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Stronger chaotic features of the complemented shift map on *m*-symbol space

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Abstract
This study aims to show that the chaoticity and related properties of the complemented shift map. In the sense of Devaney as well as generically chaotic with $\delta = diam(\sum_m)$, we verify that complemented shift map σ^c is chaotic. Few strong chaotic properties of the complemented shift map have been discussed, and it has been proved that the complemented shift map is conjugate (topologically) to the shift map. Finally, we provide an example to show that σ^c is chaotic, and for that reason, we say that σ^c is an alternative chaotic model in *m*-symbol space.

Keywords: Complemented shift map, symbol space, topologically transitive, totally transitive, generically δ-chaotic, topologically conjugacy

Introduction
Chaotic dynamical systems have been extensively discussed and investigated its characteristic in a distinguished extent of studies. Recently, the chaotic features of any dynamical system are a more demanding and challenging topic for both mathematics and physicists. Li and Yorke [1] are the first mathematicians that connect the term chaos with a map. In 1980, J. Auslander and J. A. Yorke [2] defined chaos by associating the concept of transitivity and sensitivity. In the sense of Devaney [3], we observe that the topologically transitive map is chaotic. Block and Coppel [4] proved that if a continuous interval map is strictly collective expanding in two compact intervals (non-degenerate), there is an uncountable invariant subset on which the map is topologically semi-conjugate to the full shift on two symbols. Akin [20] proposed a linkage between the sensitivity and Li-Yorke version of chaos. L. Snoha [22] introduced the concept of dense chaos as well as δ-dense chaos.

The shift map σ has many properties for which they have been studied in many papers in recent times [10, 11, 12]. The symbol space we use here has a metric that is defined naturally. Hence the study can be related to the metric space with the advantage of using the standard concepts of dynamical systems available therein. Some attractive research works on the particular property sensitive dependence on initial conditions [17, 19]. Biswas H. R. [5] elaborate the idea of the shift map to the generalized shift map σ_n in the symbol space Σ_2 and proved that some strong chaotic properties of the generalized shift map on Σ_2. In [6], Biswas, H. R. and Md. Shahidul Islam established strong chaos-related properties of the forward shift transformation on $\sum_m^+$ and further, they have proved some new results by applying the properties of topological conjugacy. They have shown that shift map on $\sum_m^+$ is topologically conjugated to $f_m(x) = mx(mod 1)$ on R/Z and concluded in that paper $f_m(x) = mx(mod 1)$ on R/Z is Devaney chaotic as well as exact Devaney chaotic. Bhaumik, I. and Choudhury discussed [9] strong chaotic properties of the complemented shift map on Σ_2. Ju H., Shao H., Choe Y. and Shi Y. [13] gave an idea that conditions for any maps to be conjugate or semi-conjugate (topologically) to subshifts of finite type, basic idea about shift map and Cantor set of logistic function. Furthermore, Ruette [18] showed that shift maps have sensitive dependence on initial conditions and dense periodic points on an invariant subset with the support of topological semi-conjugacy, so they are topologically mixing. Tarini Kumar Dutta and Anandaram Burhagohain in [21] used topological conjugacy to prove the chaoticity.

We describe the mathematical preliminaries needed for the subsequent chapters in Section 2. In Section 3, we found that $(\sum_m, \sigma^c)$ has chaotic dependence on initial conditions, which means that σ^c satisfies strong chaotic properties. We verify that σ^c is totally transitive on $\sum_m$ and prove that σ and σ^c are conjugate topologically. Finally, we also present an example that the complemented shift map $\sigma^c\colon \Sigma_{\mathrm{m}} \to \Sigma_{\mathrm{m}}$ is chaotic.

