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On the bounds of mean values of entire function in several complex variables represented by multiple Dirichlet series

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Abstract
In this paper, we make an attempt to study properties and bounds of mean values of entire functions represented by multiple Dirichlet Series.

Keywords: Entire function, several complex variables represented, Dirichlet series

Introduction

Let us consider

(1.1) $f(s_1, s_2) = \sum_{m,n=1}^{\infty} a_{m,n} \exp(s_1\lambda_m + s_2\mu_n),$

$\left((s_j = \sigma_j + it_j), j = 1,2\right)$

Where$a_{m,n} \in C$, the field of complex numbers, $\lambda_m' s, \mu_n' s$ are real, and

$0 < \lambda_1 < \lambda_2 < \cdots \lambda_m \to \infty;$

$0 < \mu_1 < \mu_2 < \cdots < \mu_n \to \infty.$

A.I. Janusauskas in his paper (Janusauskas 1977) has shown that if

(1.2) $\lim_{m\to\infty} \frac{\log m}{\lambda_m} = 0, \lim_{n\to\infty} \frac{\log n}{\mu_n} = 0,$

Then the domain of convergence of the series (1.1) coincides with its domain of absolute convergence.

The necessary and sufficient condition that the series (1.1) satisfying (1.2) to be entire shown by Sarkar [1, pp.99] is that

(1.3) $\lim_{(m,n)\to\infty} \frac{\log|a_{m,n}|}{\lambda_m + \mu_n} = -\infty$

Throughout F stands for all double Dirichlet series of the form (1.1) satisfying (1.2) and (1.3)
Then f $\in F$ denotes an entire function overC^2, the cartisian product of two copies of the Complex plane. The results can be extended to several complex variables.
Sarkar [1, pp100] has defined that Corresponding to an f$\in F$, the maximum modulus M =M_f
And the maximum term $\mu = \mu_f$ on $R^2$are defined as

$M(\sigma) = M_f(\sigma_1, \sigma_2) = \max\{|f(s_1, s_2)|: s_1, s_2 \in C, \operatorname{Re} s_1 = \sigma_1, \operatorname{Re} s_2 = \sigma_2\}$

$\mu(\sigma)=\mu_f(\sigma_1,\sigma_2)=\max_{(m,n)\in N^2}\{|a_{m,n}|\exp(\sigma_1\lambda_m+\sigma_2\mu_n)\}$

$w$here N is the set of natural numbers.

We define the mean values of $|f(s_1,s_2)|$ as

(1.4) $I_2(\sigma_1,\sigma_2\,;f)=I_2(\sigma_1,\sigma_2)=\lim_{T_1 1 T_2\to\infty}\frac{1}{4T_1T_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}|f(\sigma_1+it_1,\sigma_2+it_2)|^2dt_1\,dt_2$

And mean value $m_{2,k}(\sigma_1,\sigma_2)$ of $|f(s_1,s_2)|$ as

(1.5) $m_{2,k}(\sigma_1,\sigma_2;f)=m_{2,k}(\sigma_1,\sigma_2)=\lim_{T_1,T_2\to\infty}\frac{1}{T_1T_2e^{k\sigma_1}e^{k\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\{|f(x_1+it_1,x_2+jt_2)|^2e^{kx_1}e^{kx_2}\}dx_1dx_2dt_1dt_2$

Where k is any positive number.
From (1.4) and (1.5), we can write

(1.6) $m_{2,k}(\sigma_1,\sigma_2)=\frac{4}{e^{k\sigma_1}e^{k\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}I_2(x_1,x_2)e^{kx_1}e^{kx_2}dx_1\,dx_2$ 2.

Lemma 1
For the Dirichlet $f(s_1,s_2)$, $f\in F$, $I_2(\sigma_1,\sigma_2\,;f)$ and $m_{2,k}(\sigma_1,\sigma_2;f)$ are increasing functions of σ_1, and σ_2.

