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Deriving and testing the great circle theory

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Abstract
The primary focus of this exploration is to test the benefits of great circle routes over rhumb line routes. A distinct understanding of rhumb line routes was achieved using the derivation of logarithmic spiral arc length and research into the Mercator projection. Then an understanding of great circle theory was achieved through research into the spherical trigonometric formulae and application of these formulae into the distance formula. Data of the coordinates of different airport locations was collected and the two different distances were calculated. Finally, the values were compared in order to understand the difference between the two routes.

Keywords: Latitude, longitude, great circle, rhumb line, loxodrome, logarithmic spiral, spherical trigonometry, radian

Introduction

With the evolution of ideas about the Earth's shape, the idea of distances between two places has also evolved. Navigators in the past would follow routes that circumference continents. This only led to increased time of travel. Over the years, the organisation and division of the Earth into different zones has greatly developed. The development of the concept of latitudes and longitudes specifically helped in the mathematical development of the ideas of distances. It paved the way for development of standard formulae that helps us calculate distances between points. I came across two such methods of distance calculation: great circle theory and rhumb line. When reading up about these theories, I found that they both are considered straight lines in different depictions of the Earth. It interested me to see the distinction between the two paths and question which path is preferred and why. I found it engaging to see the mathematical concepts that were utilised to gain the standard travel times and distances today. As someone who has travelled a lot, this exploration would finally answer the question a passenger has every time they sit on a plane: "Why is the flight path an arc and not a line?" The great circle theory answers this very question. It states that the great circle route is the shortest distance of travel between any two points on Earth. Previously, the landmasses would have hindered following great circle routes. But now with the development of air travel, time of flight can be reduced through following this mathematical concept. The purpose of this exploration is to test the well-known methods of rhumb line routes and great circle theory and gain insight into the mathematical background that went into the development of these methods.

Terms to Note

Latitude: Circular lines running horizontally around the globe are called latitudes. The equator is the latitude that lies at the centre of the Earth. All latitudes are parallel to each other and decrease in radius as you move towards the poles. Two latitudes never intersect.

Longitude: Longitudes are lines that run vertically from north to south. They are all equal in length and meet at both the north and south pole. The Prime Meridian, passing through Greenwich London, is considered as the central meridian from which all other longitudes are measured.

Azimuth: It is the angular measurement in a spherical coordinate system. If the vector between the point of origin and point of interest is projected perpendicularly onto the reference plane, the angle between the projected vector and a reference vector on the plane is called the azimuth.

International Air Transport Association (IATA)- It is an international trade organisation that coordinates international affairs and establishes standards for airline companies. All airport codes and coordinates considered in this paper are according to the IATA.

Rhumb Line
A rhumb line is a line that cuts all meridians at the same angle, other than a right angle. Since the earth is a sphere, the rhumb line is a spherical spiral. Mathematically, it can be called a logarithmic spiral.

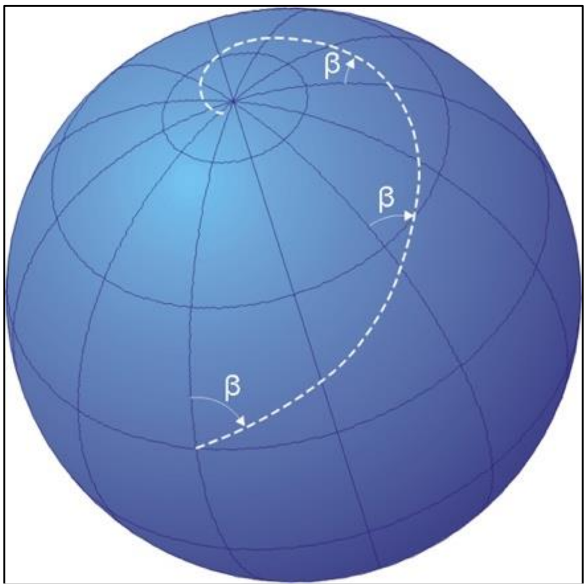

Fig 1: https://upload.wikimedia.org/wikipedia/commons/d/d6/Loxodrome.png

A Mercator Projection
Mercator Projection is a type of projection that was introduced by Gerardus Mercator in 1569. It is also known as the cylindrical orthomorphic projection. It is constantly used in navigation processes. Mercator's projection became an important tool in the 16th and 17th century because at that time, the Western European countries like France, Great Britain and Portugal needed it for efficient navigation to their imperialistic colonies. On a Mercator projection, the loxodromes (or rhumb lines) are straight lines.

