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Infinite single integral representation for the polynomial set $D_n\{(x_k), y\}$

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Abstract
In this paper, an attempt has been made to express an Infinite Single Integral Representation for the Polynomial set $D_n\{(x_k), y\}$. Many interesting new results may be obtained as particular cases a separating the parameters.
AMS subject classification: Special Function – 33 (C).

Keywords: integral representation, hypergeometric function, lauricella function

Introduction

We define the certain hypergeometric polynomial set of n- variables by means of generating functions.

$$(\xi + vt^e)^{-\sigma} F\begin{bmatrix} (\alpha_r); & \\ & \mu_1 \mathcal{Y}^{-e_1} t^{e_1} \\ (b_s) & \end{bmatrix}$$

$$F\begin{bmatrix} (A_p); (\alpha_g); (\gamma_{u_k}) & \\ & \mu x_1^{r_1} t, \mu_2 x_2^{e_2} \mathcal{Y}^{-e_2} t^{e_2} \ldots \ldots \mu_k x_k^{e_k} t^{e_k} \\ (B_q); (\beta_h); (\delta_{v_k}) & \end{bmatrix}$$

$$= \sum_{n=0}^{\infty} D^{v;\delta,\sigma;\mu;\mu_1;\mu_2;\ldots\ldots\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{\mu k});}_{n,e;e_1;e_2;\ldots\ldots e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v\kappa})} \{(x_k), \mathcal{Y}\} t^{n.} \quad \ldots (1.1)$$

Where $v, \xi, \sigma, \mu, \mu_1 \mu_{2,\ldots\ldots\ldots\ldots} \mu_k$ are real and $e, e_1, e_{2,\ldots\ldots\ldots\ldots} e_k, r_1$ are non-negative integers.
The left hand sides of (1.1) contains the products of generalized hypergeometric functions involving Lauricella functions in the notation of Burchnall and Chaundy [1]. The generalized polynomial set contains a number of parameters, for simplicity we shall denote it

$$D^{v;\delta,\sigma;\mu;\mu_1;\mu_2;\ldots\ldots\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{\mu k});}_{n,e;e_1;e_2;\ldots\ldots e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v\kappa})} \{(x_k), \mathcal{Y}\} \text{ by } D_n \{(x_k), \mathcal{Y}\}$$

where *n* denotes the order of the generalized polynomial set $D_n \{(x_k), \mathcal{Y}\}$.
After little simplification (1.1) gives

$$D_n\{(x_k),\mathcal{Y}\}=\xi^{\sigma}\sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\dots\dots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1\dots\dots-e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times\frac{[(A_p)]_{n-em-e_1m_1-(e_2-1)m_2\dots\dots(e_k-1)m_k}\,[(\alpha_g)]_{n-em-e_1m_1-e_2m_2\dots\dots e_km_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2\dots\dots-(e_k-1)m_k}\,[(\beta_h)]_{n-em-e_1m_1-e_2m_2\dots\dots e_km_k}}$$

$$\times\frac{[(\gamma_{u_k})]_{mk}\,[(\alpha_r)]_{m_1}\,(\sigma_m)_m\,(-v)^m\,\mu_1^{m_1}\,(\mu_2x_2^{e_2})^{m_2}\dots\dots(\mu_kx_x^{e_k})^{m_k}}{[(\delta_{v_k})]_{mk}\,[(b_s)]_{m_1}\,\xi^m\,m\,!\,m_1\,!\,\mathcal{Y}^{e_1m_1+e_2m_2}\,m_2\,!\,m_k\,!}$$

$$\times\frac{(\mu{x_1}^{r1})^{n-em-e1m1-e2m2-e3m3\dots\dots exmk}}{(n-em-e_1m_1-e_2m_2-.\dots\dots-e_km_k\,!}\dots\dots\dots\dots(1.2)$$

2. Notations

(i) $(n)=1,2,\dots\dots n-1,n.$

$(a_p)=\alpha_1,a_2,\dots\dots a_p.$

(ii) $[(a_p)]=a_1,a_2,\dots\dots a_p.$

$[(a_p)]=(a_1)_n,(a_2)_n\dots\dots(a_p)_n$

(iii) $\Delta_k[(\alpha;b)]=\prod_{r=1}^{a}\left(\frac{a+r-1}{a}\right)_k=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\dots\left(\frac{b+a-1}{a}\right)_k.$