Lemma 2
For the Dirichlet $f(s_1,s_2)$, $f\in F$, $\{\mu(\sigma_1,\sigma_2)\}^2\le I_2(\sigma_1,\sigma_2)\le\{M(\sigma_1,\sigma_2)\}^2$

Theorem 1
For the Dirichlet series $f(s_1,s_2), f\in F$, we have

(4.1) $\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{m_{2,k}(\sigma_1,\sigma_2)}{\{M(\sigma_1,\sigma_2)\}^2}\le\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)}\le\frac{4}{k^2}$

Proof
Since $I_2(x_1,x_2)$ is an increasing function of x_1 and x_2 from Lemma 1 and therefore from (1.6) we have

$m_{2,k}(\sigma_1,\sigma_2)\le\frac{4I_2(\sigma_1,\sigma_2)}{e^{k\sigma_1+k\sigma_2}}\int_0^{\sigma_1}\int_0^{\sigma_2}e^{kx_1}e^{kx_2}dx_1\,dx_2$

$=\frac{4I_2(\sigma_1,\sigma_2)}{e^{k\sigma_1+k\sigma_2}}\left[\left(\frac{e^{k\sigma_1}-1}{k}\right)\left(\frac{e^{k\sigma_2}-1}{k}\right)\right]$

$=\frac{4}{k^2}I_2(\sigma_1,\sigma_2)(1-e^{-k\sigma_1})(1-e^{-k\sigma_2})$

Taking limits on both sides we get

(4.2) $\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)}\le\frac{4}{k^2}$

Also, from Lemma 2, we have

(4.3) $I_2(\sigma_1,\sigma_2)\le\{M(\sigma_{11}\sigma_2)\}^2$

Therefore, from (4.2) and (4.3), it follows that

$\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{m_{2,k}(\sigma_1,\sigma_2)}{\{M(\sigma_1,\sigma_2)\}^2}\le\lim_{\sigma_1,\sigma_2\to\infty}\sup\frac{m_{2,k}(\sigma_1,\sigma_2)}{I_2(\sigma_1,\sigma_2)}\le\frac{4}{k^2}$

Theorem 2
For
$f(s_1,s_2), f\in F$, we have

(5.1) $\lim_{(\sigma_1,\sigma_2)\to\infty}\left\{\frac{1}{e^{k\sigma_1(1-\alpha_1)}e^{k\sigma_2(1-\alpha_2)}m_{2,k}(\sigma_1,\sigma_2)-m_{2,k}(\alpha_1\sigma_1,\alpha_2\sigma_2)}\right\}=0,$

Where $\alpha_1, \alpha_2 (0 < \alpha_1, \alpha_2 < 1)$ are constants. We first prove the following Lemma.

Lemma 3

Let $f(s_1, s_2)$ *be an entire function, then for*
$(0 < \sigma_1' < \bar{\sigma}_1 < \sigma_1)$ and $(0 < \sigma_2' < \bar{\sigma}_2 < \sigma_2)$

$$[I_2(\bar{\sigma}_1, \sigma_2') (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) \left(e^{k\bar{\sigma}_2} - e^{k\sigma_2'}\right) + I_2(\sigma_1', \overline{\sigma_2})\left(e^{k\overline{\sigma_1}} - e^{k\sigma_1'}\right) (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) + I_2(\bar{\sigma}_1, \bar{\sigma}_2) (e^{k\sigma_1} - e^{k\overline{\sigma_1}}) (e^{k\sigma_2} - e^{k\bar{\sigma}_2})\ ]$$

$$\leq [\ \left(\frac{k}{2}\right)^2 \{\ e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\overline{\sigma_1} + k\bar{\sigma}_2}\ m_{2,k}(\bar{\sigma}_1, \overline{\sigma_2})] \leq$$

$$[I_2(\sigma_1, \bar{\sigma}_2) (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) (e^{k\bar{\sigma}_2} - 1) + I_2(\bar{\sigma}_1, \sigma_2)(e^{k\overline{\sigma_1}} - 1) (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) + I_2(\sigma_1, \sigma_2) (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) (e^{k\sigma_2} - e^{k\bar{\sigma}_2})],$$

Where k is any positive number

Proof of Lemma 3

Since $I_2(x_1, x_2)$ is an increasing function of $x_1 and\ x_2$ and therefore from