B Logarithmic Spiral
A logarithmic spiral is a self-similar spiral that was studied first by Rene Descartes in 1638. It is also called the growth spiral or equiangular spiral. This spiral can also be related to the Fibonacci numbers and called the golden spiral.
The polar equation of a logarithmic spiral is

$$r = ae^{b\theta} \quad -(1)$$

Where
r - distance from origin
θ - angle from the x-axis
a & b - arbitrary constants

The spiral can be created using a collection of equally spaced rays. You can start at a point on any ray and create the spiral by drawing perpendiculars to the neighbouring rays, as shown in figure.

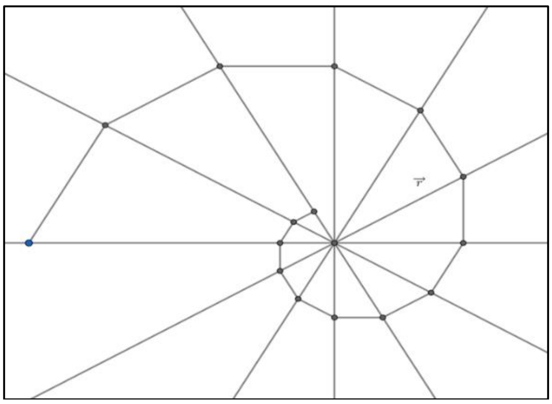

Fig 2: illustrated by author using www.geogebra.com

Differentiating r by θ, the rate of change of radius:

$$\frac{dr}{d\theta} = abe^{b\theta} = br \ -(2)$$

And the angle ψ between the tangent and the radial line at point (r, θ):

$\psi = \tan^{-1}\frac{r}{dr/_{d\theta}}$ *putting value from* (2)

$= \tan^{-1}\frac{r}{br}$ *putting value from* (1)

$= \tan^{-1}\frac{1}{b}$

$= \cot^{-1} b$

Now, If $\psi = \lim_{b\to 0} \cot^{-1} b = \cot^{-1} 0$

$\psi = {}^{\pi}/_{2}$

Angle between tangent and radial line is always perpendicular for a circle. Hence, as $b \to 0$, $\psi \to {}^{\pi}/_{2}$ and the spiral approaches a circle.

Parametrically, coordinates of a point of a logarithmic spiral it can be expressed as

$x = r\cos\theta = ae^{b\theta}\cos\theta \ -(4)$

$y = r\sin\theta = ae^{b\theta}\sin\theta \ -(5)$

Hence, the spiral can be expresses as $\left(ae^{b\theta}\cos\theta, ae^{b\theta}\sin\theta\right)$

If any point D_1 on the spiral is considered, the length of the spiral from D_1 to the origin is finite. If the point is at a distance of $\vec{r}$ from origin, the distance of P to the pole along the spiral is simply the arc length.

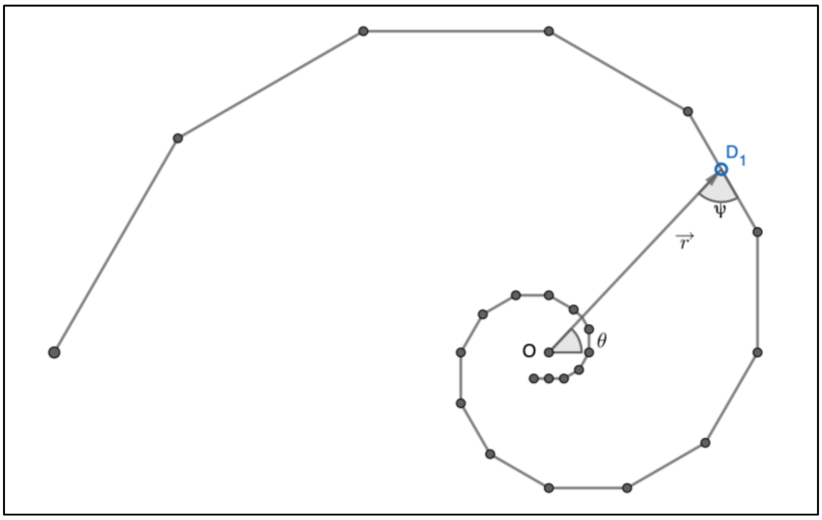

Fig 3: Illustrated by author using www.geogebra.com

The arc length is given by

$s(\theta) = \int ds$

$= \int \sqrt{x^2 + y^2}\, d\theta$

$= \int \sqrt{(ae^{b\theta}\cos\theta)^{\,2} + (ae^{b\theta}\sin\theta)^2}\, dt$

$= \frac{a\sqrt{1+b^2}}{b} e^{b\theta}$ *putting value of* (1)

