

Linear-exponential distribution

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Abstract

In this paper, we introduce a new member of a family of continuous probability distribution which is based on the product of linear and exponential functions and hence, we named it 'Linear-Exponential distribution'. Probability density function, probability distribution function and moment generating function of this distribution have been obtained. Moments about origin and hence, the first four central moments of the proposed distribution have been derived. The hazard rate function and the mean residual life function of the proposed distribution have been discussed, and show its flexibility over Lindley distribution. Estimation of parameters has been discussed by the method of moments as well as the method of maximum likelihood. Goodness of fit has been applied to different data-sets which were used by others and it has been observed that it gives better fit to the most of data-sets of same nature, having variance greater than the mean, than Lindley distribution.

Keywords: Quadratic-exponential distribution, lindley distribution, parameters, moments, goodness of fit, estimation

Introduction

The main objective of introducing Linear-Exponential distribution (LED) with a single parameter is to give a better alternative of Lindley (1958)^[2] distribution and to create another plate-form for researcher working on probability distribution theory. In the last two decades, many researchers have published different types of Quasi-Lindley distributions having two or more than two parameters which are generalised form of Lindley distribution (1958)^[2]. Lindley introduced a one parameter continuous probability distribution, known as Lindley distribution (LD), given by its probability density function

$$f(x) = \frac{\phi^2(1+x)e^{-\phi x}}{(1+\phi)}; x > 0, \phi > 0 \quad (1)$$

The r^{th} moment about origin of the Lindley distribution (1) was given by

$$\mu'_r = \frac{r!(1+r+\phi)}{\phi^r(1+\phi)} \quad (2)$$

There are very few probability distributions of single parameter which give better fit to the same nature of data-sets having variance greater than the mean. Sah (2015)^[4] introduced a one parameter continuous probability distribution, known as Mishra distribution (MD), given by its probability density function

$$f_2(x; \phi) = \frac{\phi^3}{(\phi^2 + \phi + 2)}(1+x+x^2)e^{-\phi x}; x >, \phi > 0 \quad (3)$$

One of them is Mishra distribution of Sah (2015)^[4] which gives better fit to the same nature of data-sets, having variance greater than the mean, than Lindley distribution (1958)^[2]. The r^{th} moment about origin of this distribution was obtained by

$$\mu'_r = \frac{r! [\phi^2 + (r+1)\phi + (1+r)(2+r)]}{\phi^r (\phi^2 + \phi + 2)} \quad (4)$$

The proposed distribution is a modified form of Lindley distribution replacing '1' by ' ϕ ' in the expression $(1+x)$ of the equation (1). The proposed distribution is based on the product of linear function $(\phi+x)$ and exponential function $e^{-\phi x}$ and hence, it is named as Linear-Exponential distribution (LED). It may be called modified Lindley distribution (MLD). The proposed distribution is so constructed that it follows all the basic properties of probability distribution. The different characteristics such as probability density function, probability distribution function, and moment generating function, moments about origin as well as moments about mean of the proposed distribution have been obtained. The hazard rate function and the mean residual life function of the proposed distribution have been discussed. The estimation of parameters has been discussed by the method of moments as well as the maximum likelihood methods. The proposed distribution has been fitted to some well-known data-sets which were earlier used by others and it is expected to give better alternative to the same nature of data-sets than Lindley (1958)^[2] distribution (LD).

Material and Methods

The proposed distribution is based on theoretical concept of probability distribution. Probability density function of Linear-Exponential distribution has been constructed so that it follows all the basic properties of probability distribution. Probability distribution function of LED has constructed to calculate probability of the variable under study for each interval. Estimate of the parameter has been discussed by the methods of moments as well as the maximum likelihood methods. To test validity of the theoretical work, goodness of fit has been applied to some data-sets which were earlier used by others.

