamma^*\left[S_{xy}(1+(n-1)\rho_y)^{\frac{1}{2}}(1+(n-1)\rho_x)^{\frac{1}{2}} - (1+(n-1)\rho_x)RS_x^2\right] \quad (16)$$

Squaring both sides of (15) and retaining terms to the second degree, gives

$$\left(\hat{\bar{Y}}_P^* - \bar{Y}\right)^2 = \bar{Y}^2\left[e_y^2 + 2\lambda\alpha e_x e_y + \lambda^2\alpha^2 e_x^2\right] \quad (17)$$

Taking expectation of both sides of (17) and using the results in (2), gives the MSE of $\hat{\bar{Y}}_P^*$ to the first order of approximation as:

$$V\left(\hat{\bar{Y}}_P^*\right) = \gamma^*\left\{[1+(n-1)\rho_y]S_y^2 + \lambda^2\alpha^2[[1+(n-1)\rho_x]]R^2S_x^2 + 2\lambda\alpha RS_{xy}[1+(n-1)\rho_y]^{\frac{1}{2}}[1+(n-1)\rho_x]^{\frac{1}{2}}\right\} \quad (18)$$

The $MSE\left(\hat{\bar{Y}}_P^*\right)$ in (18) is minimized when

$$\alpha\lambda^2[[1+(n-1)\rho_x]]R^2S_x^2 \geq -\lambda RS_{xy}[1+(n-1)\rho_y]^{\frac{1}{2}}[1+(n-1)\rho_x]^{\frac{1}{2}}$$

so that

$$\alpha_{P,opt} = -\tau\frac{S_{xy}}{\lambda RS_x^2} \quad (19)$$

Substituting the value of $\alpha_{P,opt}$ in (19) for α in (18), gives the MSE of an asymptotically optimum log-product estimator (AOE) $\hat{\bar{Y}}_{P..opt}^*$ (or minimum MSE of as:

$$MSE\left(\hat{\hat{Y}}^*_{P,opt}\right) = \gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \quad (20)$$

Remarks: Following from (13) and (20), it is observed that, the proposed log- ratio estimator $(\hat{\hat{Y}}_R^*)$ and the log-product estimator $(\hat{\hat{Y}}_P^*)$ have equal optimal efficiency under certain prescribed conditions.

5. Analytical Study

5.1 Efficiency comparisons

This section compares the optimal MSEs of the proposed log-ratio and log-product estimators with some existing estimators in systematic sampling listed in section 2.

Let $MSE\left(\hat{\hat{Y}}^*_{R,opt}\right) = MSE\left(\hat{\hat{Y}}^*_{P,opt}\right) = MSE\left(\hat{\hat{Y}}^*_7\right) = \gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right)$ since the proposed log-ratio and log-product estimators have equal optimal efficiency.

5.1.1 Comparison with the classical sample mean per unit estimator

$$MSE\left(\hat{\hat{Y}}^*_7\right) \leq MSE\left(\hat{\hat{Y}}^*_1\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi S_y^2$$

so that

$$S_{xy}^2 \geq 0 \quad (21)$$

If (21) holds then, the proposed estimators would be more efficient than the classical sample mean per unit estimator.

5.1.2 Comparison with the Swain classical ratio estimator

$$MSE\left(\hat{\hat{Y}}^*_7\right) \leq MSE\left(\hat{\hat{Y}}^*_2\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi\left[S_y^2 + \tau^2R^2S_x^2 - 2R\tau S_{xy}\right]$$

so that

$$S_y^2 + \tau^2R^2S_x^4 - 2R\tau S_{xy} \geq 0$$

$$\left.\begin{array}{l} \tau \geq S_{xy}/RS_x^2 \\ R \geq S_{xy}/\tau S_x^2 \\ S_x^2 \geq S_{xy}/\tau R \end{array}\right\} \quad (22)$$

If (22) holds then, the proposed estimators would be more efficient than the classical ratio estimator of mean in systematic sampling by Swain (1964) [36].

5.1.3 Comparison with the Shukla classical product estimator

Shukla (1971) [34] introduced the classical product estimator for population mean in systematic sampling as given by:

$$MSE\left(\hat{\hat{Y}}^*_7\right) \leq MSE\left(\hat{\hat{Y}}^*_3\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi\left[S_y^2 + \tau^2R^2S_x^2 + 2R\tau S_{xy}\right]$$

so that

$$S_y^2 + \tau^2R^2S_x^4 + 2R\tau S_{xy} \geq 0$$

$$\left.\begin{array}{l} \tau \geq -S_{xy}/RS_x^2 \\ R \geq -S_{xy}/\tau S_x^2 \\ S_x^2 \geq -S_{xy}/\tau R \end{array}\right\} \quad (23)$$

If (23) holds then, the proposed estimators would be more efficient than the classical product estimator of mean in systematic sampling by Shukla (1971) [34].

