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Analysis of linear difference equations with variable coefficients

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Abstract
In this paper, we studies the explicit solution of a linear difference equation of unbounded order with variable coefficients is presented. The solutions of nonhomogeneous and homogeneous linear difference equations of order N with variable coefficients are obtained. From these solutions, we also get expressions for the product of companion matrices, and the power of a companion matrix. This paper presents explicit solutions in terms of coefficients of linear difference equations with variable coefficients, for both the unbounded order case and the Nth-order case.

Keywords: explicit solution, difference equation, variable coefficients, nonhomogeneous, homogeneous, companion matrix, explicit solutions

Introduction

Asymptotics of solutions of linear recurrences with coefficients having series representations have also been studied by many authors [1-7].

Further work on convergence properties of linear recurrence sequences has been presented by Kooman and Tijdeman [8]. A survey of the literature on explicit solutions of linear recurrences reveals that in the case of linear recurrences with constant coefficients, the explicit solutions in terms of coefficients are well known. That is no longer the case when the coefficients vary with the index [5]. A method of solving linear matrix difference equations with constant coefficients by using operator identities has been presented by Verde-Star [9]. Work on the existence and construction of closed-form solutions of linear recurrences with polynomial and rational coefficients has been done by Petkovsek [10-11]. But, in the available open literature, there are no expressions in terms of coefficients for the complete solution of a linear difference equation with varying coefficients when the order is 3 or more, except for cases in which the coefficients have some special properties.

Formulations

Consider the linear difference equation

$$y_k = \sum_{i=1}^{k-1} b_{k,i} y_i + x_k, \qquad k \geq 1 \tag{1}$$

of unbounded order, with integral index k, variable complex coefficients $b_{k,1},\ldots, b_{k,k-1}$, and complex forcing term x_k. The solution of this equation, which is an expression for y_k, $k\geq 1$ in terms of only coefficients and forcing terms, is given by the proposition that follows.

Proposition 1. The solution of difference equation (1) is given by

$$y_k = \sum_{i=1}^{k-1} c_{k,i} x_i + x_k, \qquad k \geq 1, \tag{2}$$

Where

$$c_{k,i}=b_{k,i}+\sum_{j=2}^{k-i}\sum_{\substack{(l_1,\dots,l_j)\\ l_1,\dots,l_j\geq 1\\ l_1+l_2+\cdots+l_j=k-i}} b_{k,k-l_1}\left[\prod_{m=2}^{j} b_{k-\sum_{n=1}^{m-1}l_n,\,k-\sum_{n=1}^{m}l_n}\right] \tag{3}$$

for i=1,..., k–1, k≥2.

Proof. From the difference equation (1), it is clear that its solution is of the form (2). Substituting (2) in (1), we obtain, for i =1,..., k–1, k≥2,

$$\begin{aligned} y_k &= \sum_{i=1}^{k-1} b_{k,i}x_i+\sum_{i=2}^{k-1}\sum_{r=1}^{i-1} b_{k,i}c_{i,r}x_r+x_k\\ &= \sum_{i=1}^{k-1} b_{k,i}x_i+\sum_{r=1}^{k-2}\sum_{i=r+1}^{k-1} b_{k,i}c_{i,r}x_r+x_k\\ &= \sum_{i=1}^{k-2}\left[b_{k,i}+\sum_{r=i+1}^{k-1} b_{k,r}c_{r,i}\right]x_i+b_{k,k-1}x_{k-1}+x_k. \end{aligned} \tag{4}$$

Comparing (4) with (2), we get

$$c_{k,i}-b_{k,i}=\sum_{r=i+1}^{k-1} b_{k,r}c_{r,i} \qquad \text{for } i=1,\dots,k-1,\ k\geq 2. \tag{5}$$

