International Journal of Statistics and Applied Mathematics 2021; 6(5): 25-33

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2021; 6(5): 25-33

www.mathsjournal.com
Received: 18-06-2021
Accepted: 30-07-2021

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Stability analysis of sis delay models for venereal diseases with symptomatic and asymptomatic infective

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Abstract
We have proposed and analyzed the two SIS delay models for venereal diseases with susceptible, asymptomatic, and symptomatic infectives. In the first model, we have considered that asymptomatic and symptomatic have the same infection rate, and later model, symptomatic infectives are treated and have a lower infection rate than asymptomatic infectives. In these models, the asymptomatic infectives stay in the latent period to become infectious, which we considered a delay $h > 0$. The basic reproduction number is derived through the next-generation method. The endemic and disease-free equilibriums are proved to be locally stable through the Lyapunov method. The numerical simulation also confirms our theoretical findings.

Keywords: Symptomatic, latent period, asymptomatic, delay, stability, venereal diseases

Introduction

Raising cases of venereal disease is a measure of health concern in the world [1]. Venereal diseases are communicable diseases contracted and transmitted by sexual contact via semen, vaginal secretion, or blood during intercourse or using personal items such as the person's towel and clothing contracted with venereal disease [2]. In these diseases, there is a crisscross type of infection in heterosexual contact between males and females. Gonorrhea, syphilis, AIDS are some examples of venereal diseases. The highest incidence rate of venereal disease is observed in 15-29 age groups. Venereal diseases induced little or no acquired immunity following infection.

H.E. Wichmann [3] described a threshold for the ratio of removal rates and infection rates. The system will be disease-free if this ratio is large enough, and an epidemic will occur for the small values.

Many venereal diseases show an asymptomatic infective class with no overt symptom until quite late in developing the infection [4]. Gonorrhea and syphilis [5, 6] are examples of the epidemic with a latent period of becoming infectious. So, it is realistic to assume a time lag h > 0 during which infectious agents develop in the vector the latter become infectious or say infected host stay in the latent period.

This paper will study two different models of venereal diseases having two types of infective (Symptomatic and asymptomatic infective), which have a latent period for subclinical infection to become infectious.

Mathematical Models

Model-I [Where Symptomatic Infective are untreated] $\frac{dX}{dt} = -\beta X\{Y'(t-h) + W'\} + \gamma Y + \eta W, \frac{dX'}{dt} = -\beta' X'\{Y(t-h) + W\} + \gamma' Y' + \eta' W'$ $\frac{dY}{dt} = \beta X\{Y'(t-h) + W'\} - \gamma Y - \xi Y, \frac{dY'}{dt} = \beta' X'^{\{Y(t-h)+W\}} - \gamma' Y' - \xi' Y'$ ……(1)

$$\frac{dW}{dt} = \xi Y - \eta W \quad \frac{dW'}{dt} = \xi' Y' - \eta' W'$$

In the presented model $X(t)$and $X'(t)$ stands for male and female susceptible, $Y(t)$ and $Y'(t)$ stands for male and female asymptomatic infective, $W(t)$and $W'(t)$ are male and female symptomatic infective β *and* β' *are* average number of male and female

susceptible infective per day, γ and γ' denotes recovery rates for male and female asymptomatic infective per day, η and η' represents recovery rates for male and female symptomatic infective and finally ξ and ξ' stands for transfer rates from asymptomatic class to the symptomatic class. Finally, h is the latent period for subclinical infection. We consider the total population of male and female is constant, i.e., $X + Y + W = N, X' + Y' + W' = N'$,
Therefore,

$$\frac{dY}{dt} = \beta X\{Y'(t-h) + W'\} - \gamma Y - \xi Y, \frac{dY'}{dt} = \beta' X'\{Y(t-h) + W\} - \gamma' Y' - \xi' Y' \dots\dots\dots(2)$$
$$\frac{dW}{dt} = \xi Y - \eta W, \frac{dW'}{dt} = \xi' Y' - \eta' W2$$

Equilibrium classification
The present model has following equilibriums,
1. Disease-free equilibrium $(Y_0, Y_0', W_0, W_0') = (0,0,0,0)$
2. Endemic equilibrium $(Y_1, Y_1', W_1, W_1') = \left\{\frac{\alpha\varepsilon}{N'+\alpha\rho}, \frac{\alpha'\varepsilon}{N+\alpha'\rho'}, (1-\alpha)\frac{\varepsilon}{N'+\alpha\rho}, (1-\alpha')\frac{\varepsilon}{N+\alpha'\rho'}\right\}$

Noting that $X_1 = N - Y_1 - W_1$, $X_1' = N' - Y_1' - W_1'$,$X_0 = N - Y_0 - W_0$ and $X_0' = N' - Y_0' - W_0'$ Here, $\alpha = \frac{\eta}{\xi+\eta}$, $\alpha' = \frac{\eta'}{\xi'+\eta'}$ are proportion of infective that are asymptomatic and $\rho = \frac{\xi+\gamma}{\beta}$, $\rho' = \frac{\xi'+\gamma'}{\beta'}$ are total removal rates for asymptomatic infective and $\varepsilon = NN' - \alpha\alpha'\rho\rho'$.

Basic reproduction number
The reproduction number derived by next generation matrix approach given by,

$$R_0 = \frac{\beta'(N'-Y'-W')(\eta'+\xi')(\gamma+\xi)\eta e^{-\lambda h} - \beta(Y'+W')(\eta+\xi)(\gamma'+\xi')\eta' + \sqrt{\begin{array}{l}2\beta\beta'(Y'+W')(N'-Y'-W')(\gamma+\xi)(\gamma'+\xi')(\eta+\xi)(\eta'+\xi')\eta\eta' e^{-\lambda h} + \\ \{\beta(Y'+W')(\eta+\xi)(\gamma'+\xi')\eta'\}^2 + \{\beta'(N'-Y'-W')(\eta'+\xi')(\gamma+\xi)\eta e^{-\lambda h}\}^2 - \\ 4\beta\beta'(\eta+\xi)(\eta'+\xi')(\gamma+\xi)(\gamma'+\xi')\eta\eta'(Y+W)(N-Y-W)\end{array}}}{2(\gamma+\xi)(\gamma'+\xi')\eta\eta'} \dots\dots(3)$$