Results

Linear-Exponential Distribution (LED)

The nice feature of the proposed distribution, LED, is that it has a single parameter which is to be proposed as a better alternative to LD (1). It has been constructed on the basis of the product of quadratic and exponential functions. LED with parameter ϕ is defined by its probability density function as

$$f(x; \phi) = \frac{\phi^2}{(1+\phi^2)} (\phi+x) e^{-\phi x} \quad (5)$$

Where $x > 0$, $\phi > 0$

Probability distribution function of this distribution has been obtained as

$$F(x) = P(X \leq x) = \int_0^x f(x) dx = \frac{\phi^2}{(1+\phi^2)} \int_0^x (\phi+x) e^{-\phi x} dx = 1 - \frac{(1+\phi x + \phi^2)}{(1+\phi^2)} e^{-\phi x} \quad (6)$$

Moment generating function (M.G.F.) of the LED has been obtained by

$$M_X(t) = E[e^{tx}] = \int_0^{\infty} e^{tx} f(x) dx = \frac{\phi^2}{(1+\phi^2)} \int_0^{\infty} (\phi+x) e^{-(\phi-t)x} dx = \frac{\phi^2}{(1+\phi^2)} \frac{[\phi(\phi-t)+1]}{(\phi-t)^2} \quad (7)$$

The expression (7) is the M.G.F. of LED (5).

Moments and Related Measures of LED

It is necessary to obtain statistical moments of the proposed distribution to estimate parameter of LED (5) by the method of moments. The r^{th} moment about origin of the LED (5) can be obtained as

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f(x) dx = \frac{\phi^2}{(1+\phi^2)} \int_0^{\infty} x^r (\phi+x) e^{-\phi x} dx = \frac{\phi^2}{(1+\phi^2)} \frac{\Gamma(r+1)}{\phi^r} \frac{(\phi^2+r+1)}{\phi^2} = \frac{1}{(1+\phi^2)} \frac{r!}{\phi^r} (\phi^2+r+1) \quad (8)$$

The expression (8) is the general form of the r^{th} moment about origin of the LED (5). Putting $r = 1, 2, 3$ and 4 in the expression (8), we get the first four moments about origin as

$$\mu'_1 = \frac{1! (2+\phi^2)}{\phi (1+\phi^2)} \quad (9)$$

$$\mu'_2 = \frac{2! (3 + \phi^2)}{\phi^2 (1 + \phi^2)} \quad (10)$$

$$\mu'_3 = \frac{3! (4 + \phi^2)}{\phi^3 (1 + \phi^2)} \quad (11)$$

$$\mu'_4 = \frac{4! (5 + \phi^2)}{\phi^4 (1 + \phi^2)} \quad (12)$$

To know about shape and size of this distribution, it is necessary to obtain the first four moments about the mean of LED (5) have been obtained as

$$\mu_1 = 0 \quad \mu_2 = \mu'_2 - \mu'^2 = \frac{2! (3 + \phi^2)}{\phi^2 (1 + \phi^2)} - \left[\frac{1! (2 + \phi^2)}{\phi (1 + \phi^2)} \right]^2 = \frac{(\phi^4 + 4\phi^2 + 2)}{\{\phi(1 + \phi^2)\}^2} \quad (13)$$

$$\mu_3 = \mu'_3 - 3\mu'_2 \mu'_1 + 2(\mu'_1)^3 = - \left[\frac{(68 + 120\phi^2 + 60\phi^4 + 10\phi^6)}{\{\phi(1 + \phi^2)\}^3} \right] < 0 \quad (14)$$

$$\mu_4 = \mu'_4 - 4\mu'_3 \mu'_1 + 6\mu'_2 (\mu'_1)^2 - 3(\mu'_1)^4 = \frac{(9\phi^8 + 72\phi^6 + 132\phi^4 - 48\phi^2 - 6)}{\{\phi(1 + \phi^2)\}^4} \quad (15)$$

We can observe that variance (13) is greater than the mean (9). Hence, LED (5) is always over-dispersed. Shape and size of the proposed probability distribution can be studied by obtaining co-efficient of skewness and kurtosis

$$\gamma_1 = \pm \frac{[-(68 + 120\phi^2 + 60\phi^4 + 10\phi^6)]}{[(\phi^4 + 4\phi^2 + 2)]^{3/2}} \quad (16)$$

And co-efficient of kurtosis can be obtained by using

$$\beta_2 = \frac{(9\phi^8 + 72\phi^6 + 132\phi^4 - 48\phi^2 - 6)}{(\phi^4 + 4\phi^2 + 2)^2} \quad (17)$$

The Reliability Function, Hazard Rate Function and Mean Residual Life Function of LED