If we can show that $c_{k,i}$ given by (3) satisfies (5), then the proposition will be proved.
Now using (3), the right-hand side of (5) can be expressed as

$$\begin{aligned} \sum_{r=i+1}^{k-1} b_{k,r}c_{r,i} &= b_{k,i+1}b_{i+1,i}+\sum_{r=i+2}^{k-1} b_{k,r}b_{r,i}\\ &\quad+\sum_{j=2}^{k-i-1}\sum_{r=i+j}^{k-1}\sum_{\substack{(l_1,\dots,l_j)\\ l_1,\dots,l_j\geq 1\\ l_1+l_2+\cdots+l_j=r-i}} b_{k,r}b_{r,r-l_1}\\ &\quad\times\left[\prod_{m=2}^{j} b_{r-\sum_{n=1}^{m-1}l_n,\,r-\sum_{n=1}^{m}l_n}\right]. \end{aligned} \tag{6}$$

Substituting l_1=k–r and replacing the index j by j–1 in (6), we get

$$\begin{aligned} &\sum_{r=i+1}^{k-1} b_{k,r}c_{r,i}\\ &\quad= b_{k,k-(k-i-1)}b_{k-(k-i-1),\,k-(k-i-1)-1}+\sum_{l_1=1}^{k-i-2} b_{k,k-l_1}b_{k-l_1,\,k-l_1-(k-l_1-i)}\\ &\quad\quad+\sum_{j=3}^{k-i}\sum_{l_1=1}^{k-i-(j-1)}\sum_{\substack{(l_2,\dots,l_j)\\ l_2,\dots,l_j\geq 1\\ l_2+\cdots+l_j=k-i-l_1}} b_{k,k-l_1}\left[\prod_{m=2}^{j} b_{k-\sum_{n=1}^{m-1}l_n,\,k-\sum_{n=1}^{m}l_n}\right]\\ &\quad=\sum_{j=2}^{k-i}\sum_{\substack{(l_1,\dots,l_j)\\ l_1,\dots,l_j\geq 1\\ l_1+l_2+\cdots+l_j=k-i}} b_{k,k-l_1}\left[\prod_{m=2}^{j} b_{k-\sum_{n=1}^{m-1}l_n,\,k-\sum_{n=1}^{m}l_n}\right]\\ &\quad= c_{k,i}-b_{k,i} \end{aligned} \tag{7}$$

for i=1,..., k–1, k≥2, from (3). Therefore we conclude that the expression for $c_{k,i}$ given by (3) obeys (5), which proves the proposition.

Linear difference equation of order N

We know that the linear difference equation

$$y_{k+N} = \sum_{j=1}^{N} a_{k,j} y_{k+N-j} + x_{k+N}, \qquad k \geq 1 \tag{8}$$

of order N (N≥2) with variable complex coefficients $a_{k,j}$, j=1,..., N, complex forcing term x_{k+N}, and complex initial values y_1,..., y_N. The solution of this equation, which is an expression for y_{k+N}, k≥1 in terms of only coefficients, initial values, and forcing terms, is given by the following proposition.

Proposition 2. The solution of difference equation (8) with initial values y_1,..., y_N is given by

$$y_{k+N} = \sum_{j=1}^{N} d_{k,j} y_{N+1-j} + \sum_{j=2-k}^{0} d_{k,j} x_{N+1-j} + x_{k+N}, \qquad k \geq 1, \tag{9a}$$

Where

$$d_{k,j} = \sum_{r=1}^{k+j-1} \sum_{\substack{(l_1,\dots,l_r) \\ 1 \leq l_1,\dots,l_r \leq N \\ l_r \geq j \\ l_1+l_2+\cdots+l_r = k+j-1}} \left[\prod_{m=1}^{r} a_{k+l_m-\sum_{n=1}^{m} l_n,\, l_m} \right] \tag{9b}$$

for j=2–k,..., N, k≥1.