(1.6), we have

$$e^{k\sigma_1 + k\sigma_2} m_{2,k}(\sigma_1, \sigma_2) - e^{k\overline{\sigma_1} + k\bar{\sigma}_2}\ m_{2,k}(\bar{\sigma}_1, \overline{\sigma_2})$$

$$= 4\int_{\bar{\sigma}_1}^{\sigma_1} \int_0^{\bar{\sigma}_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2 +$$

$$+ 4\int_0^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2 +$$

$$+ 4\int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2$$

Hence

(5.2) $e^{k\sigma_1 + k\sigma_2}\ m_{2,k}(\sigma_1, \sigma_2)$ - $e^{k\bar{\sigma}_1 + k\bar{\sigma}_2}\ m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)$

$$\leq \frac{4}{k^2} I_2(\sigma_1, \bar{\sigma}_2) (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) (e^{k\bar{\sigma}_2} - 1) + + \frac{4}{k^2} I_2(\bar{\sigma}_1, \sigma_2) (e^{k\bar{\sigma}_1} - 1) (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) +$$

$$+ \frac{4}{k^2} I_2(\sigma_1, \sigma_2) (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) (e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

Also

(5.3) $e^{k\sigma_1 + k\sigma_2}\ m_{2,k}\ (\sigma_1, \sigma_2)$- $e^{k\bar{\sigma}_1 + k\bar{\sigma}_2}\ m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)$

$$\geq 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\sigma_2'}^{\bar{\sigma}_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2 + + 4 \int_{\sigma_1'}^{\bar{\sigma}_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2 + + 4 \int_{\bar{\sigma}_1}^{\sigma_1} \int_{\bar{\sigma}_2}^{\sigma_2} I_2(x_1, x_2) e^{kx_1 + kx_2} dx_1\, dx_2$$

$$\geq \frac{4}{k^2}\ I_2(\bar{\sigma}_1, \sigma_2') (e^{k\sigma_1} - e^{k\bar{\sigma}_1}) (e^{k\bar{\sigma}_2} - e^{k\sigma_2'}) + + \frac{4}{k^2}\ I_2\ (\sigma_1', \bar{\sigma}_2) (e^{k\bar{\sigma}_1} - e^{k\sigma_1'}) (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) + + \frac{4}{k^2}\ I_2\ (\bar{\sigma}_1, \bar{\sigma}_2) (e^{k\sigma_1} - e^{k\overline{\sigma_1}}) (e^{k\sigma_2} - e^{k\bar{\sigma}_2})$$

Combining (5.2) and (5.3) we obtain the lemma

Proof of Theorem 4

If we put

$\bar{\sigma}_1 = \alpha_1\sigma_1, \sigma_1' = \beta_1\sigma_1$
$\bar{\sigma}_2 = \alpha_2\sigma_2, \sigma_2' = \beta_2\sigma_2$
Where $\beta_1 < \alpha_1, \beta_2 < \alpha_2$ in Lemma 1, we get

$$[4I_2(\alpha_1\sigma_1, \beta_2\sigma_2) (e^{k\sigma_1} - e^{k\alpha_1\sigma_1}) (e^{k\alpha_2\sigma_2} - e^{k\beta_2\sigma_2}) +$$

$+4\ I_2(\beta_1\sigma_1, \alpha_2\sigma_2)\ (e^{k\alpha_1\sigma_1} - e^{k\beta_1\sigma_1})\ (e^{k\sigma_2} - e^{k\alpha_2\sigma_2})+$
$+4\ I_2(\alpha_1\sigma_1, \alpha_2\sigma_2)\ (e^{k\sigma_1} - e^{k\alpha_1\sigma_1})\ (e^{k\sigma_2} - e^{k\alpha_2\sigma_2})]$

$\leq k^2[\ e^{k\sigma_1+k\sigma_2}\ m_{2,k}(\sigma_1,\sigma_2)$ - $e^{k\alpha_1\sigma_1+k\alpha_2\sigma_2}\ m_{2,k}(\alpha_1\sigma_1, \alpha_2\sigma_2)] \leq$
$[4\ I_2\ (\sigma_1, \alpha_2\sigma_2)\ (e^{k\sigma_1} - e^{k\alpha_1\sigma_1})\ (e^{k\alpha_2\sigma_2} - 1) +$
$+\ 4\ I_2(\square_1\sigma_1, \sigma_2)\ (e^{k\alpha_1\sigma_1} - 1)\ (e^{k\sigma_2} - e^{k\alpha_2\sigma_2})+$
$+4\ I_2\ (\sigma_1, \sigma_2)(e^{k\sigma_1} - e^{k\alpha_1\sigma_1})\ (e^{k\sigma_2} - e^{k\alpha_2\sigma_2})]$