The Reliability Function

Let X follows LED (5) with parameter ϕ and probability density function $f(x)$. Distribution function of X is defined as

$$F(x) = 1 - \frac{(1 + \phi x + \phi^2)}{(1 + \phi^2)} e^{-\phi x} = \frac{(1 + \phi^2) - (1 + \phi x + \phi^2)e^{-\phi x}}{(1 + \phi^2)} \quad 1 - F(x) = \frac{(1 + \phi x + \phi^2)}{(1 + \phi^2)} e^{-\phi x}$$

Let the random variable X be the lifetime or the time to failure of a component. The probability that the component will survive until sometime 't' is called reliability R(t) of the component defined by

$$R(t) = P(X > t) = \int_t^{\infty} f(x) dx = 1 - F(t) \\ = \frac{(1 + \phi t + \phi^2)}{(1 + \phi^2)} e^{-\phi t} \quad (18)$$

The expression (18) is the reliability function of LED (5). The component is said to be working properly at time t=0 and no component work forever without failure i.e.

$$R(t = 0) = 1 \quad \text{and} \quad \lim_{t \rightarrow 0} R(t) = 0$$

$R(t)$ is a monotone non-increasing function of t . For $t < 0$, the reliability has no meaning.

The Hazard Rate Function

Hazard measures the conditional probability of a failure given that the system is working. The failure density (pdf) measures the overall speed of failures. Hazard rate or Instantaneous failure rate measures the dynamic speed of failures. For a continuous distribution with p.d.f. $f(x)$ and c.d.f. $F(x)$, the hazard rate function (also known as failure rate function) and the mean residual life function are respectively defined as

$$h(x) = \lim_{\Delta x \rightarrow 0} \frac{P(X < x + \Delta x / X > x)}{\Delta x} = \frac{f(x)}{1 - F(x)} \quad (19)$$

The main reason behind defining the hazard rate function [$h(x)$] is that it is more convenient to work with than the density function [$f(x)$]. Putting the value of $f(x)$ and $1-F(x)$ in the expression $h(x)$, the Hazard rate function of LED (5) has been obtained as

$$h(x) = \frac{\phi^2(\phi + x)}{(1 + \phi x + \phi^2)} \quad (20)$$

The failure rate function of LED (5) until time 't' is thus obtained as

$$h(x = t) = \frac{\phi^2(\phi + t)}{(1 + \phi t + \phi^2)} \quad (21)$$

$$\text{At } t = 0, h(x = t = 0) = \frac{\phi^3}{(1 + \phi^2)} \quad (22)$$

It is also obvious that $h(x)$ is an increasing function of x and ϕ .

Mean Residual Life Function

In reliability studies, the expected additional life time given that a component has survived until time 't' is called mean residual life time. Let a random variable X denotes the life of a component, the mean residual life function is given by

$$m(x) = E[X - x / X > x] = \frac{\int_x^{\infty} [1 - F(t)] dt}{1 - F(x)} \quad (23)$$

To obtain $m(x)$ of the LED (5), we have to calculate the following measures

$$f(x; \phi) = \frac{\phi^2}{(1 + \phi^2)} (\phi + x) e^{-\phi x}$$

Where $x > 0$, $\phi > 0$

$$F(x) = 1 - \frac{(1 + \phi x + \phi^2)}{(1 + \phi^2)} e^{-\phi x} = \frac{(1 + \phi^2) - (1 + \phi x + \phi^2) e^{-\phi x}}{(1 + \phi^2)}$$

$$1 - F(x) = \frac{(1 + \phi x + \phi^2)}{(1 + \phi^2)} e^{-\phi x}$$

$$1 - F(x = t) = \frac{(1 + \phi t + \phi^2)}{(1 + \phi^2)} e^{-\phi t}$$

Where $f(x)$ and $F(x)$ are the probability density function and probability distribution function of LED (5).

$$\int_x^\infty \{1 - F(t)\} dt = \int_x^\infty \left[\frac{(1 + \phi t + \phi^2)}{(1 + \phi^2)} e^{-\phi t} \right] dt$$

$$= \frac{(2 + \phi x + \phi^2) e^{-\phi x}}{\phi(1 + \phi^2)} \tag{24}$$

Putting the value of $\int_x^\infty [1 - F(t)] dt$ and $1 - F(x)$ in equation (23), the mean residual life function has been obtained as

$$m(x) = \frac{(2 + \phi x + \phi^2)}{\phi(1 + \phi x + \phi^2)} \tag{25}$$

$$\text{At } x = 0, m(x = 0) = \frac{(2 + \phi^2)}{\phi(1 + \phi^2)} = \mu'_1 \tag{26}$$

It can also be seen that at $x = 0$, the mean residual life function is the mean of LED (5). It can also be seen that $m(x)$ is a decreasing function of x, ϕ . The hazard rate function and the mean residual life function of the LED show its flexibility over Lindley distribution.