Proof. Difference equation (8). With initial values y_1,..., y_N can be treated as a special case of (1). In which

$$\begin{aligned} y_k &= x_k \qquad \text{for } 1 \leq k \leq N, \\ b_{k,i} &= 0 \qquad \text{for } i = 1,\dots,k-1,\ 1 \leq k \leq N, \\ b_{k,i} &= 0 \qquad \text{for } i = 1,\dots,k-N-1,\ k \geq N+2, \\ b_{k,i} &= a_{k-N,k-i} \qquad \text{for } i = k-N,\dots,k-1,\ k \geq N+1. \end{aligned} \tag{10}$$

Now the solution of (8) with initial values y_1,..., y_N is given by Proposition 1, with the expression for $c_{k,i}$ given by (3) having the additional constraints on $b_{k,i}$ in (10). The solution can therefore be expressed by using (10) as

$$\begin{aligned} y_k &= \sum_{i=1}^{N} c_{k,i} y_i + \sum_{i=N+1}^{k-1} c_{k,i} x_i + x_k, \qquad k \geq N+1, \quad \text{or} \\ y_{k+N} &= \sum_{j=1}^{N} c_{k+N,N+1-j}\, yj + \sum_{j=2-k}^{0} c_{k+N,N+1-j} x_{N+1-j} + x_{k+N}, \qquad k \geq 1. \end{aligned} \tag{11}$$

Since (3) can be rewritten as

$$c_{k,i} = \sum_{j=1}^{k-i} \sum_{\substack{(l_1,\dots,l_j) \\ l_1,\dots,l_j \geq 1 \\ l_1+l_2+\cdots+l_j = k-i}} \left[\prod_{m=1}^{j} b_{k+l_m-\sum_{n=1}^{m} l_n,\, k-\sum_{n=1}^{m} l_n} \right] \tag{12}$$

for i=1,..., k–1, k≥2, it is clear from (10) that for each term of the summation on the right-hand side of (12) to be nontrivial, the conditions

$$1 \leq l_1,\dots,l_j \leq N, \quad k + l_j - \sum_{n=1}^{j} l_n = i + l_j \geq N+1 \qquad \text{for } j = 1,\dots,k-i \tag{13}$$

Need to be satisfied. Therefore we get

$$c_{k,i} = \sum_{j=1}^{k-i} \sum_{\substack{(l_1,\ldots,l_j) \\ 1\le l_1,\ldots,l_j \le N \\ l_j \ge N+1-i \\ l_1+l_2+\cdots+l_j = k-i}} \left[\prod_{m=1}^{j} b_{k+l_m-\sum_{n=1}^{m} l_n,\, k-\sum_{n=1}^{m} l_n} \right] \tag{14}$$

for i=1,..., k–1, k≥N+1. Substituting

$$d_{k,j} = c_{k+N,N+1-j} \qquad \text{and} \qquad a_{k,j} = b_{k+N,k+N-j} \tag{15}$$

In (11) and (14) respectively, and combining the two results, we obtain the proposition.
Note that difference equation (8) with initial values y_1,..., y_N can be expressed in vector form as

$$\begin{bmatrix} y_{k+N} \\ y_{k+N-1} \\ \vdots \\ y_{k+1} \end{bmatrix} = \begin{bmatrix} a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,N-1} & a_{k,N} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \times \begin{bmatrix} y_{k+N-1} \\ y_{k+N-2} \\ \vdots \\ y_k \end{bmatrix} + \begin{bmatrix} x_{k+N} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad k \ge 1, \tag{16}$$

where the N=N matrix on the right-hand side of (16) is the companion matrix for index k. Defining

$$\mathbf{y}_k \triangleq [y_{k+N-1}, y_{k+N-2}, \ldots, y_k]^T, \qquad \mathbf{x}_k \triangleq [x_{k+N}, 0, \ldots, 0]^T, \tag{17a}$$

$$\mathbf{A}_k \triangleq \begin{bmatrix} a_{k,1} & a_{k,2} & a_{k,3} & \cdots & a_{k,N-1} & a_{k,N} \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{17b}$$

for k≥1, we can rewrite (16) as

$$\mathbf{y}_{k+1} = \mathbf{A}_k \mathbf{y}_k + \mathbf{x}_k, \qquad k \ge 1, \tag{18}$$

where $y_1 = [y_N, y_{N-1},\ldots, y_1]^T$ is the initial value vector. The solution of (18) with initial value vector y_1 can then be given in matrix form by

$$\mathbf{y}_{k+1} = \mathbf{A}_k \mathbf{A}_{k-1} \cdots \mathbf{A}_1 \mathbf{y}_1 + \sum_{l=1}^{k-1} \mathbf{A}_k \mathbf{A}_{k-1} \cdots \mathbf{A}_{l+1} \mathbf{x}_l + \mathbf{x}_k, \qquad k \ge 1. \tag{19}$$

