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# Analysis on generating transitive permutation group

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**Abstract**
Denote by f(n) the number of subgroups of the symmetric group Sym(n) of degree n, and by $f_{trans}(n)$ the number of its transitive subgroups. It was conjectured by Pyber that almost all subgroups of Sym(n) are not transitive, that is, $f_{trans}(n)/f(n)$ tends to 0 when n tends to infinity. It is still an open question whether or not this conjecture is true. The difficulty comes from the fact that, from many points of view, transitivity is not a really strong restriction on permutation groups, and there are too many transitive groups. In paper paper we solve the problem in the particular case of permutation groups of prime power degree, proving the following result.

**Keywords:** symmetric group, transitive subgroups, permutation groups, particular case

## Introduction

There exists a constant b such that $f_{trans}(n) \leqslant 2^{bn2/\sqrt{\log n}}$ for each prime power n. This can be compared with a lower bound for $f_{trans}(n)$ proved by Pyber for each prime p, there exists a constant ap such that if n is a power of p, then $f_{trans}(n) \geqslant 2^{apn2= \log n}$.

**Theorem 1.** Implies the following
**Theorem 2.** The proportion of subgroups of Sym(n) which are transitive tends to 0 as n tends to infinity through prime powers.
To prove Theorem 1, we need some information concerning the minimal number of generators of transitive permutation groups. This involves the discussion of another conjecture. In [1], Bryant, Kovacs and Robinson proved: there is a number c' such that each soluble transitive permutation group of degree n ⩾ 2 can be generated by $[c'n/\sqrt{\log n}]$ elements. It is still an open problem whether the solubility hypothesis can be removed in this result. We offer a partial solution to this question, proving the following result.

**Theorem 3.** There exists a constant c such that any permutation group of degree n ⩾ 2 containing a soluble transitive subgroup can be generated by $[cn=\sqrt{\log n}]$ elements.
Since any transitive permutation group of prime power degree contains a soluble transitive subgroup, we deduce the following corollary.
**Corollary 1.** Any transitive subgroup of $Sym(p^m)$ can be generated by $[cp^{m}= \sqrt{m}]$ elements.
It was proved by Kovacs and Newman [2] that for each prime p, there is a constant $c_p$ such that whenever n is a power of p, there is a transitive p-group of degree n which cannot generated by $[c_p n/\sqrt{\log n}]$ elements. This means that Corollary 1 is asymptotically `best possible'.
To prove Theorem 3, we employ a result due to Bryant, Kovacs and Robinson [1, Theorem 5]: there is a constant b such that, given a module of dimension a for a subgroup of index n in a finite soluble group, each sub module of the induced module can be generated by $[abn/\sqrt{\log n}]$ elements. It is an open problem whether the same result, or a similar one, remains true for arbitrary finite groups. If this were the case, then our proof of Theorem 3 could be adapted easily to prove the analogous result without the solubility hypothesis; as a consequence, an analogue of Theorem 2 for arbitrary degrees could also be proved easily.

**Generating permutation groups**

To prove Theorem 3. We use an approach to the study of the minimal number of generators d(G) of a finite group G which relies on some results proved recently by Francesca Dalla Volta and the author [3].

Let H be a finite group with a unique minimal normal subgroup, N. If N is abelian, then assume also that N has a complement in H.

For each positive integer k, let $H^k$ be the k-fold direct power of H, and define the subgroup $H^k$ by

$H_k = \{(h_1;\ldots\ldots\ldots; h_k) \in H^k \mid h_1 \equiv h_k \bmod N\}$.

Moreover, let g(H;G) be the largest integer k such that Hk is an epimorphic image of G, and let

$$t_1 = \max_R \frac{g(R,G)-2}{\dim_{Eng_H N} N}, t_2 \max_s g(S,G),$$

Where R runs over the set of finite groups with a unique minimal normal subgroup, say N, which is abelian and complemented, and S runs over the set of finite groups with a unique minimal normal subgroup which is non-abelian. The minimal number of generators can be bounded in term of $t_1$ and $t_2$. In fact, we have the following result.