Linear Stability Analysis
Theorem 1: The Endemic equilibrium is locally asymptotically stable if following conditions holds.
a. $-[\{-a_{11}(2 + a_{12}) + 2a_{12} + a_{13}\} + b_{12}\{1 - (b_{11} + b_{12} + b_{13})h\} + d_{11}] > 0.$
b. $-[\{-b_{11}(2 + b_{12}) + 2b_{12} + b_{13}\} - a_{12}\{1 - (a_{11} + a_{12} + a_{13})h\} + c_{11}] > 0.$
c. $a_{13}(1 + a_{13}h) - b_{12}(1 + b_{12}h) - (c_{11} - 2c_{12}) > 0.$
d. $b_{13}(1 + b_{13}h) - a_{12}(1 + a_{12}h) - (d_{11} - 2d_{12}) > 0.$

Proof: We use transformation $U_1 = Y - Y_1$, $U_2 = Y' - Y_1'$, $U_3 = W - W_1$, $U_4 = W' - W_1'$ in system (2), we have

$$\frac{dU_1}{dt} = \beta\{N - (U_1 + Y_1) - (U_3 + W_1)\}\{Y_1' + U_2(t-h) + U_4 + W_1'\} - \gamma(U_1 + Y_1) - \xi(U_1 + Y_1)$$
$$\frac{dU_2}{dt} = \beta'\{N' - (U_2 + Y_1') - (U_4 + W_1')\}\{Y_1 + U_1(t-h) + U_3 + W_1\} - \gamma'(U_2 + Y_1') - \xi'(U_2 + Y_1')$$
$$\frac{dU_3}{dt} = \xi(U_1 + Y_1) - \eta(U_3 + W_1), \frac{dU_4}{dt} = \xi'(U_2 + Y_1') - \eta'(U_4 + W_1').$$

Now, consider the system for the endemic equilibrium and linearizing, we have,

$$\frac{dU_1}{dt} = -[\beta(Y_1' + W_1') + \gamma + \xi]U_1 + \beta X_1 U_2(t-h) - \beta(Y_1' + W_1')U_3(t) + \beta X_1 U_4(t),$$
$$\frac{dU_2}{dt} = -[\beta'(Y_1 + W_1) + \gamma' + \xi']U_2 + \beta' X_1' U_1(t-h) + \beta' X_1' U_3(t) - \beta'(Y_1 + W_1)U_4(t), \dots\dots(4)$$
$$\frac{dU_3}{dt} = \xi U_1(t) - \eta U_3(t), \frac{dU_4}{dt} = \xi'^{U_2}(t) - \eta'^{U_4}(t).$$

Now, in (4) we assume that $a_{11} = [\beta(Y_1' + W_1') + \gamma + \xi]$, $a_{12} = \beta X_1, a_{13} = \beta(Y_1' + W_1'), b_{11} = [\beta'(Y_1 + W_1) + \gamma' + \xi']$, $b_{12} = \beta' X_1'$, $b_{13} = \beta'(Y_1 + W_1), c_{11} = \xi, c_{12} = \eta, d_{11} = \xi', d_{12} = \eta'$.
Putting these values in the system given by (4),

$$\frac{dU_1}{dt} = -a_{11}U_1 + a_{12}U_2(t-h) - a_{13}U_3(t) + a_{12}U_4(t),$$
$$\frac{dU_2}{dt} = -b_{11}U_2 + b_{12}U_1(t-h) + b_{12}U_3(t) - b_{13}U_4(t),$$
$$\frac{dU_3}{dt} = c_{11}U_1(t) - c_{12}U_3(t), \frac{dU_4(t)}{dt} = d_{11}U_2(t) - d_{12}U_4(t).$$

We can first equation of the system in the following form,

$$\frac{d}{dt}\left[U_1(t)+\int_{t-h}^{t}\beta X_1 U_2(s)ds\right]=-a_{11}U_1(t)-a_{13}U_3(t)+a_{12}U_2(t)+a_{12}U_4.\dots\dots(5)$$

Let $U_{11}(t)=\left[U_1(t)+\int_{t-h}^{t}\beta X_1 U_2(s)ds\right]^2$……(6) Differentiating (5) and putting values from (5) and (6),

$$\frac{dU_{11}}{dt}=2\left[U_1(t)+\int_{t-h}^{t}a_{12}U_2(s)ds\right][-a_{11}U_1(t)-a_{13}U_3(t)+a_{12}\{U_4(t)+U_2(t)\}],$$

$$=2[-a_{11}U_1^2(t)-a_{11}a_{12}U_1(t)\int_{t-h}^{t}U_2(s)ds-a_{13}U_1(t)U_3(t)-a_{13}U_3(t)\int_{t-h}^{t}a_{12}U_2(s)ds+$$

$$a_{12}\{U_4(t)+U_2(t)\}U_1(t)+(a_{12})^2\{U_4(t)+U_2(t)\}\int_{t-h}^{t}U_2(s)ds].$$

Using inequality $a^2+b^2\geq 2ab$,

$$\frac{dU_{11}(t)}{dt}\leq -a_{11}U_1^2(t)-\frac{a_{11}a_{12}}{2}\int_{t-h}^{t}\{U_1^2(t)+U_2^2(s)\}ds-\frac{a_{13}}{2}\{U_1^2(t)+U_3^2(t)\}-\frac{a_{12}a_{13}}{2}\int_{t-h}^{t}\{U_3^2(t)+U_2^2(s)\}ds$$

$$\dots\dots\dots.+\frac{a_{12}}{2}\{U_4^2(t)+2U_1^2(t)+U_2^2(t)\}+\frac{(a_{12})^2}{2}\int_{t-h}^{t}\{U_2^2(s)+U_4^2(t)\}ds+\frac{(a_{12})^2}{2}\int_{t-h}^{t}\{U_2^2(s)+U_2^2(t)\}ds$$

$$=[2(-a_{11}+a_{12})+a_{13}-a_{11}a_{12}h]U_1^2(t)+\{a_{12}+(a_{12})^2h\}U_2^2(t)-\{a_{13}+a_{13}a_{12}h\}U_3^2(t)+$$

$$\dots\dots\dots\dots.\{a_{12}+(a_{12})^2h\}U_4^2(t)-\{a_{12}a_{11}+a_{13}a_{12}+2(a_{12})^2\}\int_{t-h}^{t}U_2^2(s)ds.$$

We define a Lyapunov functional,

$$V_1(t)=U_{11}(t)+U_{12}(t)\dots\dots.(7)$$

Where $U_{12}(t)=-\int_{t-h}^{t}\int_{s}^{t}\{a_{11}a_{12}+a_{13}a_{12}+2(a_{12})^2\}U_2^2(l)dl\,ds$ and

$$\frac{dU_{12}}{dt}=-a_{12}\{a_{11}+a_{13}+2a_{12}\}hU_2^2(t)+a_{12}\{a_{11}+a_{13}+2a_{12}\}\int_{t-h}^{t}U_2^2(s)ds$$