Product of Companion Matrices
Consider the homogeneous case of difference equation (8) with initial values y_1,..., y_N, in which x_{k+N} =0 for k≥1. The solution of this homogeneous equation can be expressed, using (9a), as

$$y_{k+N} = \sum_{j=1}^{N} d_{k,j} y_{N+1-j}, \qquad k \ge 1, \tag{20}$$

where $d_{k,j}$, j=1,..., N are given by (9b).
Extending the definition of $d_{k,j}$ to k= – (N–1),..., 0, we can express the solution (20) as

$$y_{k+N} = \sum_{j=1}^{N} d_{k,j} y_{N+1-j}, \qquad k \ge -(N-1), \tag{21}$$

where $d_{k,1}$,..., $d_{k,N}$ are given by (9b) for k≥1, and by

$$\begin{bmatrix} d_{0,1} & d_{0,2} & \cdots & d_{0,N} \\ d_{-1,1} & d_{-1,2} & \cdots & d_{-1,N} \\ \vdots & \vdots & & \vdots \\ d_{-(N-1),1} & d_{-(N-1),2} & \cdots & d_{-(N-1),N} \end{bmatrix} = \mathbf{I}_N, \tag{22}$$

where I_N is the N×N identity matrix, for k= –(N–1),..., 0. However, (19) implies that the matrix form of the homogeneous solution is

$$\mathbf{y}_{k+1} = \mathbf{A}_k \mathbf{A}_{k-1} \cdots \mathbf{A}_1 \mathbf{y}_1, \qquad k \geq 1, \tag{23}$$

where y_k and A_k are defined in (17).
Comparing (17), (21), and (23), we find that the product of companion matrices can be expressed as

$$\mathbf{A}_k \mathbf{A}_{k-1} \cdots \mathbf{A}_1 = \begin{bmatrix} d_{k,1} & d_{k,2} & \cdots & d_{k,N-1} & d_{k,N} \\ d_{k-1,1} & d_{k-1,2} & \cdots & d_{k-1,N-1} & d_{k-1,N} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{k-N+1,1} & d_{k-N+1,2} & \cdots & d_{k-N+1,N-1} & d_{k-N+1,N} \end{bmatrix}, \quad k \geq 1, \tag{24}$$

where A_k(k≥1. is defined by (17b), and $d_{k-i+1,\,j}$, the entry in the ith row and the jth column (1≤i, j≤N. of the product A_k A_k.... A_1, which is obtained from (9b) and (22), is given by

$$d_{k-i+1,j} = \sum_{r=1}^{k-i+j} \sum_{\substack{(l_1,\ldots,l_r) \\ 1\leq l_1,\ldots,l_r\leq N \\ l_r \geq j \\ l_1+l_2+\cdots+l_r=k-i+j}} \left[\prod_{m=1}^{r} a_{k-i+1+l_m-\Sigma_{n=1}^{m} l_n,\, l_m}\right] \quad \text{if } i \leq \min(k,N), \tag{25a}$$

$$= 1 \qquad \text{if } j = i,\, i > k,\, k < N, \tag{25b}$$

$$= 0 \qquad \text{if } j \neq i,\, i > k,\, k < N. \tag{25c}$$

It can easily be shown that the characteristic equation of A_k is given by

$$\det(\mathbf{A}_k - \lambda \mathbf{I}_N) = (-1)^N \left\{ \lambda^n - \sum_{j=1}^{N} a_{k,j} \lambda^{N-j} \right\}, \tag{26}$$