2. Materials and Methods

This section is devoted to introducing some important definitions, propositions, lemmas, theorems, which are necessary to prove the main result of the paper. In this paper, we work on m-symbol sequence space, which is

$$\sum_m = \{0,1,2, \dots \dots m-1\}^N = \{(x_i)_{i=1}^{\infty} : x_i \in \{0,1,2 \dots \dots m-1\}, m \in \mathbb{N}\},$$

where $m(\geq 2) \in \mathbb{N}$, under the distance function

$$d(x,y) = \sum_{k\geq 1} \frac{|x_k - y_k|}{m^k} \text{ for } x = (x_1, x_2, x_3, \dots \dots), y = (y_1, y_2, y_3, \dots \dots) \in \sum_m$$

and discuss the chaoticity of the complemented shift map. Many works have been done on this system for the particular case of $m = 2$. Since $\sum_2$, our results are also valid in $(\sum_m, \sigma^c)$. Further, it may be mentioned that our study falls in the more excellent domain of chaotic dynamics. Here we introduce the concept of Complemented shift map on $\sum_m$.

Topological conjugacy between maps is a very powerful tool in the discussion of dynamical systems. Topological conjugacy can be used to make predictions of the behaviour of a dynamical system up to comparing it with another dynamical system whose specific properties are known [5]. The Topological conjugacy feature has an essential role in studying the chaotic behaviour of a map. With the help of this feature, we can explore the chaotic significance by comparing one map with another map. Topological conjugacy has such importance as it can protect many topological dynamical properties.

Definition 2.1 (Complemented shift map)

Consider $s = (s_0 s_1 \dots \dots \dots)$ is any point of $\sum_m$. Then $\sigma^c\colon \sum_m \to \sum_m$ is defined by $\sigma^c(s) = ({s_1}^c {s_2}^c \dots \dots \dots \dots)$, where the complement of s_i is ${s_i}^c$. Here we see that the map shifts the first element of a point and then changes all others into its complement.

Definition 2.2 (Topologically transitive) [3]

For any pair of non-empty open sets $U, V \subset A \ \exists \ k \geq 0$ such that $f^k(U) \cap V \neq \phi$, then $f\colon A \to A$ is called transitive topologically, where f is a continuous map and (A, d) is a compact metric space.

Definition 2.3 (Li -Yorke Pair) [1]

A pair $(x,y) \in A^2$ is called a Li-Yorke pair with modulus $\delta > 0$ if $\lim_{n\to\infty} \sup d(f^n(x), f^n(y)) \geq \delta$ and $\lim_{n\to\infty} \inf d\big(f^n(x), f^n(y)\big) = 0$, where f is a continuous transformation on the compact metric space (A, d). The set of all Li-Yorke pairs of modulus $\delta > 0$ is denoted by $LY(f, \delta)$.

Definition 2.4 (Strong Sensitive Dependence on initial conditions) [1]

A map $f\colon A \to A$ which is continuous has strong sensitive dependence on initial conditions if any $x \in A$ and a non-empty open set U of A (not necessarily an open neighbourhood of x), there exist $y \in U$ and $n \geq 0$ such that $d\big(f^n(x), f^n(y)\big)$ is maximum in A, where (A, d) is a compact metric space.

Definition 2.5 (Chaotic dependence on initial conditions) [8]

For any $x \in A$ and every neighbourhood $N(x)$ of x there is a $y \in N(x)$ such that the pair $(x,y) \in A^2$ is Li-Yorke then a dynamical systems (A, f) is called chaotic dependence on initial conditions.

Definition 2.6 (Topologically mixing) [7]

Given two non-empty open sets $U, V \subset X$, $\exists \ n_0$ such that $f^n(U) \cap V \neq \emptyset$ for all $n \geq n_0$ then f is topologically mixing where (X, T) is a topological space.

Definition 2.7 (Weakly topologically mixing) [7]

For every non-empty pair $U, V \subset A$, there exists a positive integer k such that $f^k(U) \cap V \neq \emptyset$ then f is weakly topologically mixing, and $f\colon A \to A$ is a map on a topological space (A, T).

Note: From the above two definitions, it follows that topologically mixing property implies weakly topologically mixing property.

Definition 2.8 (Generically δ-chaotic) [2]

If $LY(f,\delta)$ is residual in X^2 then the continuous map $f: A \to A$ on a compact metric space A is called generically δ-chaotic.
We also need the following Propositions, Lemma and Theorem.

Proposition 2.1: Consider a continuous map $f: A \to A$ on a compact metric space A. If f is topologically weak mixing, it is generically δ-chaotic on A with $\delta = diam(A)$.