$= \frac{r\sqrt{1+b^2}}{b}$

This value of arc length is further manipulated to use latitude and longitude coordinates and calculate the rhumb line distance.

Great Circles

Any circle drawn on the surface of the Earth that has the centre of the Earth as its centre and the diameter of the Earth as its diameter is called a great circle. Any great circle would divide the Earth into two halves. All longitudes are great circles; however, amongst the latitudes, only the Equator is a great circle.

The great circle is an important concept because it is considered to give the shortest distance between any two points on Earth. Hence, it becomes an important value for airplane navigations.

A great circle can be drawn between any two points on the surface of the Earth

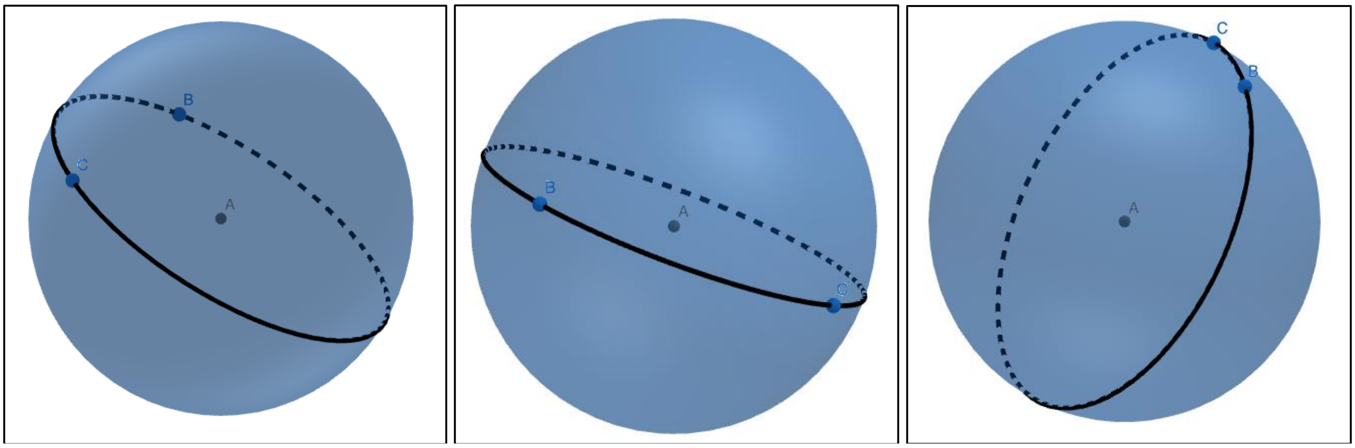

Fig 4: Illustrated by author using www.geogebra.com

A Deriving the Haversine formula for Great circle distance

All formulae are being applied to a perfectly spherical earth. It is not considered as an irregularly shaped ellipsoid and the flattening of the Earth at the poles is neglected. Haversine formula can be used to calculate the great circle distance between any two points provided that their latitude and longitude are given.

The Other Trigonometric Functions

The world of mathematics contains trigonometric formulae beyond the three commonly known sine, cosine and tangent. They are no longer required and their values can be represented by the common functions only, but at one point they were important enough to name.

$$versin\ \theta = 1 - \cos\theta$$
$$vercosin\ \theta = 1 + \cos\theta$$
$$coversin\ \theta = 1 - \sin\theta$$
$$covercosin\ \theta = 1 + \sin\theta$$
$$havercosine\ \theta = \cos^2\frac{\theta}{2}$$

For the purpose of this derivation, we need the value of haversine function only:

$$haversin\ \theta = \frac{versin\ \theta}{2}$$
$$= \frac{1-\cos\theta}{2}$$

Trigonometric identities state:

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad -(a)$$
$$\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1 \Rightarrow \cos^2\frac{\theta}{2} = 1 - \sin^2\frac{\theta}{2}$$

$putting\ value\ in\ (a)$

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$
$$1 - \cos\theta = 1 - \sin^2\frac{\theta}{2}$$

If this value is put in the haversine formula, we get:

$$haversine\ \theta = \frac{2\sin^2\frac{\theta}{2}}{2}$$
$$\boxed{haversine\ \theta = \sin^2\frac{\theta}{2}}$$

Now to apply this for distance calculation on the Earth's surface consider two points A and B. Let *d* be the great circle distance between the two points, *r* be the radius of the Earth and θ be the angle between the two points at the centre of the Earth.