which implies

$$\det(\mathbf{A}_k) = (-1)^{N+1} a_{k,N}. \tag{27}$$

Based on this result, we have the following proposition for linearly independent solutions of the homogeneous version of difference equation (8), that is, the equation

$$y_{k+N} = \sum_{j=1}^{N} a_{k,j} y_{k+N-j}, \qquad k \geq 1. \tag{28}$$

Proposition 3. Difference equation (28) with $a_{k,N}$≠/0, k≥1, has N linearly independent solutions expressed as y_{k+N} =$d_{k,j}$, j=1,..., N for k≥–(N–1), where $d_{k,1}$,..., $d_{k,N}$ are given by (9b) and (22).
Proof. If is clear from (21) that y_{k+N}=$d_{k,j}$, j=1,..., N, are N solutions of difference equation (8) for k≥ – (N–1). The Casoratian of the N sequences $\{d_{k,j}\}_{k\geq -(N-1)}$, j=1,..., N is given by the determinant

$$\begin{vmatrix} d_{k,1} & \cdots & d_{k,N} \\ d_{k+1,1} & \cdots & d_{k+2,N} \\ \vdots & & \vdots \\ d_{k+N-1,1} & \cdots & d_{k+N-1,N} \end{vmatrix}$$

$$= (-1)^{\lfloor \frac{N}{2} \rfloor} \begin{vmatrix} d_{k+N-1,1} & \cdots & d_{k+N-1,N} \\ d_{k+N-2,1} & \cdots & d_{k+N-2,N} \\ \vdots & & \vdots \\ d_{k,1} & \cdots & d_{k,N} \end{vmatrix}$$

$$= \begin{cases} (-1)^{\lfloor \frac{N}{2} \rfloor} & \text{for } k = -(N-1) \\ (-1)^{\lfloor \frac{N}{2} \rfloor} \prod_{i=1}^{k+N-1} (-1)^{N+1} a_{i,N} & \text{for } k \geq -(N-2) \end{cases}$$

$$\neq 0, \tag{29}$$

Using (22), (24), (27), and the fact that $a_{i,N} \neq 0$, $i=1,\ldots, k+N-1$, $k \geq -(N-2)$. Therefore $\{d_{k,j}\}_{k\geq -(N-1)}$, $j-1,\ldots, N$, are linearly independent sequences, which implies that $y_{k+N} = d_{k,j}$, $j=1,\ldots, N$, are N linearly independent solutions of the difference equation for $k \geq y -(N-1)$.

Power of the companion matrix for the constant coefficient case

Consider the case of the homogeneous Nth-order difference equation (28) with initial values $y_1,\ldots, y_N$ in which $a_{k,j}=a_j$ for all $k\geq 1$, $j= 1,\ldots, N$, that is, the equation

$$y_{k+N} = \sum_{j=1}^{N} a_j y_{k+N-j}, \qquad k \geq 1 \tag{30}$$

With initial values $y_1,\ldots, y_N$. We can rewrite (30). As

$$\mathbf{y}_{k+1} = \mathbf{A}\mathbf{y}_k, \qquad k \geq 1, \tag{31}$$

where y_k is defined in (17a), A is defined in (17b) with $a_{k,j}$ replaced by a_j, and y_1 is the initial value vector. From (23), the matrix form of the solution to (31) with initial value vector y_1 is given by

$$\mathbf{y}_{k+1} = \mathbf{A}^k \mathbf{y}_1, \qquad k \geq 1. \tag{32}$$

Thus the kth power of the N×N companion matrix A gives the solutions for $y_{k+1},\ldots, y_{k+N}$ of the Nth-order homogeneous linear difference equation (30) with constant coefficients $a_1,\ldots, a_N$ and initial values $y_1,\ldots, y_N$.
Using the result in (24) and (25) for the product of companion matrices, we obtain

