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Irsa Sajjad
(a) Department of Lahore Business School - The University of Lahore, Islamabad Campus, Islamabad, Pakistan
(b) Department of Mathematics and Statistics, Central South University Changsha, Hunan, China

Corresponding Author:
Irsa Sajjad
(a) Department of Lahore Business School - The University of Lahore, Islamabad Campus, Islamabad, Pakistan
(b) Department of Mathematics and Statistics, Central South University Changsha, Hunan, China

Classes of ordinary differential equation: Exponentiated exponential pareto distribution

Irsa Sajjad

Abstract
In this paper, the differentiation technique was used to obtain Classes of Ordinary Differential Equation of different order for probability density function, quantile function, hazard function, inverse hazard function, survival function, inverse survival function, of the Exponentiated Exponential Pareto distribution. Since the mentioned distribution can be differentiable easily. So different differentiable techniques like product rule, quotient rule and some algebraic solutions were applied for the simplification of respective probability distribution. The resulting solutions of ODEs are novel ways to considerate the nature of probability function that illustrate the distribution. The necessary condition was obtained for the existence of ODEs. The various probability function considered are almost in consistent. The nature of the ODE obtained are greatly affected by the values of the parameters. Thus, ODE is novel way to show the classification and approximation of probability distribution.

Keywords: differential calculus, exponentiated exponential distribution, quantile function, survival function, hazard function, inverse survival function, reversed hazard function

Introduction
Calculus in an important tool in the determination of mode along with estimation of parameters of a given probability distribution for example Maximum Likelihood Method. Differential equation emerges to understanding and modeling of real-life situations or observable situation in physical phenomena. In mathematical statistics, one of the major areas and ordinary differential are the approximation of probability functions. This approximation is useful in the retrieval of complex form of probability function specifically quantile approximation. Ordinary differential equation can be used to solve probability density function of a given probability distribution beside of determination of mode, parameter estimation and approximation. Some work available as Exponential distribution, Exponentiated exponential distribution, Pareto distribution and Exponentiated Pareto distribution.
Gupta *et al*. was the first who introduced Exponentiated distribution. Also, presented a new family of distribution known as Exponentiated Exponential distribution. Gupta and Kundu [29] introduced the Generalized Exponential distribution and some properties were discussed. Further, Nadarajah and Kotz [58] considered its estimation and inferences. Nadarajah and Kotz [60] also introduced the Exponentiated Fréchet distribution. To obtain Exponentiated distribution many authors contributions, such as Surles and Padgett [66], Exponentiated Weibull family was introduced by Pal *et al*. [50]. The Exponentiated Pareto distribution was firstly introduced by S. Nadarajah [59]. Shawky and Abu-Zinadah [17] use this distribution to estimate its different parameters by maximum likelihood methods and five other techniques. McDonald Exponentiated Generalized Weibull distribution was proposed by Oguntunde *et al*. [1]. Some contributions regarding Exponentiated and Exponential Weibull family distribution includes [13], [20], [21], [25], [27], [28], [32], [33], [36], [40], [41], [51], [52], [56], [57], [67], [70], [71] and numerical solutions of system of ordinary differential equation and some boundary conditions [2], [3], [4], [5], [6], [7], [8], [9], [10], [43], [46], [47], [53], [62], [63], [64], [65]. Others included: Exponentiated Modified Weibull Extension distribution [49], Exponentiated Lognormal distribution [23], Exponentiated Kumaraswamy and its log transformation [18], Exponentiated Generalized Inverse Gaussian distribution [19], Kumaraswamy-transmuted Exponentiated Modified Weibull distribution [11],