Let A be located at latitude φ_1 and longitude λ_1 and let B be located at latitude φ_2 and longitude λ_2

Now,

$$\Delta\varphi = \varphi_1 - \varphi_2$$
$$\Delta\lambda = \lambda_1 - \lambda_2$$

The haversine formula states that

$$haversin\ \theta = haversin\ \Delta\varphi + \cos\varphi_1 \cos\varphi_2\ haversin\ \Delta\lambda$$
$$\Rightarrow haversin\ \theta = haversin\ (\varphi_1 - \varphi_2) + \cos\varphi_1 \cos\varphi_2\ haversin\ (\lambda_1 - \lambda_2)$$

$$\Rightarrow \sin^2\frac{\theta}{2} = \sin^2\frac{(\varphi_1-\varphi_2)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(\lambda_1-\lambda_2)}{2}$$

$$\Rightarrow \sin\frac{\theta}{2} = \sqrt{\sin^2\frac{(\varphi_1-\varphi_2)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(\lambda_1-\lambda_2)}{2}}$$

$$\Rightarrow \frac{\theta}{2} = \sin^{-1}\sqrt{\sin^2\frac{(\varphi_1-\varphi_2)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(\lambda_1-\lambda_2)}{2}}$$

$$\Rightarrow \theta = 2\sin^{-1}\sqrt{\sin^2\frac{(\varphi_1-\varphi_2)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(\lambda_1-\lambda_2)}{2}}$$

We know that $arc = radius \times angle \Rightarrow angle = {}^{arc}/_{radius} \Rightarrow \theta = {}^{d}/_{r} \Rightarrow d = r\theta$

$$\boxed{\Rightarrow d = 2r\sin^{-1}\sqrt{\sin^2\frac{(\varphi_1-\varphi_2)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(\lambda_1-\lambda_2)}{2}}}$$

B Difference Between Rhumb Line and Great Circle

There is a clear distinction between a rhumb line and a great circle. While the rhumb line would be a straight line on a 2-D projection of the Earth, the great circle is a straight line on the spherical Earth itself.

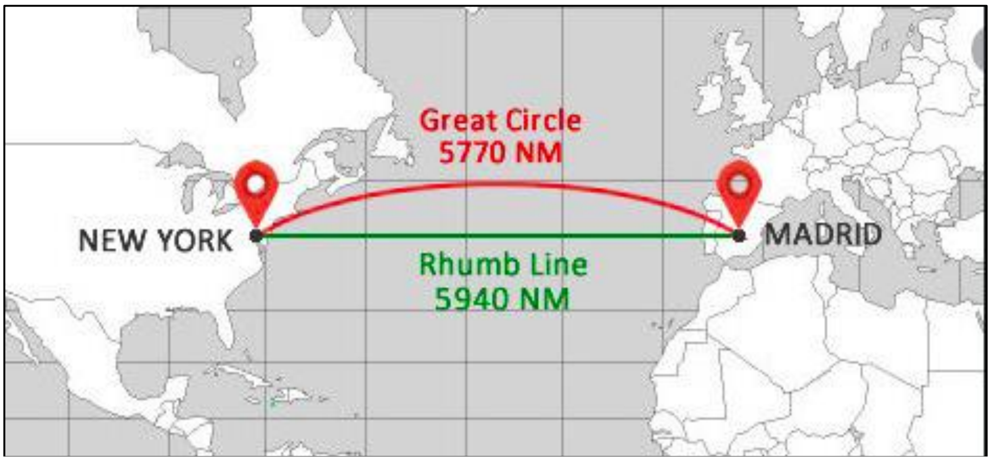

Fig 5: https://atpltuition.net/wp-content/uploads/2020/06/Screenshot-2020-06-17-at-18.10.42.png