$$\mathbf{A}^k = \begin{bmatrix} d_{k,1} & d_{k,2} & \cdots & d_{k,N-1} & d_{k,N} \\ d_{k-1,1} & d_{k-1,2} & \cdots & d_{k-1,N-1} & d_{k-1,N} \\ \vdots & \vdots & & \vdots & \vdots \\ d_{k-N+1,1} & d_{k-N+1,2} & \cdots & d_{k-N+1,N-1} & d_{k-N+1,N} \end{bmatrix}, \qquad k \geq 1, \tag{33}$$

where $d_{k-i+1,j}$, the entry in the ith row and the jth column of A^k, is expressed as

$$d_{k-i+1,j} = \sum_{r=1}^{k-i+j} \sum_{\substack{(l_1,\ldots,l_r) \\ 1\leq l_1,\ldots,l_r \leq N \\ l_r \geq j \\ l_1+l_2+\cdots+l_r = k-i+j}} \left[\prod_{m=1}^{r} a_{l_m} \right] \qquad \text{if } i \leq \min(k,N), \tag{34a}$$

$$= 1 \qquad \text{if } j = i, i > k, k < N, \tag{34b}$$

$$= 0 \qquad \text{if } j \neq i, i > k, k < N. \tag{34c}$$

Our aim is to obtain an alternative expression for $d_{k-i+1,j}$ when $i \leq$ min k, N. in terms of powers of the coefficients $a_1,\ldots, a_N$.
Now (34a). Can be rewritten as

$$d_{k-i+1,j} = \sum_{v=j}^{N} \left\{ \sum_{r=1}^{k-i+j} \sum_{\substack{(l_1,\dots,l_{r-1}) \\ 1 \le l_1,\dots,l_{r-1} \le N \\ l_1+l_2+\cdots+l_{r-1}=k-i+j-v}} (a_{l_1} \cdots a_{l_{r-1}}) \right\} a_v \tag{35}$$

for $i \le \min(k, N)$. The sum of $a_{l1} \dots a_{lr-1}$ over all $l_1,\dots, l_{r-1}$ satisfying

$$1 \le l_1,\dots,l_{r-1} \le N, \qquad l_1 + l_2 + \cdots + l_{r-1} = k - i + j - v$$

where $r=1,\dots, k-i+j$ is the same as the sum of

$$\binom{t_1 + t_2 + \cdots + t_N}{t_1,\dots,t_N} a_1^{t_1} \cdots a_N^{t_N}$$

Over all $t_1,\dots, t_N$ satisfying

$$t_1,\dots,t_N \ge 0, \qquad t_1 + 2t_2 + \cdots + Nt_N = k - i + j - v.$$

Therefore (35) implies, for $i \le \min(k, N)$,

$$d_{k-i+1,j} = \sum_{v=j}^{N} \left\{ \sum_{\substack{(t_1,\dots,t_N) \\ t_1,\dots,t_N \ge 0 \\ t_1+2t_2+\cdots+Nt_N=k-i+j-v}} \binom{t_1 + t_2 + \cdots + t_N}{t_1,\dots,t_N} \left[\prod_{m=1}^{N} a_m^{t_m} \right] \right\} a_v. \tag{36}$$

Replacing t_v by t_v-1 in (36), we get

$$d_{k-i+1,j} = \sum_{\substack{(t_1,\dots,t_N) \\ t_1,\dots,t_N \ge 0 \\ t_1+2t_2+\cdots+Nt_N=k-i+j}} \frac{t_j + t_{j+1} + \cdots + t_N}{t_1 + t_2 + \cdots + t_N} \times \binom{t_1 + t_2 + \cdots + t_N}{t_1,\dots,t_N} \left[\prod_{m=1}^{N} a_m^{t_m} \right] \tag{37}$$

for $i=1,\dots, \min(k, N)$, $j=1,\dots, N$, $k \ge 1$. This is an alternative way of expressing (34a) in terms of the coefficients $a_1,\dots, a_N$ of the companion matrix A.

Conclusion

The explicit solutions of the linear difference equations presented here utilize the combinatorial properties of the indices of the coefficients. The solution of the difference equation of unbounded order results in the solution of the Nth-order equation, which, in turn, provides expressions for the product of companion matrices and the positive integral powers of a companion matrix.

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