Proposition 2.2: If $f: A \to A$ has dense periodic points and is transitive, then f has sensitive dependence on initial conditions [16].

Lemma 2.1: Let $s, t \in \Sigma_2$ and $s_i = t_i$, $i = 0,1, \dots \dots \dots, m$. Then $d(s,t) < \frac{1}{2^m}$ and conversely, if $d(s,t) < \frac{1}{2^m}$ then $s_i = t_i$, for $i = 0,1, \dots \dots \dots, m$.

Theorem 2.1: [The Proximity Theorem]
Let $s, t \in \sum_m$ and suppose $s_i = t_i$ for $i = 0,1, \dots, n$. Then $d(s,t) \le \frac{1}{m^n}$. Conversely, if $d(s,t) < \frac{1}{m^n}$, then $s_i = t_i$ for $i \le n$.

3. Main Results
We prove the chaotic characteristics of the complemented shift map in this section. Considering a particular property of the dynamical systems, which is the total transitivity. In this section, at first, we prove that σ^c is continuous on $\sum_m$. We give an example of a function (continuous) that satisfies the vital property of chaos that is topologically transitive but not chaotic. We are representing here a compatible example that all transitive maps (topologically) are not totally transitive and establish that σ and σ^c are topologically conjugate. In the last part of this section, we solve a problem that is the dynamical systems $(\sum_m, \sigma^c)$ is chaotic.

Theorem 3.1: The dynamical systems $(\sum_m, \sigma^c)$ is continuous on the symbol space $\sum_m$.

Proof: We pick n so large that $\frac{1}{m^n} < \varepsilon$, where $\varepsilon > 0$. Consider $p = (p_0 p_1 p_2 \dots \dots \dots)$ and $q = (q_0 q_1 q_2 \dots \dots \dots)$ are any two points of $\sum_m$. Now we choose $\delta = \frac{1}{m^{n+1}}$.

Then $d(p,q) < \delta = \frac{1}{m^{n+1}} \Rightarrow d\big((p_0 p_1 \dots p_{n+1} \dots), (q_0 q_1 \dots q_{n+1} \dots)\big) < \frac{1}{m^{n+1}}$

$\Rightarrow p_i = q_i$ for $i = 0,1,2, \dots \dots ., n+1$ (using Lemma 2.1)

$\Rightarrow p_i^c = q_i^c$ for $i = 1,2, \dots \dots ., n+1$

$\Rightarrow d\big((p_1^c \dots p_{n+1}^c \dots), (q_1^c \dots q_{n+1}^c \dots)\big) < \frac{1}{m^n}$

$\Rightarrow d\big(\sigma^c(p),\ \sigma^c(q)\big) < \frac{1}{m^n} < \varepsilon$

Hence $(\sum_m, \sigma^c)$ is a continuous on $\sum_m$.

Theorem 3.2: The complemented shift map $\sigma^c: \sum_m \to \sum_m$ is topologically transitive.

Proof: To establish that the complemented shift map σ^c is topologically transitive, we have to show that for any two non-empty open sets P and Q of $\sum_m$, $\exists\ n \in \mathbb{N}$ such that $\sigma^n(P) \cap Q \neq \emptyset$. Let $u = (u_1, u_2, u_3, \dots \dots) \in P$ and $v = (v_1, v_2, v_3, \dots \dots) \in Q$ be arbitrary.
Now, $u \in P$, $v \in Q$ and P, Q are open sets. So, $\exists$ open balls $B(u, r_1) \subseteq P$ and $B(v, r_2) \subseteq Q$. If $r = \min\{r_1, r_2\}$ then $B(u,r) \subseteq P$ and $B(v,r) \subseteq Q$. We choose $n \in \mathbb{N}$ such that $\frac{1}{m^n} < r$.
Consider the point $w = (u_1, u_2, u_3, \dots \dots u_n, v_1, v_2, v_3, \dots \dots) \in \sum_m$ which agrees with u up to the n^{th} term. Therefore by the Proximity theorem, we have that
$d(u,w) \le \frac{1}{m^n} < r \Rightarrow w \in B(u,r) \subseteq P$ and consequently it follows that $\sigma^{cn}(w) \in \sigma^{cn}(P)$.