**Proposition 1.** (CFSG). If d(G) > 2; then either $d(G) \leqslant t_1 + 3$, or $t_2 \geqslant 2$ and $d(G) \leqslant [\log(t_2 - 1) + 3]$.

To use the previous result, we shall need the following lemma.

Lemma 2. Let G and H be finite groups, let N = soc H; let M be an abelian normal subgroup of G, and let $d_G(M)$ be the minimal number of generators of M as a G-module. If N is abelian, then we have

$$\frac{g(H,G)}{\dim_{end_H N} N} \leqslant \frac{g(H,G/M)}{\dim_{end_H N} N} + d_G(M)$$

**Proof.** Let $k_1$ = g(H, G=M) and $k_2$ = g(H, G). There exists a normal subgroup L of G such that $G/L \cong H_{k2}$. Since ML/L is an abelian normal subgroup of G/L, we have $G/ML \cong H_k$, with $k \leqslant k_1$, and $ML/L \cong N^{k2-k}$, hence

$$\frac{k_2 - k}{\dim_{End_H N} N} \leqslant d_G(N^{k2-k}) = d_G \frac{ML}{L} = d_G \left(\frac{M}{L \cap M}\right) \leqslant d_G(M)$$

and

$$\frac{k_2}{\dim_{End_H N} N} \leqslant d_G(M) + \frac{k}{\dim_{End_H N} N} \leqslant d_G(M) + \frac{k_1}{\dim_{End_H N} N}.$$

We shall use the following result.

**Theorem 3.** (CFSG, Pyber). There is a constant $c_0$ such that the product of the orders of the abelian composition factors of a primitive permutation group of degree n is at most $24^{-1/3} n^{1+c0}$.

This implies the following.

**Corollary 4.** The number of abelian composition factors of a primitive permutation group of degree n is at most $[(1 + c_0) \log n]$.

The proof of Theorem 3 is very similar to that of Theorem 7 in [1]. For example, the following lemma plays a role similar to that of Corollary 4.1.6 in [1]. Let W be the wreath product $R \wr Sym(s)$, and denote by $\pi : R \wr Sym(s) \to Sym(s)$ the projection over the top group. Let $R_1 \times \ldots \times R_s$ be the base subgroup of W, and consider $W_i = N_W(R_i)$; since $W_i \cong R_i \times (R \wr Sym(s-1))$, we may consider the projection $\rho_i: W_i \to R_i$. We have the following lemma.

**Lemma 5.** Let R be a finite group, and let a be the number of abelian composition factors of R. Consider a subgroup G of the wreath product $R \wr Sym(s)$; $s \geqslant 2$, which satisfies the properties:

a. G contains a subgroup $\tilde{G}$ such that $\pi(\tilde{G})$ is soluble and transitive;
b. $\rho_i(N_G(R_i)) = R_i$ for $1 \leqslant i \leqslant s$.

If H is a finite group with a unique minimal normal subgroup which is abelian and complemented, and b is the constant which appears in [1, Theorem 5], then

$$\frac{g(H,G)}{\dim_{end_H N} N} \leqslant \frac{g(H,\pi(G))}{\dim_{end_H N} N} + \left[\frac{abs}{\sqrt{\log s}}\right]$$

**Proof.** We use induction on the order of R. Let L be a minimal normal subgroup of R. We identify L with a normal subgroup $L_1$ of the factor $R_1$ of the base subgroup $R_1\times......\times R_s$ of W. Let $B_L$ be the direct product of the distinct G-conjugates of $L_1$, and consider $M = B_L \cap G$. We have $M \trianglelefteq G$ with $G/M \cong GB_L/B_L \leqslant R/L \wr Sym(s)$.

$$\frac{g(H,G/H)}{\dim_{end_H N} N} \leqslant \frac{g(H,\pi(G))}{\dim_{end_H N} N} + \left[\frac{a_1 bs}{\sqrt{\log s}}\right] \quad (1)$$

Where $a_1$ is the number of abelian composition factors of R/L. We distinguish two cases.