Exponentiated Perks distribution [22], Exponentiated reduced Kies distribution [24]. Some characterization of Exponentiated Gompertz distribution [30] and using this distribution, six methods of estimation was presented by Elbatal and Muhammad [31]. An extended Pareto distribution was introduced by Mead [35], and its important properties was discussed by Nooghabi [37]. Also available Gamma Exponentiated Exponential distribution [39], Exponentiated Kumaraswamy-power function distribution [42], Beta Exponentiated Gamma distribution [45], Beta Exponentiated Mukherijii-Islam distribution [54], Exponentiated Chen distribution [55], Exponentiated Transmuted Weibull geometric distribution [12], Transmuted Exponentiated Pareto-I distribution [14], Bivariate Exponentiated generalized Weibull distribution [15], Exponentiated Gamma distribution [16], [38], Transmuted Exponentiated Gumbel distribution [26], Exponentiated Weibull geometric distribution [44], Exponentiated Geometric distribution [61], Log logistic distribution [34], Exponentiated log logistic distribution [48]. Others include [68], Exponentiated Half Logistic distribution [69], Exponentiated Weibull Pareto distribution [34], Exponentiated Chen distribution [39] and Exponentiated Power Lindley-Poisson distribution [38].

Probability density function

The probability density function of the Exponentiated Exponential Pareto distribution is given as:

$$f(x)=\frac{\alpha\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}e^{-\beta\left(\frac{x}{p}\right)^{\theta}}\left[1-e^{-\beta\left(\frac{x}{p}\right)^{\theta}}\right]^{\alpha-1} \quad (1)$$

From the probability density function of equation (1), the first order differential equation can be obtained as:

$$f'(x)=\left[\frac{(\alpha-1)\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}\frac{e^{-\beta(x/p)^{\theta}}}{(1-e^{-\beta(x/p)^{\theta}})}-\frac{\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}+\frac{(\theta-1)}{x}\right]f(x) \quad (2)$$

The necessary condition for the existence of equation is $\alpha,\beta,\theta>0, 0<p<1$.

$$f''(x)=\left[\frac{(\alpha-1)\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}\frac{e^{-\beta(x/p)^{\theta}}}{(1-e^{-\beta(x/p)^{\theta}})}-\frac{\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}+\frac{(\theta-1)}{x}\right]f'(x)$$

$$+\left[\begin{array}{l}\frac{(\alpha-1)\beta\theta(\theta-1)}{px}\left(\frac{x}{p}\right)^{\theta-1}\frac{e^{-\beta(x/p)^{\theta}}}{(1-e^{-\beta(x/p)^{\theta}})}-\frac{(\alpha-1)\beta^2\theta^2}{px}\left(\frac{x}{p}\right)^{2\theta-1}\frac{e^{-\beta(x/p)^{\theta}}}{(1-e^{-\beta(x/p)^{\theta}})}\\ -\frac{(\alpha-1)\beta^2\theta^2}{px}\left(\frac{x}{p}\right)^{2\theta-1}\frac{e^{-2\beta(x/p)^{\theta}}}{(1-e^{-\beta(x/p)^{\theta}})}-\frac{\beta\theta(\theta-1)}{p^2}\left(\frac{x}{p}\right)^{\theta-2}-\frac{(\theta-1)}{x^2}\end{array}\right]f(x) \quad (3)$$

The necessary condition for the existence of above equation is $\alpha,\beta,\theta>0, 0<p<1$.. Equation (3) simplified using equation (2) as:

$$f''(x)=\frac{f'^2(x)}{f(x)}+\left[\begin{array}{c}\frac{1}{(\alpha-1)}\left\{\frac{f'(x)}{f(x)}+\frac{\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}-\frac{(\theta-1)}{x}\right\}^2+\left(\frac{(\theta-1)}{x}-\frac{\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta}\right)\\ \left\{\frac{f'(x)}{f(x)}+\frac{\beta\theta}{p}\left(\frac{x}{p}\right)^{\theta-1}-\frac{(\theta-1)}{x}\right\}\end{array}\right]f(x) \quad (4)$$

The necessary condition for the existence of equation (4) is $\alpha,\beta,\theta>0, 0<p<1$.