$\Delta_k[m;(a_p)]=\prod_{k=1}^{p}\prod_{r=1}^{m}\left(\frac{a_i+r-1}{m}\right)_k.$

$\Delta_{k+1}(b;a)=\Delta_k(b;a)\Delta(b;a+1).$

$\Gamma\left[a+\frac{(m)}{m}\right]=\prod_{r=1}^{m}\Gamma\left(a+\frac{r}{m}\right).$

$\Gamma\left[a+\frac{(m)+(b_q)}{m}\right]=\prod_{r=1}^{m}\prod_{i=1}^{q}\Gamma\left(a+\frac{r+b_i}{m}\right).$

(iv) $\Gamma\left[\left(m:(a_p)\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{m}\Gamma\left(\frac{a_j+r-1}{m}\right).$

$$\Gamma[(a;b)]=\prod_{r=1}^{a}\Gamma\left(\frac{b+r-1}{a}\right).$$

(v) $\Gamma*(\alpha\pm b)=\Gamma(\alpha+b)\,\Gamma(\alpha-b).$

$\Gamma**(\alpha+b)=\Gamma(\alpha+b)\,\Gamma(\alpha+b).$

3. Theorem: For $e_2>1$, ...$e_k>1$

$$D_n\{(x_k),y\}=\frac{\alpha^e\Gamma\left(\frac{\mu-v-e-2}{2}\right)\Gamma\left(\frac{v-\mu-e+2}{2}\right)}{2^{e-1}\beta\left(1-e,\frac{\mu+v+e}{2}\right)}\int_0^\infty t^{e-1}J_\mu^{(\alpha t)}J_v^{(\alpha t)}$$

$$\times F_{q+h+s:\,v_1,v_2,\dots\dots\dots\dots v_k}^{1+p+g+r:\,u_1,u_2,\dots\dots\dots\dots u_k}\left[\frac{[-n;e,e_1,e_2\dots\dots e_k],}{}\right.$$

$[(1-(B_q)-n):e,e_1,e_2-1,\dots e_k-1],[(1-(\beta_h)-n):e,e_1,e_2\dots e_k],$

$[(1-(A_p)-n):e,e_1,e_2-1,\dots e_k-1],[(1-(\alpha_g)-n):e,e_1,e_2\dots e_k],$

$[(\alpha_r):1],[(\gamma_{u_1}):1],[(\gamma_{u_2}):1]\dots[(\gamma_{u_k}):1],[\sigma:1],$

$[(b_s):1],[(\delta_{v_1}):1],[(\delta_{v_2}):1]\dots[(\delta_{v_k}):1],$

$\left(\frac{2+v+\mu-e}{2}:1\right),\left(\frac{2+v+\mu-e}{2}:1\right),\left(\frac{2+v-\mu-e}{2}:1\right),\left(\frac{2-v-\mu-e}{2}:1\right)$

$\Delta(2:1=e),$..,

$$\times\frac{-v(-1)^{e(r+s+p+q+g+h+1)}}{\xi(\mu x_1^{r1})^e},\frac{\mu_1(-1)^{e_1(r+s+p+q+g+h+1)}}{(\mu x_1^{r1}\mathcal{Y})^{e_1}},$$

$$\times\frac{\mu_2x_2^{e_2}(-1)^{e_2(r+s+p+q+g+h+1)+p+q}}{(\mu x_1^{r_1},\mathcal{Y})^{e_2}},\dots\dots\dots\dots.,$$

$$\times \frac{\mu_k \chi_k^{e_k} (-1)^e k^{(r+s+p+q+g+h+1)+p+q}}{(\mu \chi_1^{r_1})^{e_k}} \Bigg] dt \ (3.1)$$