Also $\sigma^{cn}(w) = (v_1, v_2, v_3, \dots \dots) = v \in Q$, $v = \sigma^{cn}(w) \in \sigma^{cn}(P) \Rightarrow v = \sigma^{cn}(w) \in \sigma^{cn}(P) \cap Q$.

So it follows that $\sigma^{cn}(P) \cap Q \neq \emptyset$ and hence $\sigma^c: \sum_m \to \sum_m$ is topologically transitive.
We know that the chaotic maps are Li-Yorke sensitive and all topologically transitive [2]. Here we establish an example of a continuous function which is Li-Yorke sensitive maps or topologically transitive but not chaotic in the sense of B. S. Du.

Example 3.1: Consider a function $J(x): [-1,1] \to [-1,1]$ is defined by

$$J(x) = \begin{cases} \frac{11}{10}(x+1), -1 \le x \le -\frac{1}{11} \\ -11x, -\frac{1}{11} \le x \le 0 \\ x, 0 \le x \le 1 \end{cases}$$

The above function $J(x)$ is a continuous function. We can simultaneously prove that $J(x)$ is Li-Yorke sensitive and topologically transitive [2]. However, in the sense of B. S. Du it is not chaotic because $\frac{-11}{21}$ (the period two-point) and the interval [0,1] are jumping alternatively [2] and never get close to each other.

Theorem 3.3: The complemented shift map $\sigma^c: \sum_m \to \sum_m$ is topologically mixing.

Proof: At first, we consider two non-empty open sets P and Q in the *m*-symbol space. Now we need to show that $\exists$ a non-negative integer n_0 such that $\sigma^n(U) \cap V \neq \emptyset, \forall n \ge n_0$.
Let $u = (u_1, u_2, u_3, \ldots \ldots) \in P$ and $v = (v_1, v_2, v_3, \ldots \ldots) \in Q$ be arbitrary. Then since $u \in P, v \in Q$ and P, Q are open sets in $\sum_m^+$, $\exists$ open balls $B(u, r_{,1}), B(v, r_2)$ such that $B(u, r_1) \subseteq P$ and $B(v, r_2) \subseteq Q$.
If $r = \min\{r_1, r_2\}$ then $B(u, r) \subseteq P$ and $B(v, r) \subseteq Q$ and choose $k \in N$ such that $\frac{1}{m^k} < r$. We then set up a sequence $\{w_n\}$ of points in $\sum_m^+$ with the help of *k*, *u* and *v* such that

$w_1 = (u_1, u_2, u_3, u_4 \ldots u_k, v_1, v_2, v_3, v_4 \ldots), w_2 = (u_1, u_2, u_3, u_4 \ldots u_k, a_1, v_1, v_2, v_3, v_4 \ldots),$

$w_3 = (u_1, u_2, u_3, u_4 \ldots u_k, a_1, a_2, v_1, v_2, v_3, v_4 \ldots), \ldots \ldots$

$w_i = (u_1, u_2, u_3, u_4 \ldots u_k, a_1, a_2, , \ldots \ldots, a_{i-1}, v_1, v_2, v_3, v_4 \ldots \ldots), i \ge 2,$

$a_i \in \{0,1,2, \ldots \ldots, m-1\}.$

Here, every $w_i, i \ge 2$ is constructed by using the finite word obtained by taking first (*i*-1) consecutive symbols of a fixed sequence $a = (a_1, a_2, a_3, \ldots \ldots a_{i-1}, \ldots) \in \sum_m$ chosen arbitrarily. More precisely, the first *k* letters of w_i, for each $i \ge 2$, is the finite word $u_{[1,k]} = (u_1, u_2, u_3, u_4 \ldots u_k)$ taken from $u \in P$ and then follows the word $a_{[1,i-1]} = (a_1, a_2, a_3, \ldots \ldots, a_{i-1})$ taken from a and at last the sequence representing *v*, i.e. $w_i = (u_{[1,k]}, a_{[1,i-1]}, v)$. In this case, we can also use a fixed letter from the alphabet set $\{0,1,2, \ldots, m-1\}$ repeating it for (*i*-1) times rather than using $a_{[1,i-1]}$.
Now, by using the Proximity theorem, we have, $d(u, w_i) \le \frac{1}{m^k} < r$ [since u and w_i agree up to the k^{th} digits], for all $i \in N$. So, $w_i \in B(u, r) \subseteq P$ and hence

$\sigma^{c(k+i-1)}(w_i) \in \sigma^{c(k+i-1)}(B(u, r)) \subseteq \sigma^{c(k+i-1)}(P)$ for all $i \in N$.

