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Approximate evaluation of complex hyper singular integrals

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Abstract

In this paper we develop a method for approximate evaluation of complex hyper singular integrals in the complex plane. The schemes are numerically validated using a set of conventional test integrals. A number of examples is provided to illustrate the efficiency of the method developed here.

Keywords: analytic function, cauchy principal value, Hardamard finite part integral, Taylor's coefficients

1. Introduction

Integrals of the type

$$I(f, z_0) = H \int_L \frac{f(z)}{(z-z_0)^\alpha} dz; \alpha \in \mathbb{N} - \{1\}; \quad (1)$$

are frequently appeared in contour integration, where $f(z)$ is infinitely differentiable function in $\Omega = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|, r > 1\}$;

of the complex plane \mathbb{C} and L joining the points $z_0 - h$ to $z_0 + h$ lying in the disc Ω .

It is seen that rules (Ref. [4, 7, 8, 9, 12, 15]) meant for the numerical integration of the integral $I = \int \frac{f(z)}{z-z_0} dz$;

lead to uncontrolled instability when those are applied for the approximation of the integral given in equation (1). This is due to the presence of singular point z_0 of order $\alpha > 1$ on the path of integration L . The integral defined in equation (1) is called as hyper singular integral in complex plane. The study of its real counter part has been going on for a long time and has been documented in a number of publications (Ref. [6, 13, [?], 17-22]).

$$J^* = H \int_a^b \frac{f(x)}{(x-c)^2} dx; a < c < b. \quad (2)$$

However, a very few rules in the course of numerical integration have devised for the former. Therefore, in this study we have proposed a numerical scheme for the numerical computation of the integral given in equation (1).

2. Description of the scheme for numerical evaluation of complex hyper singular integral

To establish the scheme for the numerical computation of the hyper singular integral

$$I(f, z_0) = H \int_L \frac{f(z)}{(z-z_0)^\alpha} dz; \alpha \in \mathbb{N} - \{1\};$$

we assume here that the function $f(z)$ is analytic and infinitely differentiable on the disc

$$\Omega = \{z \in \mathbb{C}: |z - z_0| < \rho = r|h|, r > 1\}.$$

With this assumption expanding $f(z)$ about the point $z = z_0$ we obtain

$$f(z) = \sum_{k=0}^{\infty} c_k (z - z_0)^k;$$

where $c_k = \frac{f^{(k)}(z_0)}{k!}$ is the Taylor's coefficient.

Since

$$\begin{aligned} f(z) &= \sum_{k=0}^{\infty} c_k (z - z_0)^k \\ &= \sum_{k=0}^{\alpha-2} c_k (z - z_0)^k + \sum_{k=\alpha-1}^{\infty} c_k (z - z_0)^k; \end{aligned}$$

Thus,

$$\begin{aligned} f(z) - \sum_{k=0}^{\alpha-2} c_k (z - z_0)^k &= \sum_{k=\alpha-1}^{\infty} c_k (z - z_0)^k \\ \frac{f(z) - \sum_{k=0}^{\alpha-2} c_k (z - z_0)^k}{(z - z_0)^{\alpha-1}} &= \sum_{k=\alpha-1}^{\infty} c_k (z - z_0)^{k-\alpha+1} \\ &= \sum_{j=0}^{\infty} c_{j+\alpha-1} (z - z_0)^j = g(z) \end{aligned}$$

for $j = k - \alpha + 1$.

As a result $g(z)$ is analytic in Ω .

Therefore,

$$\begin{aligned} I(f, z_0) &= H \int_L \frac{f(z)}{(z - z_0)^\alpha} dz \\ &= P \int_L \frac{g(z)}{z - z_0} dz + \sum_{k=0}^{\alpha-2} \int_L \frac{c_k}{(z - z_0)^{\alpha-k}} \\ &= I_g + \sum_{k=0}^{\alpha-2} I_{\alpha k}; \end{aligned} \tag{3}$$

where

$$I_g = P \int_L \frac{g(z)}{z - z_0} dz; \tag{4}$$

and

$$I_{\alpha k} = \int_L \frac{c_k}{(z - z_0)^{\alpha-k}}. \tag{5}$$

Since the function $g(z)$ is analytic in the domain Ω thus, the first integral appears in the right side of the equation (3) is a Cauchy type singular integral with singularity of order one. Hence, any quadrature rule meant for the numerical integration of the complex CPV integral may be applied for the numerical approximation of the integral given in equation (4).