Stochastic Orderings

Stochastic ordering of non-negative continuous random variables is an important tool for judging the comparative behavior. Let us consider two random variables X and Y. The random variable X is said to be smaller than Y in the

- a) Stochastic order ($X \leq_{st} Y$) if $F_X(x) \geq F_Y(x)$ for all x
- b) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x
- c) Mean residual life order ($X \leq_{mrl} Y$) if $m_X(x) \geq m_Y(x)$ for all x
- d) Likelihood ratio order ($X \leq_{lr} Y$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x

The following results due to Shaked and Shanthi Kumar (1994)^[5] establishing stochastic ordering of the distributions

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y)$$

$$\Downarrow$$

$$(X \leq_{st} Y)$$

The LED (5) is ordered with respect to the strongest ‘likelihood ratio’ ordering as shown in the following theorem

Theorem: Let $X \sim \text{LED}(\phi_1)$ and $Y \sim \text{LED}(\phi_2)$. If $\phi_1 \geq \phi_2$, then $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y), (X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\phi_1^2(1 + \phi_2^2)(\phi_1 + x)}{\phi_2^2(1 + \phi_1^2)(\phi_2 + x)} e^{-(\phi_1 - \phi_2)x} ; x > 0 \tag{27}$$

Taking log on both sides of preceding equation, we get

$$\log \left\{ \frac{f_X(x)}{f_Y(x)} \right\} = \log \left\{ \frac{\phi_1^2(1 + \phi_2^2)}{\phi_2^2(1 + \phi_1^2)} \right\} + \log \{(\phi_1 + x)\} - \log \{(\phi_2 + x)\} - (\phi_1 - \phi_2)x \tag{28}$$

D.W.R.T. x of equation (28), we get

$$\frac{d}{dx} \log \left\{ \frac{f_X(x)}{f_Y(x)} \right\} = \frac{1}{(\phi_1 + x)} - \frac{1}{(\phi_2 + x)} - (\phi_1 - \phi_2) = \frac{(\phi_2 - \phi_1)}{(\phi_1 + x)(\phi_2 + x)} - (\phi_1 - \phi_2) \tag{29}$$

If $\phi_1 \geq \phi_2$, then $\frac{d}{dx} \log \left\{ \frac{f_X(x)}{f_Y(x)} \right\} \leq 0$. It indicates that $(X \leq_{lr} Y)$ and hence $(X \leq_{hr} Y)$, $(X \leq_{mrl} Y)$ and $(X \leq_{st} Y)$

This theorem shows that the flexibility of the QED over Lindley and exponential distributions.

Estimation of Parameter of LED

LED (5) has only one parameter ' ϕ '. There are different methods of estimation of parameters in the theory of statistical inference. Here, estimate of the parameter ' ϕ ' has been obtained by using (a) Method of moments and (b) Method of maximum likelihood.

a) Method of moments: The proposed distribution has a single parameter ' ϕ '. To estimate value of the parameter, the first moment about origin is required. So, the population mean is replaced by respective sample mean and using the expression (9), we can obtain estimate of ' ϕ ' as follows. From the expression (9), we have

$$\mu'_1 = \frac{1!(2 + \phi^2)}{\phi(1 + \phi^2)}$$

After a little simplification, we get

$$f(\phi) = \mu'_1(\phi + \phi^3) - (2 + \phi^2) = 0 \quad (30)$$

The expression (30) is the polynomial in fourth degree equation. Replacing the corresponding population moment by respective sample moment and solving the expression (30) by using Regula-Falsi method, we get an estimate of ' ϕ '.

b) The method of maximum likelihood

Let (x_1, x_2, \dots, x_n) be a random sample of size n from LED (5) and let f_x be the observed frequency in the sample corresponding to $X = x$ ($x = 1, 2, \dots, k$) such that $\sum_{x=1}^k f_x = n$, where k is the largest observed value having non-zero frequency.