1. L is non-abelian. Since $\rho_1(M) \trianglelefteq \rho_1(W_1 \cap G)$ and, by (b), $\rho_1(W_1 \cap G) = R_1$, we have $\rho_1(M) \trianglelefteq R_1$. On the other hand, $\rho_1(M)$ is contained in $L_1$, which is a minimal normal subgroup of $R_1$, so we conclude that either $\rho_i(M) = 1$ or $\rho_i(M) \cong \rho_1 L$ for $1 \leqslant i \leqslant s$. (Since $\pi(G)$ is a transitive subgroup of Sym(s), $\rho_i(M) \cong \rho_1(M)$ for $1 \leqslant i \leqslant s$.) If $\rho_i(M) = 1$, then $M = 1$. If $\rho_i(M) \cong L$, then M is a subdirect product of $L^s$, so, in particular, it is a direct product of non-abelian simple groups and admits no abelian composition factor. In both cases, g(H, G/M) = g(H,G), and the conclusion follows from (1).
2. L is abelian. Let $a_2$ be the composition length of L, so that $a = a_1 + a_2$. We have that $B_L$ is isomorphic as a $\bar{G}$ -module to the induced module $L_1\uparrow^{\tilde{G}}_{\tilde{K}}$, where we define $\tilde{K} = N_{\tilde{G}}(L_1)$. Since $\pi(\tilde{G})$ is a transitive subgroup of Sym(s), $[\tilde{G}:\tilde{K}] = s$. Now M is a $\tilde{G}$ -submodule of $B_L$ so, by [1, Theorem 5],

$$d_G(M) \leqslant d_{\tilde{G}}(M) \leqslant \left[\frac{a_2 bs}{\sqrt{\log s}}\right] \quad (2)$$

By Lemma 2 and (2), we have

$$\begin{aligned}\frac{g(H,G)}{\dim_{end_H N} N} &\leqslant \frac{g(H,G/M)}{\dim_{end_H N} N} + a_G(M)\\ &\leqslant \frac{g(H,\pi(G))}{\dim_{end_H N} N} + \left[\frac{a_1 bs}{\sqrt{\log s}}\right] + \left[\frac{a_2 bs}{\sqrt{\log s}}\right]\\ &\leqslant \frac{g(H,\pi(G))}{\dim_{end_H N} N} + \left[\frac{abs}{\sqrt{\log s}}\right].\end{aligned}$$

**Lemma 6.** Let $G \leqslant Sym(n)$; $n \geqslant 2$, and let N be the socle of a finite group H with a unique minimal normal subgroup. If N is non-abelian, then g(H,G) < n=4.

**Proof.** Clearly, if S is the composition factor of N, then $g(H, G) \leqslant \mu_S(G)$, where $\mu_S(G)$ denotes the multiplicity of S as a composition factor of G. By Burnside's $p^\alpha q^\beta$-theorem, there is a prime number $p \geqslant 5$ dividing |S|. Then $\mu_S(G) \leqslant v_p(|G|) \leqslant v_p(n!)$, where, for a natural number x, $v_p(x)$ is the largest integer v such that $p^v|x$. Now, as is well known, $v_p(n!) < n/(p-1)$, so that $v_p(n!) < n/4$. Therefore g(H,G) < n/4.