Quantile Function

$$Q(t)=p\left[\frac{1}{\beta}ln\left(\frac{1}{1-t^{\frac{1}{\alpha}}}\right)\right]^{\frac{1}{\theta}} \quad (5)$$

From the Quantile function of the Exponentiated Exponential differential, the first order differential equation can be obtained by differentiating equation (5) as:

$$Q'(t)=\frac{p}{\alpha\beta\theta\left(1-t^{\frac{1}{\alpha}}\right)}\left[\frac{1}{\beta}ln\left(\frac{1}{1-t^{\frac{1}{\alpha}}}\right)\right]^{\frac{1}{\theta}-1} \quad (6)$$

$$\alpha\beta\theta\left(1-t^{\frac{1}{\alpha}}\right)Q'(t)=p\left[\frac{1}{\beta}ln\left(\frac{1}{1-t^{\frac{1}{\alpha}}}\right)\right]^{\frac{1}{\theta}-1} \quad (7)$$

The necessary condition for the existence of above equation is $\alpha,\beta,\theta>0, 0<p<1$.

$$\alpha\beta\theta(1-t^{\frac{1}{\alpha}})\,ln(1-t^{\frac{1}{\alpha}})Q'(t)+p^{\frac{1}{\alpha}}t^{\frac{1}{\alpha}-1}Q(t)=0 \quad (8)$$

Some cases of the ordinary differential equation obtained for given values of the parameters are:

Table 1: Classes of differential equations obtained for the quantile function of Exponentiated Exponential Pareto distribution for different parameters.

α	β	θ	p	ODE
1	1	1	1	$(1-t)\,ln(1-t)Q'(t)+Q(t)=0$
1	1	1	2	$2(1-t)\,ln(1-t)Q'(t)+2Q(t)=0$
1	1	2	1	$2(1-t)\,ln(1-t)Q'(t)+Q(t)=0$
1	2	1	1	$2(1-t)\,ln(1-t)Q'(t)+Q(t)=0$
1	1	2	2	$2(1-t)\,ln(1-t)Q'(t)+2Q(t)=0$
1	2	2	1	$4(1-t)\,ln(1-t)Q'(t)+Q(t)=0$
1	2	1	2	$2(1-t)\,ln(1-t)Q'(t)+2Q(t)=0$
1	2	2	2	$4(1-t)\,ln(1-t)Q'(t)+2Q(t)=0$
2	1	1	1	$2(\sqrt{t}-t)\,ln(1-t)Q'(t)+Q(t)=0$
2	2	1	1	$4(\sqrt{t}-t)\,ln(1-t)Q'(t)+Q(t)=0$
2	1	1	2	$2(\sqrt{t}-t)\,ln(1-t)Q'(t)+\sqrt{2}Q(t)=0$
2	1	2	2	$4(\sqrt{t}-t)\,ln(1-t)Q'(t)+\sqrt{2}Q(t)=0$
2	2	1	1	$4(\sqrt{t}-t)\,ln(1-t)Q'(t)+Q(t)=0$
2	2	2	1	$8(\sqrt{t}-t)\,ln(1-t)Q'(t)+Q(t)=0$
2	2	1	2	$4(\sqrt{t}-t)\,ln(1-t)Q'(t)+\sqrt{2}Q(t)=0$
2	2	2	2	$8(\sqrt{t}-t)\,ln(1-t)Q'(t)+\sqrt{2}Q(t)=0$

Survival function

The survival function of the Exponentiated Exponential Pareto distribution is given as:

$$S(t)=1-\left[1-e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\right]^{\alpha} \qquad (10)$$

To get the first order ordinary differential equation for the survival function of the given distribution, differentiate equation (10) as:

$$S'(t)=-\frac{\alpha\beta\theta}{p}e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\left[1-e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\right]^{\alpha-1} \qquad (11)$$

The necessary condition for the existence of above equation is $\alpha,\beta,\theta>0,0<p<1$.