The likelihood function, L , of the LED (5) is obtained as

$$f(x; \phi) = \frac{\phi^2}{(1 + \phi^2)} (\phi + x) e^{-\phi x} \quad L = \left(\frac{\phi^2}{1 + \phi^2} \right)^n \left[\prod_{x=1}^k (\phi + x)^{f_x} \right] e^{-n\phi\bar{x}} \quad (31)$$

and so, the log likelihood function is obtained as

$$\ln L = 2n \ln \phi - n \ln(1 + \phi^2) + \sum_{x=1}^k f_x \ln(\phi + x) - n\phi\bar{x} \quad (32)$$

The log likelihood equation is thus obtained as

$$\frac{\partial \ln L}{\partial \phi} = \frac{2n}{\phi} - \frac{2n\phi}{(1 + \phi^2)} + \sum_{x=1}^k \frac{f_x}{(\phi + x)} - n\bar{x} = 0 \quad (33)$$

We can observe that the equation (33) does not seem to be solved directly. However, Fisher's scoring method can be applied to solve this equation.

Goodness of Fit and discussion

The LED has been fitted to a number of data-sets to which earlier the Lindley distribution have been fitted by others and to almost all these data-sets this distribution provides closer fits than the Lindley distribution. The fittings of the LED to the two such data-sets have been presented in the following tables. The first data-set is regarding the survival times (in days) of guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960)^[1], and the second data-set is regarding mortality grouped data for blackbirds species, reported by Paranjpe and Rajarshi (1986)^[3]. The expected frequencies according to the Lindley distribution have also been given for ready comparison with those obtained by the LED. These data-sets have also been used Shanker, R and Mishra, A. (2013b) in the paper entitled 'A two-parameter Lindley distribution'.

Table 1: Survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli

Survival Time (in days)	Observed frequency	Expected Frequency	
		Lindley	LED
0 – 80	8	16.1	15.9
80 – 160	30	21.9	22.0
160 – 240	18	15.4	15.5
240 – 320	8	9.0	9.0
320 – 400	4	5.5	4.8
400 – 480	3	1.8	2.5
480 -560	1	2.3	2.3
Total	72	72.0	72.0
$\mu'_1 \mu'_2 \hat{\phi} \chi^2 (df)$	181.11111 43911.11111	0.011 7.77(3)	0.0110394 7.61(3)

Table 2: Mortality grouped data for blackbird’s species reporte by Paranjpe and Rajarsi (1986)

Survival Time (In Days)	Observed frequency	Expected frequency	
		Lindley	LED
0-1	192	173.5	151.3
1-2	60	98.6	99.0
2-3	50	46.5	53.4
3-4	20	20.1	26.3
4-5	12	8.1	12.3
5-6	7	3.2	5.5
6-7	6	1.4	2.4
7-8	3	0.4	1.1
>8	2	0.3	0.9
Total	352	352.0	352.0
$\mu'_1 \mu'_2 \hat{\phi} \chi^2 (df)$	1.568181 5.005682	.984 49.85(4)	0.967163125 34.67(5)

In the above tables, the expected frequencies according to the Lindley distribution have also been given for ready comparison with those obtained by the LED. From table (I), we can observe that the value of Chi-square of LED is less than the Lindley distribution with same degrees of freedom. From table (II), we can observe that the value of Chi-square of LED is also less than Lindley distribution with greater degrees of freedom. Hence, it may conclude that in most of the cases LED gives better fit to the same nature of data-sets, having variance greater than the mean, than Lindley distribution.

5. Conclusion

In this paper, we propose a single parameter continuous distribution which is named as Linear-Exponential distribution (LED). Several structural properties such as moment generating function, distribution function, moments about origin as well as mean have been derived. The reliability function, hazard rate function and mean residual life function have been obtained and discussed. The methods of estimation of parameter have been discussed. Finally, the proposed distribution has been fitted to a number of data-sets, having variance greater than mean, to test its goodness of fit and it has been observed that the LED (5) gives better fit to the most of the similar nature of the data-sets, having variance greater than the mean, than the Lindley distribution (1).

Conflict of Interest

The authors declared that there is no conflict of interest.

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