Dividing by $e^{k\alpha_1\sigma_1}e^{k\alpha_2\sigma_2}$, we get

$[4\ I_2(\alpha_1\sigma_1, \beta_2\sigma_2)\ (e^{k\sigma_1(1-\alpha_1)} - 1)\ (1 - e^{k\sigma_2(\beta_2-\alpha_2)})$

$+4\ I_2(\beta_1\sigma_1, \alpha_2\sigma_2)\ (e^{k\sigma_2(1-\alpha_2)} - 1)\ (1 - e^{k\sigma_1(\beta_1-\alpha_1)}) +$
$+\ 4\ I_2(\alpha_1\sigma_1, \alpha_2\sigma_2)(e^{k\sigma_1(1-\alpha_1)} - 1)\ (e^{k\sigma_2(1-\alpha_2)} - 1)\ ]$

$\leq k^2[\ e^{k\sigma_1(1-\alpha_1)}e^{k\sigma_2(1-\alpha_2)}\ m_{2,k}(\sigma_1,\sigma_2)$ -$m_{2,k}(\alpha_1\sigma_1, \alpha_2\sigma_2)] \leq$

$[4\ I_2(\sigma_1, \alpha_2\sigma_2)\ \left(e^{k\sigma_1(1-\alpha_1)} - 1\right)(1 - e^{-k\alpha_2\sigma_2}) +$
$+4\ I_2\ (\alpha_1\sigma_1, \sigma_2)\ (e^{k\sigma_2(1-\alpha_2)} - 1)\ (1 - e^{-k\alpha_1\sigma_1}) +$
$+4\ I_2\ (\sigma_1, \sigma_2)\ \left(e^{k\sigma_1(1-\alpha_1)} - 1\right)(e^{k\sigma_2(1-\alpha_2)} - 1)]$

Taking limits on both the sides, Theorem 4 follows from the above inequalities.

Corollary
For $f(s_1, s_2), f \in F$ we have

$[\{\mu(\bar{\sigma}_1, \sigma_2')\}^2\ (e^{k\sigma_1} - e^{k\overline{\sigma_1}})\ (e^{k\bar{\sigma}_2} - e^{k\sigma_2'}) +$
$+\{\mu(\sigma_1', \bar{\sigma}_2)\}^2\ (e^{k\bar{\sigma}_1} - e^{k\sigma_1'})\ (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) +$
$+\ \{\mu(\bar{\sigma}_1, \bar{\sigma}_2)\}^2\ (e^{k\sigma_1} - e^{k\bar{\sigma}_1})\ (e^{k\sigma_2} - e^{k\bar{\sigma}_2})]$
$\leq [\left(\frac{k}{2}\right)^2 \{e^{k\sigma_1+k\sigma_2}\ m_{2,k}(\sigma_1,\sigma_2)$- $e^{k\bar{\sigma}_1+k\bar{\sigma}_2}\ m_{2,k}(\bar{\sigma}_1, \bar{\sigma}_2)\quad\}] \leq$
$[\{M(\sigma_1, \bar{\sigma}_2)\}^2(e^{k\sigma_1} - e^{k\overline{\sigma_1}})\ (e^{k\bar{\sigma}_2} - 1) +$
$+\{M(\bar{\sigma}_1, \sigma_2)\}^2\ (e^{k\bar{\sigma}_1} - 1)\ (e^{k\sigma_2} - e^{k\bar{\sigma}_2}) +$
$+\ \{M(\sigma_{11}\sigma_2)\}^2\ (e^{k\sigma_1} - e^{k\bar{\sigma}_1})\ (e^{k\sigma_2} - e^{k\bar{\sigma}_2})]$

The result follows by using Lemma 2 and Lemma 3.

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