From (7) we have,

$$\frac{dV_1(t)}{dt}\leq\{-a_{11}(2+a_{12})+2a_{12}+a_{13}\}U_1^2(t)+a_{12}\{1-(a_{11}+a_{12}+a_{13})h\}U_2^2(t)-a_{13}(1+a_{13}h)U_3^2(t)$$
$$+a_{12}(1+a_{12}h)U_4^2(t).$$

Similarly for the remaining equations of (5) we have,

$$\frac{dV_2(t)}{dt}\leq\{-b_{11}(2+b_{12}h)+2b_{12}+b_{13}\}U_2^2(t)+b_{12}\{1-(b_{11}+b_{12}+b_{13})h\}U_1^2(t)-b_{13}(1+b_{13}h)U_4^2(t)$$
$$+b_{12}(1+b_{12}h)U_3^2(t).$$

$$\frac{dV_3(t)}{dt}=(c_{11}-2c_{12})U_3(t)+c_{11}U_1^2(t),\frac{dV_4(t)}{dt}=(d_{11}-2d_{12})U_4^2(t)+d_{11}U_2^2(t).$$

We define the Lyapunov functional for the given system,

$$V(t)=V_1(t)+V_2(t)+V_3(t)+V_4(t)\Rightarrow\frac{dV(t)}{dt}\leq-\{A_1U_1^2(t)+A_2U_2^2(t)+A_3U_3^3(t)+A_4U_4^2(t)\}.$$

Here,$A_1=-[\{-a_{11}(2+a_{12})+2a_{12}+a_{13}\}+b_{12}\{1-(b_{11}+b_{12}+b_{13})h\}+d_{11}]$, $A_2=-[\{-b_{11}(2+b_{12})+2b_{12}+b_{13}\}-a_{12}\{1-(a_{11}+a_{12}+a_{13})h\}+c_{11}]$, $A_3=a_{13}(1+a_{13}h)-b_{12}(1+b_{12}h)-(c_{11}-2c_{12})$, $A_4=b_{13}(1+b_{13}h)-a_{12}(1+a_{12}h)-(d_{11}-2d_{12})$.

Denote $A=min\{A_1,A_2,A_3,A_4\}$. Then, $V(t)+A\int_{T^*}^{t}[U_1^2(s)+U_2^2(s)+U_3^2(s)+U_4^2(s)]ds\leq V(T^*)$ for $t\geq T^*$. Therefore $\lim_{t\to+\infty}\sup\int_{T^*}^{t}[U_1^2(s)+U_2^2(s)+U_3^2(s)+U_4^2(s)]ds\leq\frac{V(T^*)}{A}<+\infty$. This implies $U_1^2(s)+U_2^2(t)+U_3^2(t)+U_4^2(t)\in L_1(T^*,+\infty)$. On other hand, we can derive the uniform continuity $U_i(t)$ from boundedness of $U_i(t)$. Hence, $U_1^2(s)+U_2^2(s)+U_3^2(s)+U_4^2(s)$ also uniformly continuous $U_i(t)$. By Barbalat's lemma, we can conclude that $\lim_{t\to\infty}[U_1^2(t)+U_2^2(t)+U_3^2(t)+U_4^2(t)]=0$. Then the endemic equilibrium point is locally asymptotically stable if $A_i>0$, i = 1, 2, 3, 4. Also from the numerical simulation we find that the perturbations $U_i, i=1,2,3,4$ are tending to zero for large time showing the asymptotic stability of endemic equilibrium point (fig 1. a).

Theorem 2: The disease-free equilibrium is locally asymptotically stable if following conditions holds.

a. $B_1=[2(\gamma+\xi)+\beta N(\gamma+\xi)h-2\beta N-\beta'N'-\beta'^2N'^2h+\beta'N'(\gamma'+\xi')h-\xi]>0.$
b. $B_2=[2(\gamma'+\xi')+\beta'N'(\gamma'+\xi')h-2\beta'N'-\beta N-\beta^2N^2h+\beta N(\gamma+\xi)h-\xi']>0.$
c. $B_3=[-\beta'N'+\beta'^2N'^2h-\xi+2\eta]>0.$
d. $B_4=[-\beta N+\beta^2N^2h-\xi'+2\eta']>0.$

Proof: - Since at the disease-free equilibrium $Y_0 = 0$, $Y_0' = 0$, $W_0 = 0, W_0' = 0$ then $X_0' = N'$, $X_0 = N$. Using these values in linear system (4) and applying the same procedure as we use for in theorem 1, we have found the Lyapunov function for each equation of model (4) at disease-free equilibrium,

$$\frac{dV_1}{dt} = [2\beta N - 2(\gamma+\xi) - \beta N(\gamma+\xi)h]U_1^2(t) + [\beta N + \beta^2 N^2 h - \beta N(\gamma+\xi)h]U_2^2(t) + [\beta N + \beta^2 N^2 h]U_4^2(t),$$
$$\frac{dV_2}{dt} = [2\beta' N' - 2(\gamma'+\xi') - \beta' N'(\gamma'+\xi')h]U_2^2(t) + [\beta' N' + \beta'^2 N'^2 h - \beta' N'(\gamma'+\xi')h]U_1^2(t) + [\beta' N' + \beta'^2 N'^2 h]U_3^2(t)$$
$$\frac{dV_3(t)}{dt} = (\xi - 2\eta)U_3(t) + \xi U_1^2(t), \frac{dV_4(t)}{dt} = (\xi' - 2\eta')U_4^2(t) + \xi' U_2^2(t)$$

Now we define the following Lyapunov functional for the system (4) at the disease-free equilibrium,

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \Rightarrow \frac{dV(t)}{dt} \le -\{B_1 {U_1}^2(t) + B_2 {U_2}^2(t) + B_3 {U_3}^2(t) + B_4 {U_4}^2(t)\}$$

Where, $B_1 = [2(\gamma+\xi) + \beta N(\gamma+\xi)h - 2\beta N - \beta' N' - \beta'^2 N'^2 h + \beta' N'(\gamma'+\xi')h - \xi]$, $B_2 = [2(\gamma'+\xi') + \beta' N'(\gamma'+\xi')h - 2\beta' N' - \beta N - \beta^2 N^2 h + \beta N(\gamma+\xi)h - \xi']$, $B_3 = [-\beta' N' + \beta'^2 N'^2 h - \xi + 2\eta]$, $B_4 = [-\beta N + \beta^2 N^2 h - \xi' + 2\eta']$. Denote $B = min\{B_1, B_2, B_3, B_4\}$, then, $V(t) + B\int_{T^*}^{t}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)]ds \le V(T^*)$ for $t \ge T^*$. Therefore $\lim_{t\to+\infty} sup \int_{T^*}^{t}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)]ds \le \frac{V(T^*)}{B} < +\infty$. This implies $[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)] \in L_1(T^*, +\infty)$. Hence, similar as endemic equilibrium, we can conclude that $\lim_{t\to\infty}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)] = 0$. Hence, disease free equilibrium, is locally asymptotically stable if all $B_i > 0$ i = 1, 2, 3. Also from the numerical simulation we find that the perturbations $U_i, i = 1,2,3,4$ are tending to zero for large time showing the asymptotic stability of disease-free equilibrium point (fig 1. b).