Also $\sigma^{c(k+i-1)}(w_i) = (v_1, v_2, v_3 \ldots \ldots) \in V$, $\sigma^{c(k+i-1)}(w_i) \in \sigma^{c(k+i-1)}(P)$ imply that $\sigma^{c(k+i-1)}(P) \cap Q \neq \emptyset$, for all $i \ge 2$.

Therefore,

$\sigma^{cn}(P) \cap Q \neq \emptyset$, for all $n \ge k$.

Hence, the complemented shift map is topologically mixing.

Theorem 3.4: The complemented shift map σ^c on $\sum_m$is generically δ −chaotic with

$\delta = diam(\sum_m) = 1.$

Proof: We have proved that σ^c on $\sum_m$ is topologically mixing. Since a topologically mixing map which is continuous on a compact metric space is also topologically weak mixing, so σ^c is topologically weak mixing. Also, using Proposition 2.1, we observe that a continuous topologically weak mixing map on a compact metric space A is generically δ −chaotic on X with $\delta = diam(A)$. So it follows that σ^con $\sum_m$ is generically δ −chaotic with $\delta = diam(\sum_m) = 1$.

Theorem 3.5: The complemented shift map $\sigma^c: \sum_m \to \sum_m$ has chaotic dependence on initial conditions.

Proof: Consider $d = (d_1, d_2, d_3, \ldots \ldots)$ be any point of $\sum_m$. Now since $d \in P$ and P is an open set, so $\exists$ an open ball $B(d, r)$ with some radius $r > 0$ s.t. $B(d, r) \subseteq P \subseteq N(d)$. Then for this > 0, we can select a large positive integer *n* such that $\frac{1}{m^n} < r$.

(i) Consider $p = (p_0 p_1 \dots \dots \dots . p_i)$ and q= $(q_0 q_1 \dots \dots \dots . q_m)$ are two finite sequences then $pq = p_0 p_1 \dots \dots \dots . p_i q_0 q_1 \dots \dots \dots . q_m$. Further, if we guess that $t_1 t_2 \dots \dots t_p$ are p finite sequences then $t_1 t_2 \dots \dots t_p$ can be defined in a like manner as above.

(ii) Let $W(p, 2n+2) = (p_{n+1} p_{n+2} \dots \dots \dots p_{2n+1} p^c_{2n+2} p^c_{2n+3} \dots \dots \dots p^c_{3n+2})$,

$(p, 2n+4) = (p_{3n+3} p_{3n+4} \dots \dots \dots p_{4n+4} p^c_{4n+5} p^c_{4n+6} \dots \dots \dots p^c_{5n+6})$,
$W(p, 2n+6) = (p_{5n+7} p_{5n+8} \dots \dots p_{6n+9} p^c_{6n+10} p^c_{6n+11} \dots \dots \dots p^c_{7n+12})$, and

so on.
Note that for any even integer $k > 0, W(p, 2n+k)$ is a finite string of length $(2n+k)$.

(iii) We take $t \in \sum_m$ such that, $t = (p_0 p_1 \dots \dots \dots . p_n \; W(p, 2n+2) W(p, 2n+4) W(p, 2n+6) \dots \dots \dots \dots \dots)$.

Using those four notations and Lemma 2.1 as above, we now prove the theorem. By making p and t agree up to

p_n. So, $d(p,t) < \frac{1}{m^n} < \varepsilon$. Therefore $t \in Q \Rightarrow t \in P$.

Now consider the following two cases, which are favourable to prove this theorem.

Case I: We consider n is an odd integer.