Proof

$$I = \int_0^\infty t^{e-1} J_\mu(\alpha t) J_V(\alpha t) \sum_{m=0}^{[\frac{n}{e}]} \sum_{m_1=0}^{[\frac{n-em}{e_1}]} \sum_{m_2=0}^{[\frac{n-em-e_1m_1}{e_2}]} \dots\dots. \sum_{m_k=0}^{[\frac{n-em-e_1m_1-e_2m_2 \dots\dots e_{k-1}m_{k-1}}{e_k}]}$$

$$\times \frac{[(A_P)]_{n-em-e_1m_1-(e_2-1)m_2\dots(e_k-1)m_k} \, [(\alpha_g)]_{n-em-e_1m_1-e_2m_2\dots e_k m_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2\dots(e_k-1)m_k} \, [(\beta_h)]_{n-em-e_1m_1-e_2m_2\dots e_k m_k}}$$

$$\times \frac{[(a_r)]_{m_1} [(\gamma_{u_1})]_{m_1} [(\gamma_{u_2})]_{m_2} \cdots [(\gamma_{u_k})]_{m_k} (\sigma)_m (-\nu)^m \mu_1^{m_1}}{[(b_s)]_{m_1} [(\delta_{V_1})]_{m_1} [(\delta_{V_2})]_{m_2} \cdots [(\delta_{V_k})]_{m_k} \xi^m \, m! \;\; m_1 \,!}$$

$$\times \frac{(\mu_2 x_2^{e_2})^{m_2} \dots (\mu_k x_k^{e_k})^{m_k} \left(\frac{2+v+\mu-e}{2}\right)_{m_1} \left(\frac{2+v+\mu-e}{2}\right)_{m_1} \left(\frac{2+v+\mu-e}{2}\right)_{m_1}}{\mathcal{Y}^{e_1m_1+e_2m_2} \, m^2\,! \dots\dots\dots\dots\dots\dots\dots\dots\dots \, m_k\,! \, \Delta(2{:}\,1 = e)}$$

$$\times \frac{\left(\frac{2-\nu-\mu-e}{2}\right)_{m_1} (\alpha t)^{2m_1} \left(\mu x_1^{r_1}\right)^{n-em-e_1m_1-e_2m_2\dots\dots e_k m_k}}{(n - em - e_1m_1 - e_2m_2 \dots\dots e_k m_k)}$$

$$= \int_0^\infty J_\mu(\alpha t) J_v(\alpha t) t^{1-2m_{1-1}} \sum_{m=0}^{[\frac{n}{e}]} \sum_{m_1=0}^{[\frac{n-em}{e_1}]} \sum_{m_2=0}^{[\frac{n-em-e_1m_1}{e_2}]} \dots\dots \sum_{m_k=0}^{[\frac{n-em-e_2m_2\dots\dots e_{k-1}m_{k-1}}{e_k}]}$$

$$\times \frac{[(A_P)]_{n-em-e_1m_1-(e_2-1)m_2\dots(e_k-1)m_k} \, [(\alpha_g)]_{n-em-e_1m_1-e_2m_2\dots\dots e_k m_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2\dots\dots(e_k-1)m_k} \, [(\beta_h)]_{n-em-e_1m_1-e_2m_2\dots\dots e_k m_k}}$$

$$\times \frac{[(\alpha_r)]_{m_1} [(\gamma_{u_1})]_{m_1} [(\gamma_{u_2})]_{m_2} \cdots [(\gamma_{u_k})]_{m_k} (\sigma)_m (-v)^m \mu_1^{m_1}}{[(b_s)]_{m_1} [(\delta_{V_1})]_{m_1} [(\delta_{V_2})]_{m_2\dots} [(\delta_{V_k})]_{m_k} \xi^m \, m! \; m_1\,!}$$

$$\times \frac{\left(\mu_2 x_2^{e_2}\right)^{m_2} \dots \left(\mu_k x_k^{e_k}\right)^{m_k} \left(\frac{2+v+\mu-e}{2}\right)_{m_1} \left(\frac{2+v+\mu-e}{2}\right)_{m_1} \left(\frac{2+v+\mu-e}{2}\right)_{m_1}}{\mathcal{Y}^{e_1m_1+e_2m_2} \, m_2\,! \; m_k\,! \, \Delta(2{:}\,1 - e)}$$

$$\times \frac{\left(\frac{2+v+\mu-e}{2}\right)_{m_1} (\alpha t)^{2m_1} \left(\mu x_1^{r_1}\right)^{n-em-e_1m_1-e_2m_2\dots\dots e_k m_k}}{(n - em - e_1m_1 - e_2m_2 \dots\dots e_k m_k)}$$

$$\times \int_0^\infty J_\mu(\alpha t) \, J_v(\alpha t) t^{l-2-m_1-1} dt \;\dots (3.2)$$