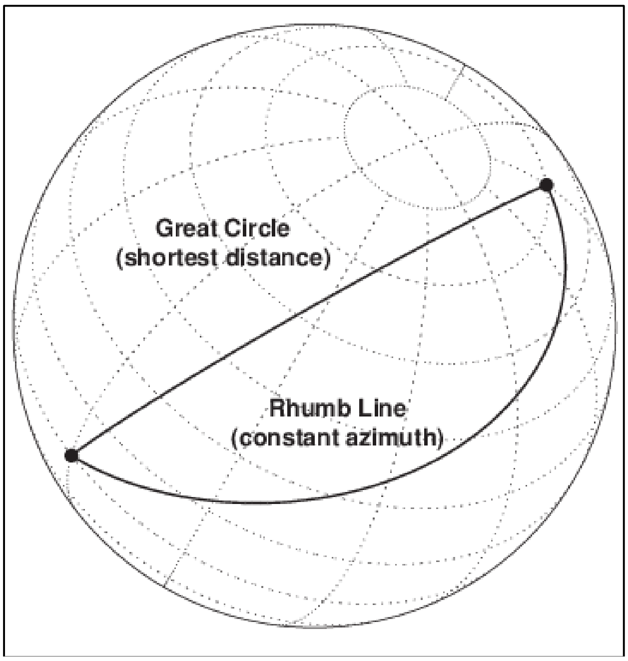

Fig 5: https://www.mathworks.com/help/map/tutor4.png

It is common knowledge that the straight line joining any two points is considered to be the shortest distance between them. However, this is only the case for a two-dimension plane. When the Earth is in question, it must be considered as the three-dimensional model it is. In reality it is an irregularly shaped ellipsoid. Mathematical calculations become really irregular if every impracticality is considered. Hence, we consider the Earth to be a sphere for the derivation of great circle distance. In a sphere, the definition of straight line changes and the rhumb line no longer exists as the shortest distance.

Distances Between Destinations

For this paper, I have taken coordinate the coordinated of airports in a few cities

Cities, Country	Airport Code	Latitude	Longitude
Mumbai, India	BOM	$19° \, 05'50.65''N$	$72° \, 52'27.28''E$
Delhi, India	DEL	$28° \, 33'22.17''N$	$77° \, 06'01.01''E$
Goa, India	GOI	$15° \, 22'49.26''N$	$73° \, 50'05.98''E$
Jaipur, India	JAI	$26° \, 49'44.20''N$	$75° \, 48'20.22''E$
London, United Kingdom	LHR	$51° \, 28'12.08''N$	$00° \, 27'15.46''W$
Vancouver, Canada	YVR	$49° \, 11'48.09''N$	$123° \, 10'53.44''W$
Singapore City, Singapore	SIN	$01° \, 21'51.91''N$	$103° \, 59'29.51''E$

Note

1. Formats for Latitude and Longitude: DMS (Degree, Minute, Second). For usage in rhumb line and great circle formulae, the coordinates must first be in the degree notation. The radian equivalents of the coordinate must be finally applied in the formula.
2. For these examples, North and East will be considered positive while South and West will be considered negative.
3. The radius of the Earth will be considered to be 6378 km

For all cases, the great circle distance is calculated using the formula described under 2.3.A. The rhumb line distances are cited from the website https://www.onboardintelligence.com/RL_Lat1Long1Lat2Long2

(1) Delhi to London

Coordinates of Delhi: $28° \, 33'22.17''N, 77° \, 06'01.01''E$

For calculations purposes: $28.556158\,°\,N, 77.010028°\,E$

In radian:

$\varphi_1 = 0.498146311$

$\lambda_1 = 1.343397155$

Coordinates of London: $51° \, 28'12.08''N, 00° \, 27'15.46''W$

For calculations purposes: $51.470022°\,N, 0.454294°\,W$

In radian:

$\varphi_2 = 0.897865939$

$\lambda_2 = -0.007924906$

Great Circle Distance $d =$

$$2r\sin^{-1}\sqrt{\sin^2\frac{(0.498146311\ - 0.897865939)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(1.343397155 + 0.007924906)}{2}}$$

$$d = 2r\sin^{-1}\sqrt{\sin^2\frac{-0.399719628}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{1.351322061}{2}}$$

$$d = 2r\sin^{-1}\sqrt{\sin^2(-0.199859814) + \cos 0.498146311 \times \cos 0.897865939 \times \sin^2 0.67566103}$$

$$d = 2r\sin^{-1}\sqrt{0.03941493 + 0.878468759 \times 0.623280218 \times 0.391141734}$$

$$d = 2r\sin^{-1}\sqrt{0.03941493 + 0.21462694}$$

$$d = 2r\sin^{-1}\sqrt{0.25404187}$$

$$d = 2r\sin^{-1}(0.504025664)$$

$$d = 2 \times 6378 \times 0.528253483$$

$$d = 6738.401433\ km$$

Rhumb Line Distance = $\mathbf{6992.0408\ km}$

(2) Mumbai to Vancouver:

Coordinates of Mumbai: $19° \, 05'50.65''N, 72° \, 52'27.28''E$

For calculations purposes: $19.097402°\,N, 72.874244°\,E$

In radian:

$\varphi_1 = 0.333143568$

$\lambda_1 = 1.271250701$

Coordinates of Vancouver: $49° \, 11'48.09''N, 123° \, 10'53.44''W$

For calculations purposes: $49.196691°\,N, 123.181511°\,W$

In radian:

$\varphi_2 = 0.858208943$

$\lambda_2 = -2.148833025$

Great Circle Distance $d =$

$$2r\sin^{-1}\sqrt{\sin^2\frac{(0.333143568 - 0.858208943)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(1.271250701 + 2.148833025)}{2}}$$

$$d = 2r\sin^{-1}\sqrt{\sin^2\frac{-0.525065375}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{3.420083726}{2}}$$

$d = 2r\sin^{-1}\sqrt{\sin^2(-0.262826975) + \cos 0.333143568 \times \cos\ 0.858208943 \times \sin^2 1.710041863}$

$d = 2r\sin^{-1}\sqrt{0.067502005 + 0.945019019 \times\ 0.65379376 \times 0.980735672}$

$d = 2r\sin^{-1}\sqrt{0.067502005 + 0.60594512}$

$d = 2r\sin^{-1}\sqrt{0.673447125}$

$d = 2r\sin^{-1}(0.820638242)$

$d = 2 \times 6378 \times 0.96252701$

$d = 12277.99454\ km$

Rhumb Line Distance = $\mathbf{15151.21}\ \boldsymbol{km}$

(3) Delhi to Goa

Coordinates of Delhi: $28°\ 33'22.17''N, 77°\ 06'01.01''E$

For calculations purposes: $28.556158\ °\ N, 77.010028°\ E$

In radian:

$\varphi_1 = 0.498146311$

$\lambda_1 = 1.343397155$

Coordinates of Goa: $15°\ 22'49.26''N, 73°\ 50'05.98''E$

For calculations purposes: $15.380350°\ N, 73.834994°\ W$

In radian:

$\varphi_2 = 0.268301661$

$\lambda_2 = 1.288010451$

Great Circle Distance $d =$

$$2r\sin^{-1}\sqrt{\sin^2\frac{(0.498146311 - 0.268301661)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(1.343397155 - 1.288010451)}{2}}$$

$$d = 2r\sin^{-1}\sqrt{\sin^2\frac{0.22984465}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{0.055386704}{2}}$$

$d = 2r\sin^{-1}\sqrt{\sin^2(0.114922325) + \cos 0.498146311 \times \cos 0.268301661 \times \sin^2 0.027693352}$

$d = 2r\sin^{-1}\sqrt{0.0131491 + 0.878469759 \times\ 0.964222506 \times 0.000766725}$

$d = 2r\sin^{-1}\sqrt{0.0131491 + 0.000649446}$

$d = 2r\sin^{-1}\sqrt{0.013798546}$

$d = 2r\sin^{-1}(0.117467212)$

$d = 2 \times 6378 \times 0.117739049$

$d = 1501.879309\ km$

Rhumb Line Distance = $\mathbf{1497.1568}$

(4) Jaipur to Mumbai

Coordinates of Jaipur: $26°\ 49'44.20''N, 75°\ 48'20.22''E$

For calculations purposes: $26.828944°\ N, 75.805616°\ E$

In radian:

$\varphi_1 = 0.468016023$

$\lambda_1 = 1.322386857$

Coordinates of Mumbai: $19°\ 05'50.65''N, 72°\ 52'27.28''E$

For calculations purposes: $19.097402°\ N, 72.874244°\ E$

In radian:

$\varphi_1 = 0.333143568$

$\lambda_1 = 1.271250701$

Great Circle Distance $d =$

$$2r\sin^{-1}\sqrt{\sin^2\frac{(0.468016023 - 0.333143568)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(1.322386857 - 1.271250701)}{2}}$$

$$d = 2r\sin^{-1}\sqrt{\sin^2\frac{0.13872455}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{0.051136156}{2}}$$
$$d = 2r\sin^{-1}\sqrt{\sin^2(0.067436227) + \cos 0.468016023\times \cos\ 0.333143568 \times \sin^2 0.025568078}$$
$$d = 2r\sin^{-1}\sqrt{0.004540755 +\ 0.892465048 \times 0.945019019 \times 0.000653584}$$
$$d = 2r\sin^{-1}\sqrt{0.004540755 + 0.00055123}$$
$$d = 2r\sin^{-1}\sqrt{0.005091985}$$
$$d = 2r\sin^{-1}(0.071358146)$$
$$d = 2 \times 6378 \times 0.071418844$$
$$d = 911.0187777\ km$$

Rhumb Line Distance = $\mathbf{907.2948}$ $\boldsymbol{km}$

(5) London to Singapore

Coordinates of London: $51°\ 28'12.08''N, 00°\ 27'15.46''W$

For calculations purposes: $51.470022°\ N, 0.454294°\ W$

In radian:

$\varphi_1 = 0.897865939$

$\lambda_1 = -0.007924906$

Coordinates of Singapore: $01°\ 21'51.91''N, 103°\ 59'29.51''E$

For calculations purposes: $1.364419°\ N, 103.99130°\ W$

In radian:

$\varphi_2 = 0.023801531$

$\lambda_2 = 1.814070456$

Great Circle Distance $d =$

$$2r\sin^{-1}\sqrt{\sin^2\frac{(0.897865939 - 0.023801531)}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(-0.00792490 - 1.814070456)}{2}}$$
$$d = 2r\sin^{-1}\sqrt{\sin^2\frac{0.874064408}{2} + \cos\varphi_1\cos\varphi_2\sin^2\frac{(-1.821995356)}{2}}$$
$$d = 2r\sin^{-1}\sqrt{\sin^2(0.437032204) + \cos 0.897865939 \times \cos 0.023801531 \times \sin^2(-0.906099767)}$$
$$d = 2r\sin^{-1}\sqrt{0.179142657 + 0.623280218 \times 0.999716756 \times 0.619532691}$$
$$d = 2r\sin^{-1}\sqrt{0.179142657 + 0.386033098}$$
$$d = 2r\sin^{-1}\sqrt{0.565175755}$$
$$d = 2r\sin^{-1}(0.75178172)$$
$$d = 2 \times 6378 \times 0.850759916$$
$$d = 10852.29349\ km$$

Rhumb Line Distance = $\mathbf{11353.5008}$ $\boldsymbol{km}$

A Evaluation

- For most of the cases, there was a significant difference between rhumb line distance and great circle distance. This proves that the great circle route would be the ideal route for travel.
- It is also seen that the difference between the rhumb line distance and the great circle distance is proportional to the distance between them. Due to the large radius of the Earth, if the points considered are close by, the surface of the Earth can be considered as a flat surface. The moment the Earth becomes a flat surface, the great circle route is nothing but a deviation from the straight line and rhumb line route becomes a shorter distance. For Jaipur to Mumbai or Mumbai to Goa, it was seen that the value of Rhumb line was lesser. The difference though, remained minute and not as significant as in cases of London to Singapore or Mumbai to Vancouver.

Limitations

- The earth has been assumed to be a perfect sphere. The flattening of the Earth at the poles is not taken into consideration.
- A scope for human error exists considering the lengthy calculations. Some decimal digits could be slightly inaccurate.
- The coordinates for airport locations were taken from different sources on the internet. The precision of the coordinates is not guaranteed or uniform.

Conclusion

The aim of the experiment was achieved as a reasonable understanding of the great circle theory and rhumb line routes was gained. The values for great circle routes closely resembled the internationally accepted values and this proved the accuracy of the formula derived. A clear distinction between rhumb line distance and great circle distance was also seen.

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