Now $\sigma^{cn+1}(p) = (p_{n+1} p_{n+2} \dots \dots \dots p_{2n+1} \dots \dots \dots)$ and $\sigma^{cn+1}(t) = (p_{n+1} p_{n+2} \dots \dots \dots p_{2n+1} \dots \dots \dots)$.

Note that t formation of infinitely many finite sequences of the type $W(p, 2n+k)$. We get,

$$\underset{n\to\infty}{Lt\, inf}\; d(\sigma^{cn}(p), \sigma^{cn}(t)) \;\leq \underset{n\to\infty}{Lt}\; d((p_{n+1} p_{n+2} \dots \dots \dots p_{2n+1} \dots \dots), (p^c_{n+1} p^c_{n+2} \dots \dots \dots p^c_{2n+1} \dots \dots))$$

$$\leq \underset{n\to\infty}{Lt} \left(\frac{0}{m} + \frac{0}{m^2} + \;\dots \dots \dots + \frac{0}{m^{n+1}}\right) = 0$$

So, $\underset{n\to\infty}{Lt\, inf}\; d(\sigma^{cn}(p), \sigma^{cn}(t)) = 0$. (3.1)

Similarly, $\sigma^{c2n+2}(p) = (p_{2n+2} p_{2n+3} \dots \dots \dots p_{3n+2} \dots \dots)$ and

$\sigma^{c2n+2}(t) = (p^c_{2n+2} p^c_{2n+3} \dots \dots \dots p^c_{3n+2} \dots \dots)$.

Hence,

$$\underset{n\to\infty}{Lt\, sup}\; d(\sigma^{cn}(p), \sigma^{cn}(t)) \;\geq \underset{n\to\infty}{Lt}\; d((p_{2n+2} p_{2n+3} \dots \dots \dots p_{3n+2} \dots \dots), (p^c_{2n+2} p^c_{2n+3} \dots \dots \dots p^c_{3n+2} \dots \dots))$$

$$\geq \underset{n\to\infty}{Lt} \left(\frac{1}{m} + \frac{1}{m^2} + \;\dots \dots \dots .. + \frac{1}{m^{n+1}}\right) = \frac{1}{m-1} = \delta > 0$$

Hence, $\underset{n\to\infty}{Lt\, sup}\; d\left(\sigma^{cn}(s), \sigma^{cn}(t)\right) \geq \delta$. (3.2)

Case II: Here n is an even integer.

Then $\sigma^{cn+1}(p) = (p^c_{n+1} p^c_{n+2} \dots \dots \dots p^c_{2n+1} \dots \dots)$ and $\sigma^{cn+1}(t) = (p^c_{n+1} p^c_{n+2} \dots \dots \dots p^c_{2n+1} \dots \dots)$.

In this case, also t formation of infinitely many finite sequences of the type $W(p, 2n+k)$.

So we get, $\underset{n\to\infty}{Lt\, inf}\; d(\sigma^{cn}(p), \sigma^{cn}(t))$

$$\leq \underset{n\to\infty}{Lt}\; d((p^c_{n+1} p^c_{n+2} \dots \dots \dots p^c_{2n+1} \dots \dots), (p^c_{n+1} p^c_{n+2} \dots \dots \dots p^c_{2n+1} \dots \dots)) \leq \underset{n\to\infty}{Lt} \left(\frac{0}{m} + \frac{0}{m^2} + \;\dots \dots \dots .. + \frac{0}{m^{n+1}}\right) = 0$$

Hence, $\underset{n\to\infty}{Lt\, inf}\; d(\sigma^{cn}(p), \sigma^{cn}(t)) = 0$. (3.3)

Similarly, $\sigma^{c2n+2}(p) = (p_{2n+2} p_{2n+3} \dots \dots \dots p_{3n+2} \dots \dots)$ and $\sigma^{c2n+2}(t) = (p^c_{2n+2} p^c_{2n+3} \dots \dots \dots p^c_{3n+2} \dots \dots)$.So we get,