However, it is well known that the integral

$$I_{\alpha k} = \int_L \frac{c_k}{(z - z_0)^{\alpha-k}};$$

is analytically a diverging integral and diverges for $\alpha - k > 1$. Moreover it is a hyper singular integral and its finite part (Hadamard finite part) can be evaluated by transforming the integral onto the real axis with the help of the transformation

$$z = z_0 + ht; -1 \leq t \leq 1$$

(Ref. Davis and Rabinowitz, ^[10], Kai Diethelm ^[11], Ang, W.T ^[1], Ang, W.T and Clements, D.L ^[2], Hasegawa.T ^[14] and A.R. Krommer and C.W. Ueberhuber ^[16])

Now, by using this transformation the integral given in equation (5) is reduced into

$$\begin{aligned}
I_{\alpha k} &= \int_{z_0-h}^{z_0+h} \frac{c_k}{(z-z_0)^{\alpha-k}} dz \\
&= \frac{c_k}{h^{\alpha-k-1}} \int_{-1}^1 \frac{dt}{t^{\alpha-k}} \\
&= \begin{cases} \frac{2c_k}{h^{\alpha-k-1}} \int_0^1 \frac{dt}{t^{\alpha-k}}, & \text{for } \alpha - k \text{ is even} \\ 0, & \text{for } \alpha - k \text{ is odd.} \end{cases}
\end{aligned} \tag{6}$$

The integral appears for $\alpha - k$ as even i.e.

$$\int_0^1 \frac{dt}{t^{\alpha-k}};$$

is a hyper singular integral in real axis. To evaluate this integral, we consider the convergent integral

$$\int_{\epsilon}^1 \frac{dt}{t^{\alpha-k}}; \tag{7}$$

The value of the convergent integral can be simply obtained as

$$\frac{1 - \epsilon^{k-\alpha+1}}{k - \alpha + 1} = \frac{1}{k - \alpha + 1} - \frac{1}{k - \alpha + 1} \frac{1}{\epsilon^{\alpha-k-1}}.$$

Now letting $\epsilon \rightarrow 0$; of course the limit does not exist and so Hadamard suggested to simply ignore the unbounded contribution of $\lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon}$ (Ref. Kai Diethelm ^[11], pp.233) and to assign the value of remaining finite expression.

As a result, the finite part value of our original integral

$$I_{\alpha k} = \begin{cases} \frac{2c_k}{h^{\alpha-k-1}(k - \alpha + 1)}, & \text{for } \alpha - k \text{ is even} \\ 0, & \text{for } \alpha - k \text{ is odd} \end{cases}$$

It is noteworthy to mention here that “as far as equality is concerned, the common rules for ordinary integrals are also valid for finite part integrals, but rules concerning inequalities are not applicable”. (Ref. pp.13, Ang, W.T ^[1], Ang, W.T and Clements, D.L ^[2], Davis and Rabinowitz, ^[10])

Therefore, the integral

$$I(f, z_0) \approx R_g + \sum_{k=0}^{\alpha-2} \frac{2c_k}{h^{\alpha-k-1}(k - \alpha + 1)};$$

where R_g is the quadrature rule (constructed in chapter-ch:comp) meant for the numerical integration of the complex CPV integral I_g .

To verify the accuracy of the proposed scheme numerically, we have applied the scheme over some standard test integrals already considered by different researchers. The results of their numerical approximations are given in Table- 1 and Table-2 in the section of Numerical experiments.

3. Numerical experiments

3.1 Evaluation of complex strongly singular integrals

In this subsection we have considered the hyper singular integrals

$$\begin{aligned}
\Gamma_1 &= H \int_{-i}^i \frac{e^z}{z^2} dz; & \Gamma_2 &= H \int_{-i}^i \frac{e^z}{z^3} dz; \\
\Gamma_3 &= H \int_{-i}^i \frac{e^z}{z^4} dz; & \Gamma_4 &= H \int_{-i}^i \frac{e^z}{z^5} dz;
\end{aligned}$$

Since the scheme (as constructed in Subsection-2) has established by incorporating the rules meant for the numerical integration of complex CPV integrals therefore, here we have employed the scheme with the help of the rules for CPV integrals in order of their increasing algebraic degree of precession for the approximate evaluation of these above integrals. The approximate value of the integrals with their absolute errors are reflected in Table-1 and Table-2.

The basic purpose of numerically evaluating an integral (either singular or nonsingular) by a sequence of rules of increasing precession is to establish the fact that the final approximation corresponding to the rule of highest precession is possibly the best available approximation at hand. Because, from the trend of approximate values resulting from numerical integration of any integral (either definite or CPV integral) by a sequence of rules from lowest precession to highest precession will suggest to

accept or not to accept the results of numerical integration by the highest precession as the value (Approximate value) of an integral; which one cannot be assured of by applying a single quadrature rule.