**Lemma 7.** There exists a constant c1 such that if G is a subgroup of Sym(n), $n \geqslant 2$, containing a soluble transitive subgroup, and H has a unique minimal normal subgroup N; which is abelian, then.

$$\frac{g(H,G)-2}{\dim_{end_H N} N} + 3 \leqslant \frac{c_1 n}{\sqrt{\log n}}.$$

**Proof.** Let b and $c_0$ be the constants which appear, respectively, in [1, Theorem 5] and in Corollary 4. We choose $c_1$ such that

$$b(1+c_0)x\sqrt{x+1} \leqslant c_1(2^x - \sqrt{x+1}) \text{ whenever } x \geqslant 1, \quad (3)$$

$$1+(1+c_0)\log n \leqslant \frac{c_1 n}{\sqrt{\log n}} \text{ whenever } n \geqslant 1, \quad (4)$$

and we prove the lemma using induction on the degree n.

If G is a primitive permutation group, then

$$\frac{g(H,G)-2}{\dim_{end_H N} N}+3 \leqslant g(H, G)-2$$

$$\leqslant 1+\text{number of abelian composition factors of G}$$

$$\leqslant 1+(1+c_0)\log n \text{ (by Corollary 4)}$$

$$\leqslant \frac{c_1 n}{\sqrt{\log n}}$$

So we may assume that G is imprimitive: choose a block of imprimitivity, say B, of minimal size, and let R be the permutation group induced by the action of the stabilizer $St_G(B)$ in G of B on the set B. By the minimality of B, R is a primitive permutation group of degree r = |B|. We may identify G with a subgroup of the wreath product R ≀ Sym(s) (where n = rs and s ⩾ 2) satisfying the hypothesis of Lemma 5. So we have

$$\frac{g(H,G)-2}{\dim_{end_H N} N}+3 \leqslant \frac{g(H,\pi(G))-2}{\dim_{end_H N} N}+3\left[\frac{abs}{\sqrt{\log s}}\right],$$

where a is the number of abelian composition factors of the group R. On the other hand, π (G) is a subgroup of Sym(s) containing a soluble transitive subgroup, so

$$\frac{g(H,\pi(G))-2}{\dim_{end_H N} N}+3 \leqslant \frac{c_1 s}{\sqrt{\log s}}.$$

Furthermore, by Corollary 4, a ⩽ $(1 + c_0)$ log r, so

$$\frac{g(H,G)-2}{\dim_{end_H N} N}+3 \leqslant \frac{c_1 s}{\sqrt{\log s}}+\frac{(1+c_0)bs\log r}{\sqrt{\log s}}$$

Let x = logr and y = logs. Then

$$\frac{g(H,G)-2}{\dim_{end_H N} N}+3 \leqslant \frac{2^y(c_1+(1+c_0)bx)}{\sqrt{y}}$$

$$=\frac{2^y}{\sqrt{x+y}}(c_1+(1+c_0)bx)\sqrt{\frac{x+y}{y}}$$

$$\leqslant \frac{2^y}{\sqrt{x+y}}(c_1+(1+c_0)bx)\sqrt{x+1} \text{ (since } y>1)$$

$$=\frac{2^y}{\sqrt{x+y}}(c_1\sqrt{x+1}+(1+c_0)bx)\sqrt{x+1}$$

$$\leqslant \frac{2^x 2^y c_1}{\sqrt{x+y}} \text{ (by 2)}$$

$$=\frac{c_1 rs}{\sqrt{\log r+\log s}}$$

$$=\frac{c_1 n}{\log n}$$

**Proof of Theorem 3.** Let G be a permutation group of degree n ⩾ 2, and let $t_1$, $t_2$ be the integer numbers defined in Proposition 1. If d(G) ⩾ 2, then either d(G) ⩽ $t_1$ +3 ⩽ $c_1 n/\sqrt{\log n}$ (by Lemma 7), or $t_2$ ⩾ 2, and in this latter case, n ⩾ 9 and d(G) ⩽ $\log(t_2 - 1) + 3$ ⩽ log(n/4 − 1) + 3 (by Lemma 6). There exists $c_2$ such that, for each n ⩾ 9,

$$\log\left(\frac{n}{4}-1\right)+3\leqslant\frac{c_2 n}{\sqrt{\log n}}$$

We take c = max ($c_1$, $c_2$).

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