$\underset{n\to\infty}{Lt\ sup}\ d(\sigma^{cn}(p), \sigma^{cn}(t))$

$\geq \underset{n\to\infty}{Lt}\ d((p_{2n+2}p_{2n+3} \dots \dots .p_{3n+2} \dots \dots), (p^c_{2n+2}p^c_{2n+3} \dots \dots .p^c_{3n+2} \dots \dots)) \geq \underset{n\to\infty}{Lt} (\frac{1}{m} + \frac{1}{m^2} + \dots \dots \dots + \frac{1}{m^{n+1}})$
$= \frac{1}{m-1} = \delta > 0$

Hence, $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma^{cn}(s), \sigma^{cn}(t)\big) \geq \delta.$ (3.4)

By virtue of (3.1), (3.2), (3.3), (3.4) in the above two cases above, we get

$\underset{n\to\infty}{Lt\ inf}\ d(\sigma^{cn}(p), \sigma^{cn}(t)) = 0$ and $\underset{n\to\infty}{Lt\ sup}\ d\big(\sigma^{cn}(s), \sigma^{cn}(t)\big) \geq \delta.$ (3.5)

By virtue of (3.5), it is proved that the pair (p, t) is Li-Yorke. Hence the dynamical system $(\sum_m, \sigma^c)$ has chaotic dependence on initial conditions.
he following example establishes that any map which is continuous has strong sensitive dependence on initial conditions, then it has sensitive dependence on initial conditions, but the converse is not always true.

Example 3.2: Let $M: [-1,1] \to [-1,1]$ be a map defined by

$$M(x) = \begin{cases} \frac{8}{7}x + \frac{8}{7}, -1 \leq x \leq -\frac{1}{8} \\ -8x, -\frac{1}{8} \leq x \leq 0 \\ x, 0 \leq x \leq 1 \end{cases}$$

Here M is a continuous map, and also it can be quickly showed that the function has sensitive dependence on initial conditions. We observe that the maximum distance between any two points of [-1,1] is equal to 2. We consider the point $-\frac{8}{15}$ and the open interval $U = (0,1)$. Then there exists no point $y \in U$ such that $d\big(M^n(x), M^n(y)\big) = 2$, for any $n \geq 0$. Hence $M(x)$ does not strong sensitive dependence on initial conditions [2].

Theorem 3.6: The complemented shift map $\sigma^c: \sum_m \to \sum_m$ and the shift map $\sigma: \sum_m \to \sum_m$ are conjugate topologically.

Proof: Consider a map $g: \sum_m \to \sum_m$ by $g(a) = (a_0a_1^ca_2a_3^c \dots \dots)$, where $a = (a_0a_1a_2 \dots \dots)$ is any point of $\sum_m$. At first, we prove the continuity of the function g.

Let $s = (s_0s_1s_2 \dots \dots)$ and $t = (t_0t_1t_2 \dots \dots)$ be any points of $\sum_m$, choose $\epsilon_1 > 0$ be arbitrary and an even integer n so large that $\frac{1}{m^n} < \epsilon_1$.

Then $d(s,t) < \delta_1 = \frac{1}{m^n} \Rightarrow d\big((s_0s_1s_2 \dots \dots s_n \dots), (t_0t_1t_2 \dots \dots t_n \dots)\big) < \frac{1}{m^n}$

$\Rightarrow s_i = t_i$ for $i = 0,1,2, \dots \dots ., n$ using Lemma 2.1

$\Rightarrow s_i^c = t_i^c$ for $i = 1,2, \dots \dots ., n$

$\Rightarrow d\big((s_0s_1^cs_2s_3^c \dots \dots s_n \dots), (t_0t_1^ct_2t_3^c \dots \dots t_n \dots)\big)$

