International Journal of Statistics and Applied Mathematics 2021; 6(5): 155-166

International Journal of Statistics and Applied Mathematics

**ISSN:** 2456-1452
Maths 2021; 6(5): 155-166

www.mathsjournal.com
Received: 10-06-2021
Accepted: 30-07-2021

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# Inference for the generalized inverse Lindley distribution under type-II censored data

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**DOI:** https://doi.org/10.22271/maths.2021.v6.i5b.739

**Abstract**
In this paper, we consider the classical and the Bayesian inferences for the generalized inverse Lindley (GIL) distribution and their corresponding reliability characteristics (reliability function and hazard rate function) under the type-II censoring scheme. In the classical setup, first we obtain the maximum likelihood estimator for the unknown parameters of the distribution and their corresponding reliability characteristics. Further, we consider symmetric (squared error) and asymmetric (LINEX and general entropy) loss functions for the estimation of parameters and their corresponding reliability characteristics under the Bayesian paradigm. The performances of various derived estimators were recorded using Markov chain Monte Carlo (MCMC) simulation technique in Open BUGS for different sample sizes under type-II censoring schemes. Finally, a real data set is provided to illustrate the computation of various estimators.

**Keywords:** Generalized inverse Lindley distribution, Type-II censoring, maximum likelihood estimator (MLE), Bayesian estimation, Symmetric and asymmetric loss functions, MCMC

## Introduction
In the random phenomenon of life, there are several life-time models are available for analysing uncertainty of various fields. Initially, exponential distribution played an extensive role to analyse the life-time data because of its simplicity and analytical flexibility. Although, the exponential distribution has limitations in the study of life-time models due to its constant hazard rate, which is not appropriate to analyse the life-time models. Therefore, many of the researchers have proposed new life-time distributions which overcome the limitation of constant hazard rate. Such few specific distributions are Weibull distribution, gamma distribution, Lindley distribution, lognormal distribution which are extension of exponential distribution. The Lindley distribution was proposed by Lindley (1958) [17], in the context of Bayes theorem as a counter example of fiducial statistics and also shows that non constant hazard rate distribution, Ghitany *et al.* (2008) [8] worked on statistical properties of Lindley distribution. Nadrajadeh *et al.* (2011) and Ghitany *et al.* (2013) [9] proposed power Lindley and the extension of Lindley distribution which is also called generalized Lindley distribution respectively. Sharma *et al.* (2016) [21] extended the inverse Lindley distribution by applying the power transformation of given method by Gupta and Kundu (2009) [11] and found the generalized inverse Lindley (GIL) distribution. In the recent past, Kumar *et al.* (2020) [12] discussed GIL distribution based on generalized order statistics.
The probability density function (PDF) $f(x)$ and Cumulative distribution function (CDF) $F(x)$ of the GIL distribution are given, respectively, by

$$f(x;\alpha,\theta)=\frac{\alpha\theta^2}{(1+\theta)}\left[\frac{(1+x^{\alpha})}{x^{2\alpha+1}}\right]e^{-\frac{\theta}{x^{\alpha}}}, x>0, \sim\alpha,\theta>0 \quad (1.1)$$

and

$$F(x)=\left[1+\frac{\theta}{(1+\theta)}\frac{1}{x^{\alpha}}\right]e^{-\frac{\theta}{x^{\alpha}}} \quad (1.2)$$

The corresponding reliability characteristics $R(t)$ and $H(t)$ of this distribution at same time $t > 0$ are given, respectively, by

$$R(t) = 1 - \left[1 + \frac{\theta}{(1+\theta)}\frac{1}{t^{\alpha}}\right] e^{-\frac{\theta}{t^{\alpha}}}, \tag{1.3}$$

And

$$H(t) = \frac{\alpha\theta^2(1+t^{\alpha})e^{-\frac{\theta}{t^{\alpha}}}}{t^{\alpha+1}\left[(1+\theta)t^{\alpha}\left(e^{-\frac{\theta}{t^{\alpha}}}-1\right)-\theta\right]} \tag{1.4}$$