Model-II [Where Symptomatic Infective are treated and have lower infection rate than untreated Infective]
We have the following system of ordinary differential equation,

$$\frac{dX}{dt} = -\beta XY'(t-h) - BXW' + \gamma Y + \eta W, \frac{dX'}{dt} = -\beta' X' Y(t-h) - B' X' W + \gamma' Y' + \eta' W'$$
$$\frac{dY}{dt} = \beta XY'^{(t-h)} + BXW' - \gamma Y - \xi Y, \frac{dY'}{dt} = \beta' X' Y(t-h) + B' X' W - \gamma' Y' - \xi' Y' \ldots\ldots (9)$$
$$\frac{dW}{dt} = \xi Y - \eta W, \frac{dW'}{dt} = \xi' Y' - \eta' W'$$

Total population of male and female $X + Y + W = N, X' + Y' + W' = N'$ respectively. Here, $B \le \beta$ and $B' \le \beta'$ are the number of male and female susceptible infected by symptomatic infective per day and rest of the parameters are same as in model I. The above model can be reduced to,

$$\frac{dY}{dt} = (\beta Y' + BW')(N - Y - W) - (\gamma+\xi)Y,$$
$$\frac{dY'}{dt} = (\beta' Y + B' W)(N' - Y' - W') - (\gamma'+\xi')Y' \ldots (9)$$
$$\frac{dW}{dt} = \xi Y - \eta W, \frac{dW'}{dt} = \xi' Y' - \eta' W'$$

Equilibrium Classification
We can see that our model has two equilibrium points

1. Disease free equilibrium $(Y_0, Y_0', W_0, W_0') = (0,0,0,0)$.
2. Endemic equilibrium$(Y_1, Y_1', W_1, W_1') = \left[\frac{\alpha\alpha'\varepsilon}{N'\alpha' + A'\alpha\rho}, \frac{\alpha\alpha'\varepsilon}{N\alpha + A\alpha'\rho'}, \frac{(1-\alpha)\alpha'\varepsilon}{N'\alpha' + \alpha\rho A'}, \frac{\alpha(1-\alpha')\varepsilon}{N\alpha + A\alpha'\rho'}\right]$

Noting that $X_1 = N - Y_1 - W_1$, $X_1' = N' - Y_1' - W_1'$, $X_0 = N - Y_0 - W_0$ and $X_0' = N' - Y_0' - W_0'$. Here, $\frac{1}{\alpha} = 1 + \frac{\xi}{\eta}, \frac{1}{\alpha'} = 1 + \frac{\xi'}{\eta'}$ are reciprocal proportions of asymptomatic infective, $\frac{1}{A} = 1 + \frac{B'\xi}{\beta'\eta}, \frac{1}{A'} = 1 + \frac{B\xi'}{\beta\eta'}$ are the modification $\rho = \frac{\xi+\gamma}{\beta}, \rho' = \frac{\xi'+\gamma'}{\beta'}$. Total removal rate of asymptomatic infective and $\varepsilon = NN' - AA'\rho\rho'$.

Basic Reproduction Number
We use next generation matrix approach to derive the following reproduction numbers,

$$R_0 = \frac{\beta'(N'-Y'-W')(\eta'+\xi')(\gamma+\xi)\eta e^{-\lambda h} - \beta\left(Y'+\frac{B}{\beta}W'\right)(\eta+\xi)(\gamma'+\xi')\eta' + \sqrt{\begin{array}{l} 2\beta\beta'\left(Y'+\frac{B}{\beta}W'\right)(N'-Y'-W')(\gamma+\xi)(\gamma'+\xi')(\eta+\xi)(\eta'+\xi')\eta\eta' e^{-\lambda h} + \\ \left\{\beta\left(Y'+\frac{B}{\beta}W'\right)(\eta+\xi)(\gamma'+\xi')\eta'\right\}^2 + \{\beta'(N'-Y'-W')(\eta'+\xi')(\gamma+\xi)\eta e^{-\lambda h}\}^2 - \\ 4\beta\beta'(\eta+\xi)(\eta'+\xi')(\gamma+\xi)(\gamma'+\xi')\eta\eta'\left(Y+\frac{B\prime}{\beta\prime}W\right)(N-Y-W)\end{array}}}{2(\gamma+\xi)(\gamma'+\xi')\eta\eta'} \ldots (10)$$

Linear Stability Analysis

Theorem 3: The endemic equilibrium of the system (9) is locally asymptotically stable if following condition hold:

a. $D_1 = 3a_{12} + 2a_{14} - a_{11}(1-(a_{12}+a_{14})h) - a_{13} - b_{11}(b_{11}h + \{1 + b_{11} - (2b_{12} + b_{14} - b_{13})\}) > 0.$
b. $D_2 = -a_{11}(a_{11}h + \{1 + a_{11} - (2a_{12} + a_{14} - a_{13})\}) + 3b_{12} + 2b_{14} - b_{11}(1-(b_{12}+b_{14})h) - b_{13} > 0.$
c. $D_3 = a_{12}(1 + a_{11}h) - b_{13}(1 + b_{11}h) > 0.$
d. $D_4 = -a_{13}(1 + a_{11}h)b_{12}(1 + b_{11}h) > 0.$

Proof: We take the following perturbation $U_1 = Y - Y_1, U_2 = Y' - Y_1{}'$, $U_3 = W - W_1, U_4 = W' - W_1{}'$ in system (9) for endemic equilibrium. After linearizing the resultant nonlinear system, we have following equation,

$$\begin{aligned} \frac{dU_1}{dt} &= \beta X_1 U_2(t-h) - (\beta Y_1' + BW_1')(U_1+U_3) + BX_1U_4(t) - (\gamma+\xi)U_1 \\ \frac{dU_2}{dt} &= \beta' X_1'^{U_1(t-h)} - (\beta' Y_1 + B'W_1)(U_2+U_4) + B'X_1'^{U_3} - (\gamma'+\xi')U_2 \ldots\ldots (11) \\ \frac{dU_3}{dt} &= \xi U_1 - \eta U_3, \frac{dU_4}{dt} = \xi' U_2 - \eta' U_4 \end{aligned}$$