5.1.4 Comparison with the Kadillar and Cingi classical chain ratio estimator

$$MSE\left(\hat{\hat{Y}}_7^*\right) \leq MSE\left(\hat{Y}_4^*\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi\left[S_y^2 + 4\tau^2R^2S_x^2 - 4R\tau S_{xy}\right]$$

so that

$$S_y^2 + 4\tau^2R^2S_x^4 - 4R\tau S_{xy} \geq 0$$

$$\left.\begin{array}{l}\tau \geq S_{xy}/2RS_x^2 \\ R \geq S_{xy}/2\tau S_x^2 \\ S_x^2 \geq S_{xy}/2\tau R\end{array}\right\} \quad (24)$$

If (24) holds then, the proposed estimators would be more efficient than the classical chain ratio estimator of mean in systematic sampling by Kadillar and Cingi (2003) [19].

5.1.5 Comparison with the Kadillar and Cingi classical chain product estimator

$$MSE\left(\hat{\hat{Y}}_7^*\right) \leq MSE\left(\hat{Y}_5^*\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi\left[S_y^2 + 4\tau^2R^2S_x^2 + 4R\tau S_{xy}\right]$$

so that

$$S_y^2 + 4\tau^2R^2S_x^4 + 4R\tau S_{xy} \geq 0$$

$$\left.\begin{array}{l}\tau \geq -S_{xy}/2RS_x^2 \\ R \geq -S_{xy}/2\tau S_x^2 \\ S_x^2 \geq -S_{xy}/2\tau R\end{array}\right\} \quad (25)$$

If (25) holds then, the proposed estimators would be more efficient than the classical chain product estimator of mean in systematic sampling by Kadillar and Cingi (2003) [19].

5.1.6 Comparison with the classical regression estimator

$$MSE\left(\hat{\hat{Y}}_7^*\right) \leq MSE\left(\hat{Y}_6^*\right)$$

$$\gamma^*\varphi\left(S_y^2 - \frac{S_{xy}^2}{S_x^2}\right) \leq \gamma^*\varphi S_y^2\left(1 - \rho_{xy}^2\right)$$

so that

$$\rho_{xy} \leq \frac{S_{xy}}{S_xS_y} \quad (26)$$

If (26) holds then, the proposed estimators would be more efficient than the classical regression estimator by Hansen et al. (1953).

5.2 The percent relative efficiency (PRE)

The percent relative efficiency (PRE) of an estimator $\hat{\hat{Y}}_i^*$ with respect to the classical sample mean per unit estimator in systematic sampling $(\bar{y}_{sy})$ is defined by:

$$PRE\left(\hat{\hat{Y}}_i^*, \bar{y}_{sy}\right) = \frac{MSE(\bar{y}_{sy})}{MSE\left(\hat{\hat{Y}}_i^*\right)} \times 100 \qquad i = 1,2,3,4,5,6,7 \quad (27)$$

6. Empirical Study

In this section, we investigate our theoretical results, as well as, test the optimality and efficiency performances of our proposed estimators over other existing ones considered in this study, using live data from four Populations.

(i) Data Statistics under Positive Correlation

Population I

The details of population parameters are:

$N = 15, n = 3, \bar{Y} = 80, \bar{X} = 44.47, S_x^2 = 149.55, S_y^2 = 426.56, \rho_y = 0.6652, \rho_x = 0.7070, \rho_{xy} = 0.9446, S_{xy} = 238.57$

Population II

The details of population parameters are:

$N = 176, n = 16, \bar{Y} = 28.2614, \bar{X} = 6.9943, S_x^2 = 241.1289, S_y^2 = 8.7616, \rho_y = 0.6342, \rho_x = 0.6986, \rho_{xy} = 0.6741, S_{xy} = 30.9842$

(ii) Data Statistics under Negative Correlations

Population III

The details of population parameters are:

$N = 96, n = 34, \bar{Y} = 58.6834, \bar{X} = 36.4283, S_x^2 = 46.3822, S_y^2 = 52.6421, \rho_y = 0.8236, \rho_x = 0.7876, \rho_{xy} = -0.8628, S_{xy} = -42.6336$