Where, $\alpha$ is a shape parameter and $\theta$ is a scale parameter of the GIL distribution.
In reliability or the process of carrying out a life test, the data are frequently censored due to the restrictions of time and cost. Among the different censoring schemes, Type-I (time) and Type-II (failure) are the most popular censoring schemes. Here, we are using type-II censored sample for the estimation purpose because the experiment will be terminated in this censoring scheme after obtaining a prefixed number of failure $(r)$ out of $n$ units put on the life testing experiment. The type-I and type-II censoring schemes was discussed by Nelson (2003) [19] and Lawless (2011) [16] for the problem of estimation. Singh *et al.* (2005) [22] and Kundu and Howlader (2010) [15] discussed different lifetime models under type-II censoring for the parameter estimation in the Bayesian inference. Goyal *et al.* (2019) discussed a new lifetime model in both classical and Bayesian technique under type-II censored data.
The rest of the paper is organized as follows, In Section 2, the Maximum likelihood estimations are obtained for the parameters and the reliability characteristics of GIL distribution are presented. The Bayes estimation under symmetric (squared error) loss function and asymmetric (LINEX, general entropy) loss functions are obtained in Section 3. In Section 4, interval estimation under type-II censoring scheme for classical (confidence interval) and the Bayesian (credible and HPD intervals) paradigms. A simulation results under type-II censored sample is elaborated for both paradigms in Section 5. Section 6 deals with a real data set analysis for illustration purpose. Lastly, the conclusions appear in section 7.

**Maximum Likelihood Function**

Suppose $n$ items are put on test and a test is terminated after the first $r$ ordered observations are recorded. Let $X_{(1)} \le X_{(2)} \le X_{(3)} \le \cdots . \le X_{(r)}, 0 < r < n$, be the failure time of first r ordered observations and the time started at $X_{(1)} \ge 0$. Obviously $(n-r)$ items survived until $X_{(r)}$. The likelihood function based on the type-II censored sample is given by-

$$l(\alpha,\theta|X) = \frac{n!}{(n-r)!}\prod_{i=1}^{r} f\left(x_{(i)};\alpha,\theta\right)\left[1 - F\left(x_{(r)};\alpha,\theta\right)\right]^{n-r} \tag{2.1}$$

Substituting Equations (1.1) and (1.2) into Equation (2.1), the likelihood function is-

$$l(\alpha,\theta|X) = \frac{n!}{(n-r)!}\prod_{i=1}^{r}\left[\frac{\alpha\theta^2}{(1+\theta)}\left(\frac{\left(1+x_{(i)}^{\alpha}\right)}{x_{(i)}^{2\alpha+1}}\right)e^{-\frac{\theta}{x_{(i)}^{\alpha}}}\right]\left[1 - \left(1+\frac{\theta}{(1+\theta)}\frac{1}{x_{(r)}^{\alpha}}\right)e^{-\frac{\theta}{x_{(r)}^{\alpha}}}\right]^{n-r} \tag{2.2}$$

After taking logarithm, Equation (2.2) will be:

$$L = \ln l(\alpha,\theta|\mathbf{x})$$

$$= A + r\ln\alpha\,\theta^2 - r\ln(1+\theta) + \sum_{i=1}^{r}\left(1+x_{(i)}^{\alpha}\right) - (2\alpha+1)\sum_{i=1}^{r}\ln x_{(i)} - \theta\sum_{i=1}^{r}x_{(i)}^{-\alpha} + (n-r)ln\left[1 - \left(1+\frac{\theta}{(1+\theta)}\frac{1}{x_{(r)}^{\alpha}}\right)e^{-\frac{\theta}{x_{(r)}^{\alpha}}}\right]$$
(2.3)

where,

$$A = \ln\left(\frac{n!}{(n-r)!}\right)$$

Further, partial derivative of the Equation (2.3) under the parameter $\alpha$ and $\theta$ and equate as zero to find the ML estimates of the unknown parameters as