Now in the above system assume that $a_{11} = \beta X_1,\ a_{12} = (\beta Y_1{}' + BW_1{}'), a_{13} = BX_1, a_{14} = (\gamma+\xi), b_{11} = \beta' X_1{}'$,$b_{12} = (\beta' Y_1 + B'W_1), b_{13} = B'X_1{}', b_{14} = (\gamma'+\xi'),\ c_{11} = \xi, c_{12} = \eta,\ d_{11} = \xi', d_{12} = \eta'.$
Putting these values in the system given by (11),

$$\begin{aligned} \frac{dU_1}{dt} &= a_{11}U_2(t-h) - a_{12}(U_1+U_3) + a_{13}U_4(t) - a_{14}U_1(t), \\ \frac{dU_2}{dt} &= b_{11}U_1(t-h) - b_{12}(U_2+U_4) + b_{13}U_3(t) - b_{14}U_2(t), \\ \frac{dU_3}{dt} &= c_{11}U_1(t) - c_{12}U_3(t), \frac{dU_4}{dt} = d_{11}U_2(t) - d_{12}U_4(t), \ldots\ldots\ldots\ldots (12) \end{aligned}$$

The first equation of (12) can be written as

$\frac{d}{dt}\left[U_1(t) + \int_{t-h}^{t} a_{11}U_2(t)ds\right] = a_{11}U_2(t) - a_{12}(U_3+U_1) + a_{13}U_4(t) - a_{14}U_1(t) \ldots.$ (13)

Let $U_{11}(t) = \left[U_1(t) + \int_{t-h}^{t} a_{11}U_2(S)\,ds\right]^2$ .... (14)

Differentiating (14) and putting values from (13) and (14),

$\frac{dU_{11}}{dt} = 2\left[U_1(t) + \int_{t-h}^{t} a_{11}U_2(s)ds\right][a_{11}U_2(t) - a_{12}(U_1+U_3) - a_{13}U_4(t) - a_{14}U_1(t)]$

$= 2[a_{11}U_1(t)U_2(t) + (a_{11})^2U_2(t)\int_{t-h}^{t} U_2(s)ds - (a_{12}+a_{14})U_1^2(t) + (a_{12}+a_{14})a_{11}U_1(t)\int_{t-h}^{t} U_2(s)ds -$
$a_{12}U_1(t)U_3(t) - a_{11}a_{12}U_3(t)\int_{t-h}^{t} U_2(s) - a_{13}U_4(t)U_1(t) + a_{13}a_{11}U_4(t)\int_{t-h}^{t} U_2(s)].$ Using inequality $a^2 + b^2 \geq 2ab$,

$$\begin{aligned} \frac{dU_{11}}{dt} \leq\ & 2[-(a_{12}+a_{14})U_1^2(t) + \frac{a_{11}}{2}\{U_1^2(t) + U_2^2(t)\} + \frac{(a_{11})^2}{2}\int_{t-h}^{t}\{U_2^2(t) + U_2^2(s)\}ds - \\ & \frac{a_{11}(a_{12}+a_{14})}{2}\int_{t-h}^{t}\{U_1^2(t) + U_2^2(t)\}\,ds - \frac{a_{12}}{2}\{U_1^2(t) + U_3^2(t)\} - \frac{a_{11}a_{12}}{2}\int_{t-h}^{t}\{U_2^2(s) + U_3^2(t)\}ds \\ & + \frac{a_{13}}{2}\{U_4^2(t) + U_1^2(t)\} + \frac{a_{13}a_{11}}{2}\int_{t-h}^{t}\{U_4^2(t) + U_2^2(s)\}]. \\ \leq\ & [\{-2(a_{12}+a_{14}) + a_{11} - a_{12} + a_{13} - a_{13} - a_{11}(a_{12}+a_{14})h\}U_1^2(t) + \{a_{11} + (a_{11})^2h\}U_2^2(t) \\ & + \{-a_{12} - a_{11}a_{12}h\}U_3^2(t) + \{a_{13} + a_{13}a_{11}h\}U_4^2(t) + \{(a_{11})^2 - a_{11}(a_{12}+a_{14}) - a_{11}a_{12} + a_{11}a_{13}\} \\ & \times \int_{t-h}^{t} U_2^2(s)ds]. \end{aligned}$$

Now we define a Lyapunov functional,

$V_1(t) = U_{11}(t) + U_{12}(t).$ ...... (14)

Where, $U_{12} = \{(a_{11})^2 - a_{11}(2a_{12} + a_{14} - a_{13})\}\int_{t-h}^{t}\int_{s}^{t} U_2^2(l)dlds$

$\Rightarrow \frac{dU_{12}}{dt} = \{(a_{11})^2 - a_{11}(2a_{12} + a_{14} - a_{13})U_2^2(t)\} - \{(a_{11})^2 - a_{11}(2a_{12} + a_{14} - a_{13})\}\int_{t-h}^{t} U_2^2(s)ds$

From equation (14) we have,
$\frac{dV_1(t)}{dt} \leq \{-3a_{12} - 2a_{14} + a_{11} - a_{11}(a_{12} + a_{14})h + a_{13}\}U_1^2(t) + [(a_{11})^2 h + a_{11}\{1 + a_{11} - \cdots \ldots \ldots \ldots (2a_{12} + a_{14} - a_{13})\}]U_2^2(t) - a_{12}(1 + a_{11}h)U_3^2(t) + a_{13}\{1 + a_{11}h\}U_4^2(t)$.