$\Rightarrow d\big(g(s), g(t)\big) < \frac{1}{m^n} < \epsilon_1$

Which proves that on the symbol space$\sum_m$, our assuming map $g: \sum_m \to \sum_m$ is continuous. Similarly, we can prove that the inverse of g is also continuous.
Next, we prove that the map $g: \sum_m \to \sum_m$ is bijective. Let $g(s) = g(t)$, then we get $(s_0s_1^cs_2s_3^c \dots \dots) = (t_0t_1^ct_2t_3^c \dots \dots)$. Hence $s_m = t_m$, for $m = 0,2,4, \dots \dots$ and $s_m^c = t_m^c$, for $m = 1,3,5, \dots ....$ So we get $s_m = t_m$ for al $m \geq 0$, that is $s = t$, which proves that $g: \sum_m \to \sum_m$ is injective.
To prove that g is a homomorphism, we are only to prove that g is surjective on $\sum_m$. Let $b = (b_0b_1b_2b_3 \dots \dots)$ be any point of $\sum_m$. Then $b^* = (b_0^cb_1b_2^cb_3 \dots \dots)$ is a point of $\sum_m$, such that $g(b^*) = b$. Hence the mapping $g: \sum_m \to \sum_m$ is surjective. Hence we conclude that the map $g: \sum_m \to \sum_m$ is a homomorphism. Now we try to show that the map g is a conjugacy between σ and σ^c.

Let $s = (s_0 s_1 s_2 \dots\dots)$ be the point defined as above. Then $\sigma \circ g(s) = \sigma(s_0 s_1^c s_2 s_3^c \dots\dots) = (s_1^c s_2 s_3^c \dots\dots)$. On the other hand, we get $g \circ \sigma^c(s) = g(s_1^c s_2^c s_3^c \dots\dots) = (s_1^c s_2 s_3^c \dots\dots)$.

Hence $\sigma \circ g(s) = g \circ \sigma^c(s)$. So $\sigma^c: \sum_m \to \sum_m$ and $\sigma: \sum_m \to \sum_m$ are conjugate topologically.

The study of transitive maps on compact spaces is widely motivated. It is essential to know if there exists some iterate that is not transitive for a transitive map. The following example gives that all topologically transitive maps are not totally transitive.

Example 3.3: Consider $G(x)$ from [0,1] onto itself is defined by

$$G(x) = \begin{cases} 6x + \frac{1}{7}, 0 \le x \le \frac{1}{7} \\ -6x + \frac{13}{7}, \frac{1}{7} \le x \le \frac{2}{7} \\ \frac{1}{5} - \frac{1}{5}x, \frac{2}{7} \le x \le 1 \end{cases}$$

Since *G* is transitive, it has an orbit that is dense in [0,1] and hence, the interval is the unique invariant compact with a non-empty interior. It can be spontaneously shown that the map *G* is topologically transitive on [0,1], and we can see that the subintervals $\left[0, \frac{2}{7}\right]$and $\left[\frac{2}{7}, 1\right]$are invariant under G^2, so G^2 is not topologically transitive on [0,1]. Hence $G(x)$ is not totally transitive.

Problem 3.1: Prove that $\sigma^c: \sum_m \to \sum_m$ is chaotic.

Solution: In the significance of Li- Yorke, the complemented shift map $\sigma^c: \sum_m \to \sum_m$ is chaotic because, according to the definition of Devaney, it is chaotic, and we know that Devaney chaos implicates Li- Yorke. By the scrambled set *S* consisting of points, Γ_α where Γ_α is defined by,

$$\Gamma_\alpha = \alpha_0 01 \alpha_0 \alpha_0 \alpha_1 \alpha_1 0011 \alpha_0 \alpha_0 \alpha_0 \alpha_1 \alpha_1 \alpha_1 \alpha_2 \alpha_2 \alpha_2 000111 \dots\dots\dots\dots,$$

$\forall \alpha = (\alpha_0 \alpha_1 \alpha_2 \dots\dots\dots)$ in $\sum_m$, we can prove it directly. So in the symbol $\sum_m$, σ^c is a strong chaotic map. As the shift map is frequently used to model the chaoticity of a dynamical system, we can now use the complemented shift map in the position of the shift map to ideal the chaoticity of a dynamical system. Hence for chaotic dynamical systems, we accomplish that σ^c is a new model.

4. Conclusion

In this work, we have established that some chaos-related properties of the complemented shift map and proved that σ^c on $\sum_m$ is Devaney as well as Auslander-York Chaotic. It is generally δ-chaotic with $\delta = diam(\sum_m) = 1$. We have shown that an example that is topologically transitive but not totally transitive. In this paper, we have proved that σ^c is conjugate topologically to σ. Finally, we proved that $\sigma^c: \sum_m \to \sum_m$ is chaotic.

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