$$\frac{\partial L}{\partial\alpha} = \frac{r}{\alpha} + \sum_{i=1}^{r}\frac{\ln\left(x_{(i)}\right)x_{(i)}^{\alpha}}{\left(1+x_{(i)}^{\alpha}\right)} - 2\sum_{i=1}^{r}\ln x_{(i)} + \theta\sum_{i=1}^{r}\ln x_{(i)}\,x_{(i)}^{-\alpha}$$

$$-\frac{(n-r)e^{-\frac{\theta}{x_{(r)}^{\alpha}}}\ln\left(x_{(r)}\right)\left(x_{(r)}^{\alpha}+1\right)\theta^2\ln x_{(r)}^{-2\alpha}}{(1+\theta)} \tag{2.4}$$

And

$$\frac{\partial L}{\partial \theta}=\frac{2r}{\theta}-\frac{r}{(1+\theta)}-\sum_{i=1}^{r} x_{(i)}^{-\alpha}+\frac{(n-r)e^{-\frac{\theta}{x_{(r)}^{\alpha}}}\theta\left(1+\theta+x_{(r)}^{\alpha}(2+\theta)\right)\ln x_{(r)}^{-2\alpha}}{(1+\theta)^2} \tag{2.5}$$

Because of its abstract nature, the likelihood equations could not generate the MLEs of α and θ, as show by Equations (2.4) and (2.5). For that reason, heuristically obtaining the MLEs of the parameters is impossible. As a result, any numerical approximation approach, such as Newton-Raphson (N-R) method, fixed point iterations and so on can be used. We have been evaluating the MLE of the parameters using the N-R technique so far.

Following are the expressions for the MLEs of reliability characteristics:

$$\widehat{R(t)}=1-\left[1+\frac{\hat{\theta}}{(1+\hat{\theta})}\frac{1}{t^{\hat{\alpha}}}\right]e^{-\frac{\hat{\theta}}{t^{\hat{\alpha}}}} \tag{2.6}$$

and

$$\widehat{H(t)}=\frac{\hat{\alpha}\hat{\theta}^2(1+t^{\hat{\alpha}})e^{-\frac{\hat{\theta}}{t^{\hat{\alpha}}}}}{t^{\hat{\alpha}+1}\left[(1+\hat{\theta})t^{\hat{\alpha}}\left(e^{-\frac{\hat{\theta}}{t^{\hat{\alpha}}}}-1\right)-\hat{\theta}\right]} \tag{2.7}$$

**Bayesian Inference**

In this section, we obtain Bayes estimation of unknown parameters of the distribution and their reliability characteristics under the symmetric (squared error) and asymmetric (LINEX and general entropy) loss functions. Some authors used symmetric/asymmetric or both loss functions for parameter estimation in Bayesian inference (see, Soliman *et al.* (2012) [24], Soliman *et al.* (2013) [25] and Goyal *et al.* (2019) [10].

For GIL distribution, we consider both parameters shape as well as scale are unknown. To estimate these parameters, we take shape parameter α as informative prior distribution Gamma, ie., $\alpha \sim Gamma(a,b)$ and non-informative prior distribution for scale parameter θ. Therefore, the prior distribution for α is

$$\pi(\alpha)=\frac{a^b\alpha^{b-1}e^{-a\alpha}}{\Gamma b}, \alpha>0, a,b>0 \tag{3.1}$$

and the prior distribution for θ is,

$$\pi(\theta)=\frac{1}{\theta}, \theta>0. \tag{3.2}$$

The joint prior of α and θ is,

$$\pi(\alpha,\theta)=\frac{a^b\alpha^{b-1}e^{-a\alpha}}{\theta\Gamma b}, \alpha,\theta>0, a,b>0 \tag{3.3}$$