Similarly for remaining equation of system (12) we have

$$\frac{dV_2(t)}{dt} \leq \{-3b_{12} - 2b_{14} + b_{11} + b_{13} - b_{11}(b_{12} + b_{14})h\}U_2^2(t) + [(b_{11})^2 h + b_{11}\{1 + b_{11} - \cdots \ldots \ldots \ldots (2b_{12} + b_{14} - b_{13})\}]U_1^2(t) - b_{12}(1 + b_{11}h)U_4^2(t) + b_{13}(1 + b_{11}h)U_3^2(t)$$

$$\frac{dV_3(t)}{dt} = (c_{11} - 2c_{12})U_3^2(t) + c_{11}U_1^2(t), \frac{dV_4(t)}{dt} = (d_{11} - 2d_{12})U_4^2(t) + d_{11}U_2^2(t)$$

Now we define a Lyapunov function for the given system,

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \Rightarrow \frac{dV}{dt} = \frac{dV_1}{dt} + \frac{dV_2}{dt} + \frac{dV_3}{dt} + \frac{dV_4}{dt}$$

$$\Rightarrow \frac{dV(t)}{dt} \leq -[3a_{12} + 2a_{14} - a_{11} + a_{11}(a_{12} + a_{14})h - a_{13} - (b_{11})^2 h - b_{11}\{1 + b_{11} - (2b_{12} + b_{14} - b_{13})\}]U_1^2(t) - [-(a_{11})^2 h - a_{11}\{1 + a_{11} - (2a_{12} + a_{14} - a_{13})\} + 3b_{12} + 2b_{14} - b_{11} - b_{13} + b_{11}(b_{12} + b_{14})h]U_2^2(t) - \{a_{12}(1 + a_{11}h) - b_{13}(1 + b_{11}h)\}U_3^2(t) - \{-a_{13}(1 + a_{11}h)b_{12}(1 + b_{11}h)\}U_4^2(t)$$

$\ldots \ldots \ldots \leq -\{A_1U_1^2(t) + A_2U_2^2(t) + A_3U_3^2(t) + A_4U_4^2(t)\}$.
Here, $D_1 = 3a_{12} + 2a_{14} - a_{11} + a_{11}(a_{12} + a_{14})h - a_{13} - (b_{11})^2 h - b_{11}\{1 + b_{11} - (2b_{12} + b_{14} - b_{13})\}$, $D_2 = -(a_{11})^2 h - a_{11}\{1 + a_{11} - (2a_{12} + a_{14} - a_{13})\} + 3b_{12} + 2b_{14} - b_{11} - b_{13} + b_{11}(b_{12} + b_{14})h$, $D_3 = a_{12}(1 + a_{11}h) - b_{13}(1 + b_{11}h)$, $D_4 = -a_{13}(1 + a_{11}h)b_{12}(1 + b_{11}h)$.
Denote $D = min\{D_1, D_2, D_3, D_4\}$. Then, $V(t) + D\int_{T^*}^{t}[U_1^2(s) + U_2^2(s) + U_3^2(s) + U_4^2(s)]ds \leq V(T^*)$for $t \geq T^*$. Therefore $\lim_{t\to+\infty} sup \int_{T^*}^{t}[U_1^2(s) + U_2^2(s) + U_3^2(s) + U_4^2(s)]ds \leq \frac{V(T^*)}{D} < +\infty$. This implies$U_1^2(s) + U_2^2(s) + U_3^2(s) + U_4^2(s) \in L_1(T^*, +\infty)$.
Hence, we can conclude that $\lim_{t\to\infty}[U_1^2(t) + U_2^2(t) + U_3^2(t) + U_4^2(t)] = 0$. Therefore, the endemic equilibrium is locally asymptotically stable if $D_i > 0$, $i = 1,2,3,4$. Also from the numerical simulation we find that the perturbations $U_i, i = 1,2,3,4$ are tending to zero for large time showing the asymptotic stability of endemic Equilibrium point (fig. c).

Theorem 4: The disease-free equilibrium is locally asymptotically stable if following conditions holds:
a. $H_1 = -\beta N - BN + 2(\gamma + \xi) + \beta N(\gamma + \xi)h - \beta' N' - 2\beta'^2 N'^2 h - \beta' B' N'^2 h + \beta' N'(\gamma' + \xi')h - \xi > 0$.
b. $H_2 = [-\beta N - 2\beta^2 N^2 h - \beta B N^2 h + \beta N(\gamma + \xi)h - \beta' N' - B' N' + 2(\gamma' + \xi') + \beta' N'(\gamma' + \xi')h - \xi'] > 0$.
c. $H_3 = [-B' N' - \beta' B' N'^2 h - \xi + 2\eta] > 0$.
d. $H_4 = [-BN - \beta B N^2 h - \xi' + 2\eta'] > 0$.

Proof: Since at the disease-free equilibrium $Y_0 = 0, Y_0' = 0, W_0 = 0, W_0' = 0$ then $X_0' = N', X_0 = N$. Using these values in linear system (11) and applying the same procedure as we use for in theorem 3, we have found the Lyapunov functions for each equation of model (9) at disease-free equilibrium,

$$\frac{dV_1}{dt} = [\beta N + BN - 2(\gamma + \xi) - \beta N(\gamma + \xi)h]U_1^2(t) + [\beta N + 2\beta^2 N^2 h + \beta B N^2 h - \beta N(\gamma + \xi)h]U_2^2(t) + [BN + \beta B N^2 h]U_4^2(t),$$

$$\frac{dV_2}{dt} = [\beta' N' + B' N' - 2(\gamma' + \xi') - \beta' N'(\gamma' + \xi')h]U_2^2(t) + [\beta' N' + 2\beta'^2 N'^2 h + \beta' B' N'^2 h - \beta' N'(\gamma' + \xi')h]U_1^2(t) + [B' N' + \beta' B' N'^2 h]U_3^2(t),$$

$$\frac{dV_3(t)}{dt} = (\xi - 2\eta)U2_3(t) + \xi U_1^2(t), \frac{dV_4(t)}{dt} = (\xi' - 2\eta')U_4^2(t) + \xi'^{U_2^2}(t).$$

Now we define the following Lyapunov functional for the system (9) at the disease-free equilibrium,
$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t) \Rightarrow \frac{dV}{dt} \leq -\{B_1 {U_1}^2(t) + B_2 {U_2}^2(t) + B_3 {U_3}^2(t) + B_4 {U_4}^2(t)\}$
Where, $H_1 = [-\beta N - BN + 2(\gamma + \xi) + \beta N(\gamma + \xi)h - \beta' N' - 2\beta'^2 N'^2 h - \beta' B' N'^2 h + \beta' N'(\gamma' + \xi')h - \xi]$, $H_2 = [-\beta N - 2\beta^2 N^2 h - \beta B N^2 h + \beta N(\gamma + \xi)h - \beta' N' - B' N' + 2(\gamma' + \xi') + \beta' N'(\gamma' + \xi')h - \xi']$, $H_3 = [-B' N' - \beta' B' N'^2 h - \xi + 2\eta]$, $H_4 = [-BN - \beta B N^2 h - \xi' + 2\eta']$.
Denote $H = min\{H_1, H_2, H_3, H_4\}$, $V(t) + H\int_{T^*}^{t}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)]ds \leq V(T^*)$ for $t \geq T^*$. Therefore $\lim_{t\to+\infty} sup \int_{T^*}^{t}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)]ds \leq \frac{V(T^*)}{H} < +\infty$. This implies $[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)] \in L_1(T^*, +\infty)$. On other hand we can derive the uniform continuity of $U_1(t), U_2(t), U_3(t), U_4(t)$ from boundedness of $U_1(t), U_2(t), U_3(t), U_4(t)$. Hence ${U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)$ is also uniform continuous. By Barbalat's lemma, we can conclude that is uniformly continuous $\lim_{t\to\infty}[{U_1}^2(s) + {U_2}^2(s) + {U_3}^2(s) + {U_4}^2(s)] = 0$. Thus disease free equilibrium is locally asymptotically stable if all $H_i > 0$ i=1,2,3,4. Also from the numerical simulation we find that the perturbations $U_i, i = 1,2,3,4$ are tending to zero for large time showing the asymptotic stability of disease-free Equilibrium point (fig 1. d).