$$pS'(t)+\alpha\beta\theta(1-(1-S(t)^{\frac{1}{\alpha}})(1-(1-S(t))^{\frac{1}{\alpha}-1} \qquad (12)$$

With initial condition

$$S(1)=1-\left[1-e^{-\beta\left(\frac{1}{p}\right)^{\theta}}\right]^{\alpha} \qquad (13)$$

Table 2: For different values of parameters, the ODE can be obtained as:

α	ODE	Initial Condition
1	$pS'(t)+\beta\theta S(t)=0$	$S(1)=e^{-\beta\left(\frac{1}{p}\right)^{\theta}}$
2	$pS'(t)+2\beta\theta(\sqrt{1-S(t)})-2\beta\theta(1-S(t)=0$	$S(1)=1-\left[1-e^{-\beta\left(\frac{1}{p}\right)^{\theta}}\right]^{2}$

Inverse survival function

$$Q(t)=p\left[\frac{1}{\beta}ln\left(\frac{1}{1-(1-t)^{\frac{1}{\alpha}}}\right)\right]^{\frac{1}{\theta}} \qquad (14)$$

$$Q'(t)=-\frac{p(1-t)^{\frac{1}{\alpha}-1}}{\alpha\beta\theta(1-(1-t)^{\frac{1}{\alpha}})}\left[\frac{1}{\beta}ln\left(\frac{1}{1-(1-t)^{\frac{1}{\alpha}}}\right)\right]^{\frac{1}{\theta}-1} \qquad (15)$$

$$\alpha\beta\theta(1-(1-t)^{\frac{1}{\alpha}})\, ln(1-(1-t)^{\frac{1}{\alpha}})Q'(t) - \beta(1-t)^{\frac{1}{\alpha}-1}Q(t) = 0 \quad (16)$$

Table 3: Classes of differential equations obtained for the Inverse Survival function of Exponentiated Exponential Pareto distribution for different parameters.

α	β	θ	ODE
1	1	1	$t\, ln\, t \,.\, Q'(t) - Q(t) = 0$
1	1	2	$2(1-t)\, ln(1-t)Q'(t) + 2Q(t) = 0$
1	2	1	$2(1-t)\, ln(1-t)Q'(t) + Q(t) = 0$
1	2	2	$2(1-t)\, ln(1-t)Q'(t) + Q(t) = 0$
1	1	2	$2(1-t)\, ln(1-t)Q'(t) + 2Q(t) = 0$
1	2	2	$4(1-t)\, ln(1-t)Q'(t) + Q(t) = 0$
1	2	1	$2(1-t)\, ln(1-t)Q'(t) + 2Q(t) = 0$
1	2	2	$4(1-t)\, ln(1-t)Q'(t) + 2Q(t) = 0$
2	1	1	$2(\sqrt{t}-t)\, ln(1-t)Q'(t) + Q(t) = 0$
2	2	1	$4(\sqrt{t}-t)\, ln(1-t)Q'(t) + Q(t) = 0$
2	1	1	$2(\sqrt{t}-t)\, ln(1-t)Q'(t) + \sqrt{2}Q(t) = 0$
2	1	2	$4(\sqrt{t}-t)\, ln(1-t)Q'(t) + \sqrt{2}Q(t) = 0$
2	2	1	$4(\sqrt{t}-t)\, ln(1-t)Q'(t) + Q(t) = 0$
2	2	2	$8(\sqrt{t}-t)\, ln(1-t)Q'(t) + Q(t) = 0$
2	2	1	$4(\sqrt{t}-t)\, ln(1-t)Q'(t) + \sqrt{2}Q(t) = 0$
2	2	2	$8(\sqrt{t}-t)\, ln(1-t)Q'(t) + \sqrt{2}Q(t) = 0$