Combining the likelihood function (2.2) and prior density (3.3), the posterior density is

$$\Pi(\alpha,\theta|x)=\frac{\pi(\alpha,\theta)l(\alpha,\theta|x)}{\int_0^\infty\int_0^\infty \pi(\alpha,\theta)l(\alpha,\theta|x)d\alpha d\theta} \tag{3.4}$$

$$\Pi(\alpha,\theta|x)=\frac{K_1}{K_0}$$

where,

$$K_1=\prod_{i=1}^{r}\left[\frac{\alpha\theta^2}{(1+\theta)}\left(\frac{\left(1+x_{(i)}^{\alpha}\right)}{x_{(i)}^{2\alpha+1}}\right)e^{-\frac{\theta}{x_{(i)}^{\alpha}}}\right]\left[1-\left(1+\frac{\theta}{(1+\theta)}\frac{1}{x_{(r)}^{\alpha}}\right)e^{-\frac{\theta}{x_{(r)}^{\alpha}}}\right]^{n-r}\frac{a^b\alpha^{b-1}e^{-a\alpha}}{\theta\Gamma b}$$

&

$$K_0=\int_0^\infty\int_0^\infty K_1\, d\alpha d\theta$$

**Bayesian estimate under asymmetric loss function**

The asymmetric loss function was employed when the amount of losses due to underestimation and overestimation were not identical. A useful asymmetric loss function known as the LINEX loss function and a suitable alternative to the modified LINEX loss function known as general entropy loss function (GELF).

$L(\hat{\delta},\delta) = (\hat{\delta}-\delta)^2, \hat{\delta} \in D, \delta \in \Theta,$

where, $\hat{\delta}$ is the Bayes estimator of $\delta$. $D$ is a decision space and $\Theta$ is parameter space.
The Bayes estimators $\tilde{\alpha}_{BS}$ and $\tilde{\theta}_{BS}$ of parameters $\alpha$ and $\theta$, respectively, are

$$\tilde{\alpha}_{BS} = \int_0^\infty \int_0^\infty \alpha\Pi(\alpha,\theta|x)d\alpha d\theta$$

$$\tilde{\alpha}_{BS} = \frac{1}{K_0}\int_0^\infty \alpha\left(\int_0^\infty K_1 d\theta\right)d\alpha \tag{3.1.1}$$

and

$$\tilde{\theta}_{BS} = \int_0^\infty \int_0^\infty \theta\Pi(\alpha,\theta|x)d\alpha\, d\theta$$

$$\tilde{\theta}_{BS} = \frac{1}{K_0}\int_0^\infty \theta\left(\int_0^\infty K_1 d\alpha\right)d\theta. \tag{3.1.2}$$

The Bayes estimator $\tilde{R}_{BS}$ and $\tilde{H}_{BS}$ of the reliability function $R(t)$ and hazard rate function $H(t)$, respectively, are

$$\tilde{R}_{BS} = \int_0^\infty \int_0^\infty \left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)\Pi(\alpha,\theta)d\alpha\, d\theta$$

$$\tilde{R}_{BS} = \frac{1}{K_0}\int_0^\infty \int_0^\infty \left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)K_1\, d\alpha d\theta \tag{3.1.3}$$

and

$$\tilde{H}_{BS} = \int_0^\infty \int_0^\infty \left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)\Pi(\alpha,\theta|x)d\alpha\, d\theta$$

$$\tilde{H}_{BS} = \frac{1}{K_0}\int_0^\infty \int_0^\infty \left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)K_1 d\alpha\, d\theta \tag{3.1.4}$$

**Bayesian estimate under asymmetric loss function**

When the magnitude of overestimation and underestimation are not equal then we used the asymmetric loss function. In the asymmetric loss function, we consider LINEX loss function and General Entropy loss function.
*LINEX Loss function*: The LINEX loss function is defined as

$$L(\hat{\delta}-\delta) \propto e^{c(\hat{\delta}-\delta)} - c(\hat{\delta}-\delta) - 1, c \neq 0, \hat{\delta} \in D, \delta \in \Theta.$$