Keeping this in mind, we have evaluated numerically the above integrals by a sequence of rules instead of a single quadrature rule. The approximate values corresponding to the rule of highest precession have been taken into account for calculation of final approximate value of integrals.

Table 1: Numerical evaluation of complex hyper singular integrals

Rules	Approximate Value of Γ_1	Absolute Error	Approximate Value of Γ_2	Absolute Error
$Q_1(f)$	2.972769271793576i	1.5×10^{-6}	2.327856195900890i	1.6×10^{-7}
$Q_2(f)$	2.972770934810439i	1.8×10^{-7}	2.327856381435858i	2.0×10^{-8}
$Q_3(f)$	2.972770804264940i	5.2×10^{-8}	2.327856366882020i	5.8×10^{-9}
$R_1(f)$	2.972770688067246i	6.4×10^{-8}	2.327856355188614i	5.8×10^{-9}
$R_2(f)$	2.972770776296036i	2.4×10^{-8}	2.327856363211525i	2.2×10^{-9}
$R_3(f)$	2.972770763520918i	1.1×10^{-8}	2.327856362049843i	1.0×10^{-9}
$Q_{32}(f)$	2.972770749298414i	3.2×10^{-9}	2.327856360754087i	2.9×10^{-10}
$Q_{13}(f)$	2.972770749533820i	2.9×10^{-9}	2.327856360775551i	2.7×10^{-10}
$Q_{12}(f)$	2.972770750030788i	2.4×10^{-9}	2.327856360820862i	2.2×10^{-10}
$R_{12}(f)$	2.972770752342519i	1.3×10^{-10}	2.327856361033359i	9.9×10^{-12}
$R_{13}(f)$	2.972770752342596i	1.3×10^{-10}	2.327856361033365i	9.8×10^{-12}
$R_{23}(f)$	2.972770752342690i	1.3×10^{-10}	2.327856361033371i	9.8×10^{-12}
$T_1(f)$	2.972770752411771i	5.9×10^{-11}	2.327856361038685i	4.5×10^{-12}
$T_2(f)$	2.972770752415014i	5.6×10^{-11}	2.327856361038938i	4.3×10^{-12}
$T_3(f)$	2.972770752421825i	4.9×10^{-11}	2.327856361039460i	3.8×10^{-12}

Table 2: Numerical evaluation of complex hyper singular integrals

Rules	Approximate Value of Γ_3	Absolute Error	Approximate Value of Γ_4	Absolute Error
$Q_1(f)$	0.415750566541269i	1.7×10^{-8}	-0.316797848137200i	1.5×10^{-9}
$Q_2(f)$	0.415750585151556i	2.0×10^{-9}	-0.316797846441370i	1.9×10^{-10}
$Q_3(f)$	0.415750583692510i	5.9×10^{-10}	-0.316797846574279i	5.4×10^{-11}
$R_1(f)$	0.415750582614445i	4.9×10^{-10}	-0.316797846665897i	3.7×10^{-11}
$R_2(f)$	0.415750583283161i	1.8×10^{-10}	-0.316797846614448i	1.4×10^{-11}
$R_3(f)$	0.415750583186333i	8.4×10^{-11}	-0.316797846621900i	6.5×10^{-12}
$Q_{32}(f)$	0.415750583078175i	2.4×10^{-11}	-0.316797846630241i	1.9×10^{-12}
$Q_{13}(f)$	0.415750583079966i	2.2×10^{-11}	-0.316797846630098i	1.7×10^{-12}
$Q_{12}(f)$	0.415750583083746i	1.9×10^{-11}	-0.316797846629796i	1.4×10^{-12}
$R_{12}(f)$	0.415750583101609i	7.0×10^{-13}	-0.316797846628416i	4.6×10^{-14}
$R_{13}(f)$	0.415750583101609i	7.0×10^{-13}	-0.316797846628418i	4.8×10^{-14}
$R_{23}(f)$	0.415750583101609i	7.0×10^{-13}	-0.316797846628420i	5.0×10^{-14}
$T_1(f)$	0.415750583101969i	3.4×10^{-13}	-0.316797846628412i	4.1×10^{-14}
$T_2(f)$	0.415750583102004i	3.1×10^{-13}	-0.316797846628342i	2.7×10^{-14}
$T_3(f)$	0.415750583101921i	3.9×10^{-13}	-0.316797846628402i	3.1×10^{-14}

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