Population IV

The details of population parameters are

$N = 120, n = 60, \bar{Y} = 63.4321, \bar{X} = 41.3242, S_x^2 = 56.4286, S_y^2 = 42.3823, \rho_y = 0.7645, \rho_x = 0.8432, \rho_{xy} = -0.9242, S_{xy} = -45.1968$

Table 2: Efficiency Comparisons with different Estimators under positive correlations

$\hat{\bar{Y}}_i^*$	Optimality conditions	Population 1	Population 1I
$\hat{\bar{Y}}_1^*$	$S_{xy}^2 \geq 0$	$56{,}915.6449 > 0$	$960.0206 > 0$
$\hat{\bar{Y}}_2^*$	$\tau \geq S_{xy}/RS_x^2$ $R \geq S_{xy}/\tau S_x^2$ $S_x^2 \geq S_{xy}/\tau R$	$1.0178 > 0.8867$ $1.7990 > 1.5674$ $149.55 > 130.2933$	$1.0449 > 0.8752$ $4.0406 > 3.3844$ $8.7616 > 7.3382$
$\hat{\bar{Y}}_3^*$	$\tau \geq -S_{xy}/RS_x^2$ $R \geq -S_{xy}/\tau S_x^2$ $S_x^2 \geq -S_{xy}/\tau R$	$1.0178 > -0.8867$ $1.7990 > -1.5674$ $149.55 > -130.2933$	$1.0449 > -0.8752$ $4.0406 > -3.3844$ $8.7616 > -7.3382$
$\hat{\bar{Y}}_4^*$	$\tau \geq S_{xy}/2RS_x^2$ $R \geq S_{xy}/2\tau S_x^2$ $S_x^2 \geq S_{xy}/2\tau R$	$1.0178 > 0.4434$ $1.7990 > 0.7837$ $149.55 > 65.1467$	$1.0449 > 0.4376$ $4.0406 > 1.6922$ $8.7616 > 3.6694$
$\hat{\bar{Y}}_5^*$	$\tau \geq -S_{xy}/2RS_x^2$ $R \geq -S_{xy}/2\tau S_x^2$ $S_x^2 \geq -S_{xy}/2\tau R$	$1.0178 > -0.4434$ $1.7990 > -0.7837$ $149.55 > -65.1467$	$1.0449 > -0.4376$ $4.0406 > -1.6922$ $8.7616 > -3.6694$
$\hat{\bar{Y}}_6^*$	$\rho_{xy} \leq \frac{S_{xy}}{S_x S_y}$	$0.9446 = 0.9446$	$0.6741 = 0.6741$

Table 3: MSEs and PREs for the estimators under positive correlations

S/No	Estimator	Population I		Population II	
		MSE	PRE	MSE	PRE
1.	$\hat{\bar{Y}}_1^*$	309.2617	100	157.5365	100
2.	$\hat{\bar{Y}}_2^*$	39.3464	7.86	88.6509	177.70
3.	$\hat{\bar{Y}}_3^*$	1306.1679	23.68	430.5146	36.59
4.	$\hat{\bar{Y}}_4^*$	496.4221	62.30	971.5822	16.21
5	$\hat{\bar{Y}}_5^*$	3030.0651	10.21	907.5849	17.36
6.	$\hat{\bar{Y}}_6^*$	33.3170	928.24	85.9502	183.29
7.	$\hat{\bar{Y}}_7^*$	33.3240	928.24	85.9502	183.29

Table 4: Efficiency Comparisons with different Estimators under negative correlations