The Bayes estimator $\hat{\delta}_{BL}$ of $\delta$ under the LINEX loss function is

$$\hat{\delta}_{BL} = -\frac{1}{c}\ln\left[E_\delta\left(exp(-c\delta)\right)\right], \text{(3.2.1)}$$

provided, $E_\delta\left(exp(-c\delta)\right)$ exists and finite.
The Bayes estimators $\tilde{\alpha}_{BL}$ and $\tilde{\theta}_{BL}$ of parameters $\alpha$ and $\theta$ under LINEX loss function Equation (3.2.1), respectively, are

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty \int_0^\infty exp(-c\alpha)\right]\Pi(\alpha,\theta|x)d\alpha d\theta$$

$$\tilde{\alpha}_{BL} = -\frac{1}{c}\ln\left[\frac{1}{K_0}\int_0^\infty exp(-c\alpha)\left(\int_0^\infty K_1 d\alpha\right)d\theta\right] \tag{3.2.2}$$

and

$$\tilde{\theta}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty \int_0^\infty exp(-c\theta)\right]\Pi(\alpha,\theta|x)d\alpha d\theta$$

$$\tilde{\theta}_{BL} = -\frac{1}{c}\ln\left[\frac{1}{K_0}\int_0^\infty exp(-c\theta)\left(\int_0^\infty K_1 d\theta\right)d\alpha\right] \tag{3.2.3}$$

The Bayes estimators $\tilde{R}_{BL}$ and $\tilde{H}_{BL}$ of the reliability function $R(t)$ and hazard function $H(t)$, respectively, are

$$\tilde{R}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty\int_0^\infty exp\left(-c\left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)\right)\right]\Pi(\alpha,\theta|x)\mathrm{d}\alpha d\theta$$

$$\tilde{R}_{BL} = -\frac{1}{c}\ln\left[\frac{1}{K_0}\int_0^\infty\int_0^\infty exp\left(-c\left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)\right)K_1\right]\mathrm{d}\alpha d\theta \quad (3.2.4)$$

and

$$\tilde{H}_{BL} = -\frac{1}{c}\ln\left[\int_0^\infty\int_0^\infty exp\left(-c\left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)\right)\right]\Pi(\alpha,\theta|x)\mathrm{d}\alpha d\theta$$

$$\tilde{H}_{BL} = -\frac{1}{c}\ln\left[\frac{1}{K_0}\int_0^\infty\int_0^\infty exp\left(-c\left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)\right)K_1\right]\mathrm{d}\alpha d\theta. \quad (3.2.5)$$

*General Entropy Loss Function* (*GELF*): The GELF is expressed as

$$L(\hat{\delta},\delta) \propto \left(\frac{\hat{\delta}}{\delta}\right)^q - q\ln\left(\frac{\hat{\delta}}{\delta}\right) - 1, q \neq 0, \hat{\delta} \in D, \delta \in \Theta.$$

The Bayes estimator $\hat{\delta}_{BG}$ of $\delta$ under GELF is

$$\hat{\delta}_{BG} = [E_\delta(\delta^{-q})]^{-\frac{1}{q}}, \quad (3.2.6)$$

provided, $E_\delta(\delta^{-q})$ exists and finite.
The Bayes estimators $\tilde{\alpha}_{BG}$ and $\tilde{\theta}_{BG}$ of parameters $\alpha$ and $\theta$ under GELF Equation (3.2.2), respectively, are

$$\tilde{\alpha}_{BG} = \left[\int_0^\infty\int_0^\infty(\alpha^{-q})\Pi(\alpha,\theta|x)d\alpha\, d\theta\right]^{-\frac{1}{q}}$$

$$\tilde{\alpha}_{BG} = \left[\frac{1}{K_0}\int_0^\infty \alpha^{-q}\left(\int_0^\infty K_1 d\theta\right)d\alpha\right]^{-\frac{1}{q}} \quad (3.2.7)$$

and

$$\tilde{\theta}_{BG} = \left[\int_0^\infty\int_0^\infty(\theta^{-q})\Pi(\alpha,\theta|x)d\alpha\, d\theta\right]^{-\frac{1}{q}}$$

$$\tilde{\theta}_{BG} = \left[\frac{1}{K_0}\int_0^\infty \theta^{-q}\left(\int_0^\infty K_1 d\alpha\right)d\theta\right]^{-\frac{1}{q}} \quad (3.2.8)$$