Again, we have [3; ch 3 p. 78]

$$\int_0^\infty t^{e-1} J_\mu(\alpha t) \, J_v(\alpha t) dt = \frac{2^{e-1}\alpha^{-e}\beta\left(1 - e, \frac{\mu+v+e}{2}\right)}{\Gamma\left(\frac{v-\mu-e}{2} + 1\right)\Gamma\left(\frac{\mu-v-e}{2} + 1\right)} \;\dots (3.3)$$

Where $-\mathrm{Re}(\mu + v) < \mathrm{Re}\,(l) < 1$ and $\alpha > 0$. this using the equation (3.2) and (3.3), we obtain

$$I = \frac{2^{e-1}\beta\left(1 - e, \frac{\mu+v+e}{2}\right)}{\alpha^e \left(\frac{\mu-v-e+2}{2}\right)_{m_1} \left(\frac{v-\mu-e+2}{2}\right)_{m_1}} D_n\{(\mathcal{X}_\kappa), \mathcal{Y}\}$$

Hence the proof

4. Particular cases of (3.1)

Separating the term corresponding to $e_1 = 0 = \mu_1$ in (3.1) we obtain a number of results specializing the remaining parameters:

(i) On making the substitution p=0 = q=g=h=u_2;v_2=1= σ =v=ξ=r_1=μ=e=y;δ_1=$\lambda+\frac{1}{2}$, $\mu_2=\frac{1}{4}$,e_2=2 and writing $\frac{x}{\sqrt{x^2-1}}$ for x_1, we obtained

$$C_n^{\lambda}(x)=\frac{(2\lambda)_n\,\alpha^e\,\Gamma\left(\frac{\mu+v-e+2}{2}\right)\Gamma\left(\frac{v-\mu-e+2}{2}\right)x^n}{2^{e-1}\beta\left(1-e,\frac{\mu+v+e}{2}\right)n!}$$

$$\times\int_0^{\infty}t^{e-1}J_\mu(\alpha t)J_v(\alpha t)F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2},+\frac{1}{2},\frac{v+\mu-e+2}{2},\frac{2+\mu-v-e}{2},\\ \frac{2+v-\mu-e}{2},\frac{2-v-\mu-e}{2}\\ \frac{x^2-1}{(x+1)^2}\\ \lambda+\frac{1}{2},\Delta(2;1-e);\end{matrix}\right]dt$$

(ii) For p=o=q=g=h=u_2=v_2;σ=1=ξ=v=r_1=y,μ=2=e_2;μ_2=-1 in (3.1), we get

$$H_n(x)=\frac{\alpha^e\,\Gamma\left(\frac{\mu-v-e+2}{2}\right)\Gamma\left(\frac{v-\mu-e+2}{2}\right)(2x)^n}{n!\;2^{e-1}\beta\left(1-e,\frac{\mu+v+e}{2}\right)}$$

$$\times\int_0^{\infty}t^{e-1}J_\mu(\alpha t)J_v(\alpha t)\,F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\frac{v+\mu-e+2}{2},\frac{2+\mu-v-e}{2},\\ \frac{2+v-\mu-e}{2},\frac{2-v-\mu-e}{2};\\ \frac{-1}{(2\alpha t)^2}\\ \Delta(2;1-e);\end{matrix}\right]dt$$

(iii) On taking p=0=q=g=h=u_2;v_2=1=ξ=σ=v=r_1=e=δ,=y=μ;$\mu_2=\frac{1}{4}$, e_2=2 and $\frac{x}{\sqrt{x^2-1}}$ for x_1 we achieve

$$P_n(x)=\frac{\alpha^e\,\Gamma\left(\frac{\mu-v-e+2}{2}\right)\Gamma\left(\frac{v-\mu-e+2}{2}\right)x^n}{n!\;2^{e-1}\beta\left(1-e,\frac{\mu+e+v}{2}\right)}$$

$$\times\int_0^{\infty}t^{e-1}J_\mu(\alpha t)J_v(\alpha t)\,F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},\frac{v+\mu-e+2}{2},\frac{2+\mu-v-e}{2},\\ \frac{2+v-\mu-e}{2},\frac{2-v-\mu-e}{2};\\ \frac{x^2-1}{(x\times t)^2}\\ 1,\Delta(2;1-e);\end{matrix}\right]dt$$