$\bar{Y}_{\theta}^{*}$	Optimality conditions	Population 1	Population 1I
$\bar{Y}_1^*$	$S_{xy}^2 \geq 0$	$1{,}817.6238 > 0$	$2{,}042.2751 > 0$
$\bar{Y}_2^*$	$\tau \geq S_{xy}/RS_x^2$ $R \geq S_{xy}/\tau S_x^2$ $S_x^2 \geq S_{xy}/\tau R$	$0.9787 > -0.5675$ $1.6109 > -0.9392$ $46.3822 > -27.0417$	$1.0491 > -0.6947$ $1.5350 > -1.0165$ $42.3823 > -28.0661$
$\bar{Y}_3^*$	$\tau \geq -S_{xy}/RS_x^2$ $R \geq -S_{xy}/\tau S_x^2$ $S_x^2 \geq -S_{xy}/\tau R$	$0.9787 > 0.5675$ $1.6109 > 0.9392$ $46.3822 > 27.0417$	$1.0491 > 0.6947$ $1.5350 > 1.0165$ $42.3823 > 28.0661$
$\bar{Y}_4^*$	$\tau \geq S_{xy}/2RS_x^2$ $R \geq S_{xy}/2\tau S_x^2$ $S_x^2 \geq S_{xy}/2\tau R$	$0.9787 > -0.2838$ $1.6109 > -0.4696$ $46.3822 > -13.5209$	$1.0491 > -0.3474$ $1.5350 > -0.5083$ $42.3823 > -14.0331$
$\bar{Y}_5^*$	$\tau \geq -S_{xy}/2RS_x^2$ $R \geq -S_{xy}/2\tau S_x^2$ $S_x^2 \geq -S_{xy}/2\tau R$	$0.9787 > 0.2838$ $1.6109 > 0.4696$ $46.3822 > 13.5209$	$1.0491 > 0.3474$ $1.5350 > 0.5083$ $42.3823 > 14.0331$
$\bar{Y}_6^*$	$\rho_{xy} \leq \frac{S_{xy}}{S_x S_y}$	$-0.8628 = -0.8628$	$-0.9242 = -0.9242$

Table 5: MSEs and PREs for the estimators under negative correlations

S/No	Estimator	Population III		Population IV	
		MSE	PRE	MSE	PRE
1.	$\hat{\bar{Y}}_1^*$	43.1747	100	42.9998	100
2.	$\hat{\bar{Y}}_2^*$	247.9823	17.41	237.6842	18.09
3.	$\hat{\bar{Y}}_3^*$	27.4734	157.15	15.8339	271.57
4.	$\hat{\bar{Y}}_4^*$	121.9402	35.41	119.4022	36.01
5	$\hat{\bar{Y}}_5^*$	200.8786	21.49	156.1865	27.53
6.	$\hat{\bar{Y}}_6^*$	11.0344	391.27	6.2717	685.62
7.	$\hat{\bar{Y}}_7^*$	11.0344	391.27	6.2717	685.62

7. Discussion of results

Analytically, it has been shown that the proposed log-ratio and log-product estimators have equal optimal efficiency. The results of efficiency comparisons using Tables 2 and 4 showed that all of the optimality conditions are satisfied in accordance with section 5.1for the four populations considered in the study. It also showed that the proposed log-ratio and log-product estimators have same efficiency as the classical regression estimator of population mean in systematic sampling in all the four populations considered but are always more efficient than all existing estimators under review.

Numerical results for the percent relative efficiency (*PREs*) in Tables 3 and 5 showed that the proposed log-ratio and log-product estimators have the highest percent gains in efficiency in all the four populations considered in the study and are therefore more efficient than all others estimators under review,

The numerical results in Table 3 showed that all ratio-type estimators considered are more efficient than all corresponding product-type estimators with appreciable gains in efficiency under populations where the study and auxiliary variables are positively correlated while numerical results in Table 5 showed that all product-type estimators considered are more efficient than all corresponding ratio-type estimators with appreciable gains in efficiency under populations where the study and auxiliary variables are negatively correlated. These findings support a well-known fact in sampling survey literature that the ratio-type estimators are more effective when there is a highly positive linear relationship between the study and the auxiliary variables while the product-type estimators are more effective when there is a highly negative linear relationship between the study and the auxiliary variables [Solanki et al (2012) [30], Izunobi and Onyeka (2019) [18]].

Following the numerical results in Tables 3 and 5 and the results of the efficiency comparisons in Tables 2 and 4, it is observed that the proposed log-ratio and log-product estimators do not discriminate between negatively and positively correlated populations as they always have the highest percent efficiency gains in all the four populations considered in the study. This showed that they are substantially adaptable to both negatively and positively correlated populations. These results proved the relevance of using the concept of inverse exponentiation in estimation theory.

8. Conclusion

Sequel to the discussion of results above, we conclude that the proposed log-estimators at optimal conditions have same efficiency as the classical regression estimator of population mean in systematic sampling and are always more efficient than all existing estimators with appreciable gains in efficiency; thus providing a better and efficient alternative estimators in practical situations.

The proposed log-ratio and log-product estimators do not discriminate between negatively and positively correlated populations. Therefore, the new estimation technique (inverse exponentiation) should be used by survey researchers as it gives consistent and more precise estimates of the population parameters.

9. References

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