The Bayes estimators $\tilde{R}_{BG}$ and $\tilde{H}_{BG}$ of the reliability function $R(t)$ and hazard function $H(t)$, respectively, are

$$\tilde{R}_{BG} = \left[\int_0^\infty\int_0^\infty\left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)^{-q}\Pi(\alpha,\theta|x)d\alpha\, d\theta\right]^{-\frac{1}{q}}$$

$$\tilde{R}_{BG} = \left[\frac{1}{K_0}\int_0^\infty\int_0^\infty\left(1-\left[1+\frac{\theta}{(1+\theta)}\frac{1}{t^\alpha}\right]e^{-\frac{\theta}{t^\alpha}}\right)^{-q}K_1 d\alpha\, d\theta\right]^{-\frac{1}{q}} \quad (3.2.9)$$

and

$$\tilde{H}_{BG} = \left[\int_0^\infty\int_0^\infty\left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)^{-q}\Pi(\alpha,\theta|x)d\alpha\, d\theta\right]^{-\frac{1}{q}}$$

$$\widetilde{H}_{BG} = \left[\frac{1}{K_0}\int_0^\infty \int_0^\infty \left(\frac{\alpha\theta^2(1+t^\alpha)e^{-\frac{\theta}{t^\alpha}}}{t^{\alpha+1}\left[(1+\theta)t^\alpha\left(e^{-\frac{\theta}{t^\alpha}}-1\right)-\theta\right]}\right)^{-q} K_1 d\alpha\, d\theta\right]^{-\frac{1}{q}} \quad (3.2.10)$$

Equations (3.1.1) to (3.1.4), Equations (3.2.2) to (3.2.5) and Equations (3.2.7) to (3.2.10) are not computed analytically. Therefore, for these kinds of equations, we generate the samples from equation (3.4) from one of the simulation technique Markov Chain Monte Carlo (MCMC) and compute these Bayes estimators under symmetric and asymmetric loss functions (see, El-Din *et al.* (2017) [6], Riad *et al.* (2020) [20] and Almongy *et al.* (2021) [2].

**Interval estimation under type-II censoring scheme**

In this section, we deal with the confidence intervals (CIs) for the parameters under the classical setup. In the Bayesian estimation, we obtained credible intervals and highest posterior density (HPD) for both the parameters. The intervals under classical and Bayesian setup are as follows

**Confidence interval**

In the classical setup, the CIs can be obtained the diagonal elements of the inverse Fisher information matrix $I^{-1}(\hat{\alpha},\hat{\theta})$ that gives the asymptotic variance for the parameters α and θ respectively. Thus, the two sided $100(1-\eta)\%$ confidence interval for α and θ can be defined respectively as

$$\left[\hat{\alpha} - Z_{\eta/2\sqrt{var(\hat{\alpha})}}, \hat{\alpha} + Z_{\eta/2\sqrt{var(\hat{\alpha})}}\right],$$

and

$$\left[\hat{\theta} - Z_{\eta/2\sqrt{var(\hat{\theta})}}, \hat{\theta} + Z_{\eta/2\sqrt{var(\hat{\theta})}}\right].$$

Where, $Z_{\eta/2}$ is a standard normal variate.