Numerical Simulation

We see that for the set of parameters $N = 2, N' = 1.8, \beta = .5, \beta' = .5, \gamma = .03, \gamma' = .03, \eta, = .03, \eta' = .03, \xi = .01, \xi' = .01, h = 15$ the endemic equilibrium of the model I exist and stable as the perturbation are tending to zero as shown in fig 1.(a). Again, the disease-free equilibrium of the model I is exist and stable for the set of parameters $N = 2, N' = 1.8, \beta = .02, \beta' = .02, \gamma = .08, \gamma' = .08, \eta, = .01, \eta' = .01, \xi = .01, \xi' = .01, h = 15$ as the perturbation are tending to zero as shown in fig 1.(b). Further, the set of parameter $N = 2, N' = 1.8, \beta = .05, \beta' = .07, B = .03, B' = .04, \gamma = .01, \gamma' = .04, \eta, = .03, \eta' = .03, \xi = .01, \xi' = .01, h = 15$ ensures the existence and stability of the endemic equilibrium of the model II as the perturbation are tending to zero as shown in fig 1.(c) and for the parameter set $N = 2, N' = 1.8, \beta = .05,\ \beta' = .065, B = .02, B' = .02, \gamma = .08, \gamma' = .08, \eta = .02, \eta' = .02, \xi = .01, \xi' = .01, h = 15$ the disease-free equilibrium exist and stable as the perturbation are tending to zero as shown in fig 1.(d).

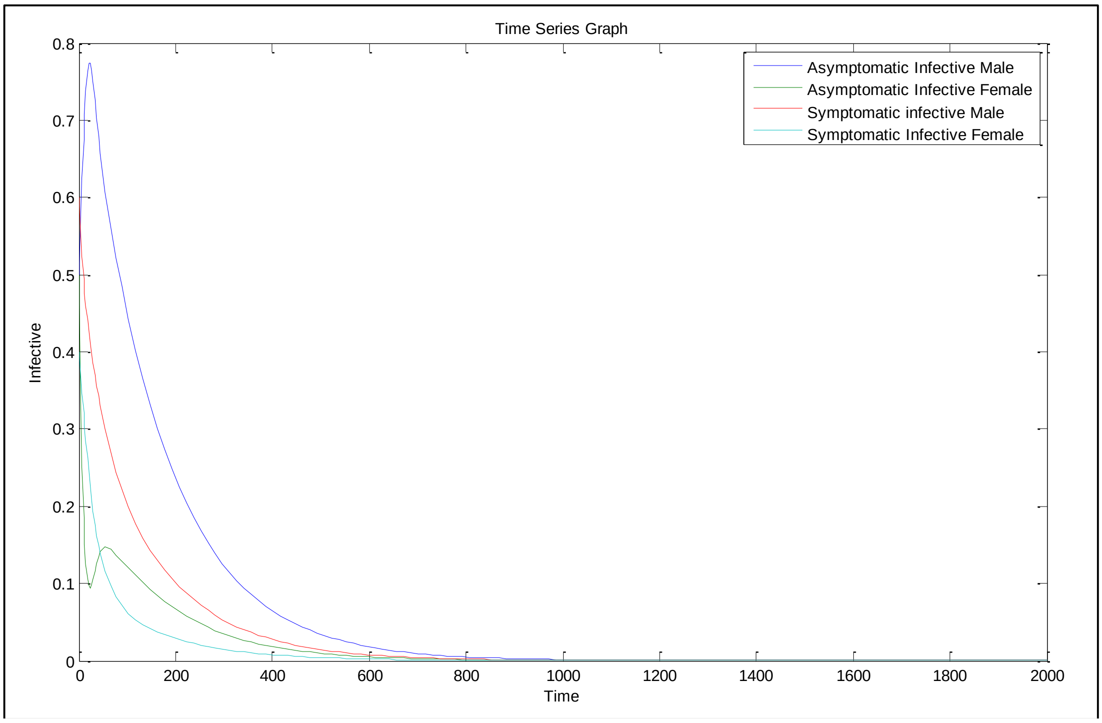

Fig 1 (a): Stable population distribution for Asymptomatic Infective male and female, Symptomatic Infective male and female in endemic case against time

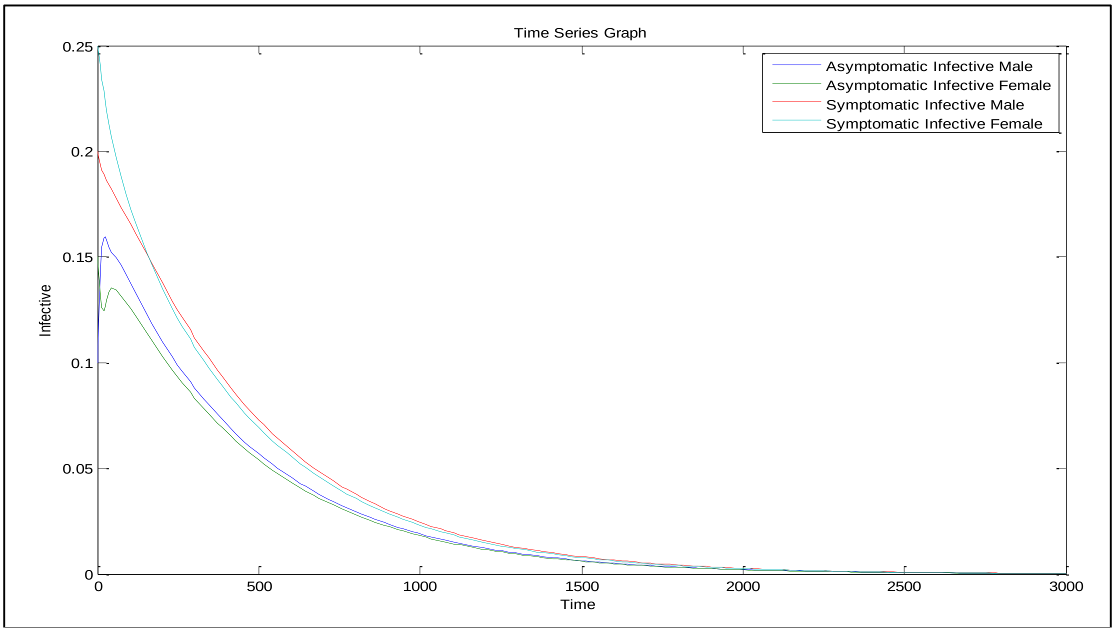

Fig 1 (b): Stable population distribution for Asymptomatic Infective male and female and symptomatic infective male and female in disease-free case