(iv) On making the substitution p=o=q=u_2=v_2; h=1=y=v=ξ=e=r_1=σ; e_2=2=μ; g={1,2}, μ_2=1;α_1=α; α_2=β; $\beta_1=\alpha+\beta$ in (3.1), we arrive at

$$G_n(\alpha,\beta;x)=\frac{(\alpha)_n(\beta)_n(2x)^n\,\alpha^e\,\Gamma\left(\frac{\mu-v-e+2}{2}\right)\Gamma\left(\frac{v-\mu-e+2}{2}\right)}{(\alpha+\beta)_n\,n!\;2^{e-1}\beta\left(1-e;\frac{\mu+e+v}{2}\right)}$$

$$\times\int_0^{\infty}t^{e-1}J_\mu(\alpha t)J_v(\alpha t)F\left[\begin{matrix}-\frac{n}{2},-\frac{n}{2}+\frac{1}{2},1-\alpha-\beta-n,\frac{v+\mu-e+2}{2},\\ \frac{2+\mu-v-e}{2},\frac{2+v-\mu-e}{2},\frac{2-v-\mu-e}{2};\\ \frac{1}{(x\alpha t)^2}\\ 1-\alpha-n,1-\beta-n,\Delta(2;1-e);\end{matrix}\right]dt$$

(v) On putting q=0=h=u_2=p;v_2=1=ξ=σ = v=r_1=e=y= ξ;g={1,2}, α_1=1,α_2= v, δ_1= v;e_2=2= μ; μ_2=-1 and z for x_1 in (3.1), we get

$$R_{n,v}\left(\frac{1}{z}\right) = \frac{(v)_n (2z)^n\, a^e\, \Gamma\left(\frac{\mu-v-e+2}{2}\right)\, \Gamma\left(\frac{v-\mu-e+2}{2}\right) x^n}{n!\; 2^{e-1}\, \beta\left(1-e;\frac{\mu+e+v}{2}\right)}$$

$$\times \int_0^\infty t^{e-1} J_u(\alpha t) J_v(\alpha t)\, F\left[\begin{matrix} -\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}, \frac{v+\mu-e+2}{2}, \frac{2+\mu-v-e}{2}, \\ \frac{2+v-\mu-e}{2}, \frac{2-v-\mu-e}{2}; \\ \frac{-1}{(z\alpha t)^2} \\ -n, 1-v-3, \Delta(2;1-e); \end{matrix}\right] dt$$

Where $R_{n,v}\left(\frac{1}{z}\right)$ are the lommel polynomials [4; p. 112(4)]

(vi) On putting q=o=h=y=u_2=v_2= σ;p=1=r_1=e= ν= ξ=y;e_2=3= μ; μ_2=-1$\lambda_1 = \nu$ and x_1=x in (3.1), we get

$$h_n(x) = \frac{(v)_n\, (3x)^n\, a^e\, \Gamma\left(\frac{\mu-v-e+2}{2}\right)\, \Gamma\left(\frac{v+\mu-e+2}{2}\right) x^n}{n!\; 2^{e-1}\, \beta\left(1-e;\frac{\mu+e+v}{2}\right)}$$

$$\times \int_0^\infty t^{e-1} J_\mu(\alpha t) J_v(\alpha t)\, F\left[\begin{matrix} \Delta(3;n), \frac{v+\mu-e+2}{2}, \frac{2+\mu-v-e}{2}, \\ \frac{2+v-\mu-e}{2}, \frac{2-v-\mu-e}{2}; \\ \frac{-1}{4x^3(\alpha t)^2} \\ \Delta(2,1-v-n), \Delta(2;1-e); \end{matrix}\right] dt$$

Were h_n (x) are the humbert polynomials [2].

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