Hazard function

$$h(t) = \frac{\frac{\alpha\beta\theta}{p}\left(\frac{t}{p}\right)^{\theta-1} e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\left(1-e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\right)^{\alpha-1}}{1-\left(1-e^{-\beta\left(\frac{t}{p}\right)^{\theta}}\right)^{\alpha}} \quad (17)$$

Differentiate the above equation to obtain an ordinary differential equation which is independent of the power of the parameters.

$$h'(t) = \frac{h(t)}{1-(1-e^{-\beta(t/p)^{\theta}})^{\alpha}}\left[\frac{\alpha\beta\theta}{p}.\frac{e^{-\beta(t/p)^{\theta}}\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha-1}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)} + \frac{\alpha\beta\theta}{p}.\frac{e^{-\beta(t/p)^{\theta}}\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha-1}}{1-\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha}}\right] \quad (18)$$

$$h'(t) = \left[\frac{\alpha\beta\theta}{p}.\frac{e^{-\beta(t/p)^{\theta}}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)} + \left(\frac{t}{p}\right)^{1-\theta}\frac{h(t)}{1-\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha}} + \frac{\theta-1}{t} - \frac{\beta\theta}{p}\right]h(t) \quad (19)$$

$$h'' \frac{h'^2(t)}{h(t)}\left[-\frac{\alpha^2\beta^2\theta}{p^2}\frac{e^{-\beta(t/p)^{\theta}}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)}\left\{1-\frac{e^{-\beta(t/p)^{\theta}}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)}\right\}+\frac{(1-\theta)\left(\frac{p}{t}\right)^{\theta}}{\left(1-\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha}\right)}+\left(\frac{t}{p}\right)^{2(1-\theta)}+\frac{1-\theta}{t^2}\right] \quad (20)$$

Reversed hazard function

$$j(t) = \frac{\frac{\alpha\beta\theta}{p}\left(\frac{t}{p}\right)^{\theta-1} e^{-\beta\left(\frac{t}{p}\right)\theta}\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha-1}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)^{\alpha}} \quad (21)$$

The above equation can be simplified by the following way:

$$j(t) = \frac{\frac{\alpha\beta\theta}{p}\left(\frac{t}{p}\right)^{\theta-1} e^{-\beta\left(\frac{t}{p}\right)\theta}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)} \quad (22)$$

$$j'(t) = \left[\frac{\beta\theta}{p}\left\{\frac{\theta-1}{t} - \frac{\beta\theta}{p}\right\} - \frac{e^{-\beta\left(\frac{t}{p}\right)\theta}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)}\right]j(t) \quad (23)$$

$$j'' \frac{j'^2(t)}{j(t)}\left[\frac{\beta\theta}{p}\left\{\frac{1-\theta}{t^2}-\frac{e^{-2\beta\left(\frac{t}{p}\right)\theta}}{\left(1-e^{-\beta(t/p)^{\theta}}\right)^2}\right\}+\frac{j(t)}{\alpha}\left(\frac{t}{p}\right)^{1-\theta}\right] \quad (24)$$

Conclusion

In this work, classes of ordinary differential equation have been used to get probability density function (pdf), quantile function (QF), survival function (SF), Inverse survival function (ISF), Hazard function (HF) and reversed hazard function (RHF) of the Exponentiated Exponential Pareto distribution. The differentiation technique involves modified product rule, quotient rule and some efficient algebraic simplifications were used to derive the specific classes of ODE. In all, the parameters that characterize the nature, direction and distinctiveness of ODEs, and the range regulates the existence of ODEs. Moreover, the convolution of the ODEs highly based on the values of the parameters. Further, several methods can be used to obtain desirable solutions to the ODEs. This method may not be applicable to the distribution whose pdf or CDF are either the distribution domain having singular points or not differentiable.

For all the probability functions the ode can be obtained for some specific values of the distribution. Numerous analytic, semi analytics and numerical methods can be applied for the solution of respective differential equation. Further, in order to create link between ode and the probability distribution comparison can be made with two or more solution methods.

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