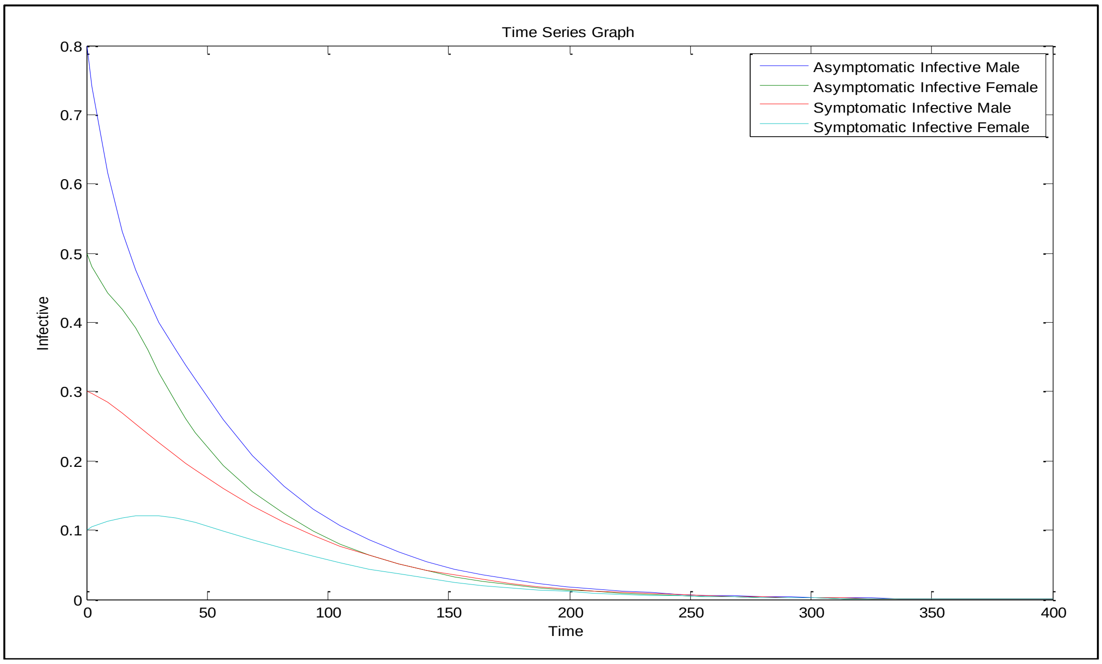

Fig 1 (c): Stable population distribution for Asymptomatic Infective male and female and, symptomatic infective male and female in endemic case

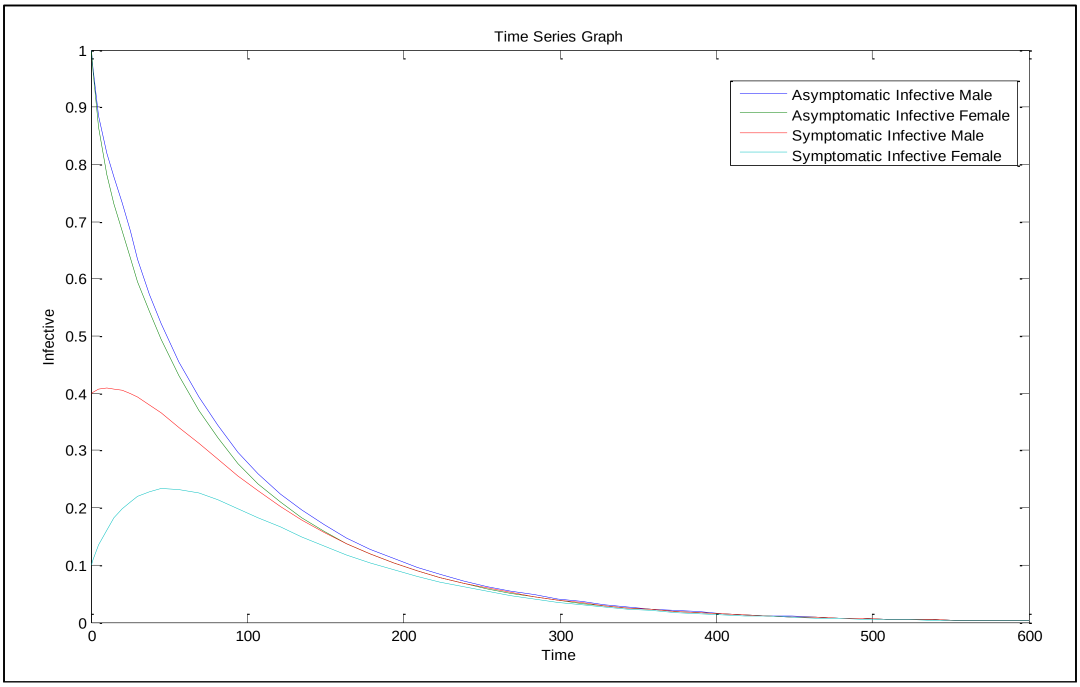

Fig 1 (d): Stable population distribution for Asymptomatic Infective male and female and Symptomatic Infective male and female in Disease free case

Conclusion

Here, we analyzed two venereal disease models with delay having asymptomatic infective and symptomatic infective. After analysis, we have obtained two steady states, disease-free and endemic, and conclude that these states are linearly asymptotically stable under the condition involving the disease-related parameters. Thus, we can say that since the disease-free equilibrium point is stable, therefore disease will not remain in the population; hence it will die out. The stability condition of the endemic equilibrium point can say that disease will always persist in the population. From the reproduction number in these two models, we can see that as $h \to \infty$, $R_0 \to 0$. Thus, from this result, we can conclude that if the latent period for subclinical infection of any disease is too large, the disease will die out.

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