The Fisher information matrix can be defined as

$$I(\hat{\alpha},\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 L}{\partial\alpha^2} & -\frac{\partial^2 L}{\partial\alpha\,\partial\theta} \\ -\frac{\partial^2 L}{\partial\theta\,\partial\alpha} & -\frac{\partial^2 L}{\partial\theta^2}\end{bmatrix}_{(\hat{\alpha},\hat{\theta})}$$

where,

$$\frac{\partial^2 L}{\partial\alpha^2} = -\frac{r}{\alpha^2} + \sum_{i=1}^r \frac{\ln^2(x_{(i)})x_{(i)}^\alpha}{\left(1+x_{(i)}^\alpha\right)^2} - \theta\sum_{i=1}^r \ln^2 x_{(i)}\, x_{(i)}^{-\alpha} + \frac{(n-r)e^{-\frac{\theta}{x_{(r)}^\alpha}}\ln(x_{(r)})^2\left(x_{(r)}^{2\alpha}-x_{(r)}^\alpha(\theta-2)-\theta\right)\theta^2 \ln x_{(r)}^{-2\alpha}}{(1+\theta)}, \quad (4.1.1)$$

$$\frac{\partial^2 L}{\partial\theta^2} = -\frac{2r}{\theta^2} + \frac{r}{(1+\theta)^2} + \frac{(n-r)e^{-\frac{\theta}{x_{(r)}^\alpha}}\theta\left(2x_{(r)}^{2\alpha}-\theta(1+\theta)^2-x_{(r)}^\alpha(3\theta^2+\theta^2+\theta-1)\right)\ln x_{(r)}^{-2\alpha}}{(1+\theta)^2} \quad (4.1.2)$$

and

$$\frac{\partial^2 L}{\partial\alpha\partial\theta} = \frac{\partial^2 L}{\partial\theta\partial\alpha} = \sum_{i=1}^r \ln x_{(i)}\, x_{(i)}^{-\alpha} + \frac{(n-r)e^{-\frac{\theta}{x_{(r)}^\alpha}}\ln(x_{(r)})\left(x_{(r)}^\alpha+1\right)\theta\left(\theta(1+\theta)-x_{(r)}^\alpha(2+\theta)\right)\ln x_{(r)}^{-2\alpha}}{(1+\theta)^2} \quad (4.1.3)$$

**4.2 *Credible interval and HPD interval***

In the Bayesian inference, let parameter τ is a random variable and the probability for this parameter τ lies within the specified intervals. The credible and the highest posterior density (HPD) intervals addressed by Edwards *et al.* (1963) [6]. All the Bayesian credible intervals, the HPD interval is the shortest interval. The HPD interval for parameter τ based on the simulation method MCMC samples, ie., $\tau_{(1)}, \tau_{(2)}, \ldots, \tau_{(M)}$ discussed by Chen and Saho (1999) [5]. For the parameter τ, the credible interval $100(1-\eta)\%$ is obtained as

$$((\tau\_\{(1)\}, \tau\_\{[(1-\eta)M]+1\}), \ldots, (\tau\_\{[M\eta]\}, \tau\_\{M\}),$$

where, $[K]$ defines the largest integer value which is less than or equal to $K$. Therefore, for the parameter τ, the shortest length interval is the HPD interval. There are few rechears who discussed HPD interval in very detailed form (see, Edward *et al.* (1963), Box and Tiao (1973) [3] and Sinha (1987) [23].

**Simulation study**

In this section, we perform Markov chain Monte Carlo (MCMC) simulation method for the GIL distribution under the type-II censored sample with the help of OpenBUGS software (see, Thomas *et al.* (2006) [28], Thomas (2010) [27] and Kumar *et al.* (2010) [14]. The propose of this simulation data analysis is to validate the performance of the different estimator presented in this article. In the classical setup, the estimators are measured by MLE with their mean square error (MSE) and the estimators along with their MSEs under different two loss functions which are symmetric (squared error) and asymmetric (LINEX, general entropy) loss functions are obtained in the Bayesian paradigm. In the interval estimation, the confidence interval in the classical setup and the credible and HPD intervals under the Bayesian paradigm are also obtained. For the parameters $(\alpha, \theta)$ ~ (1.5,1.5), we generate different posterior samples of the different values on n and r with the Gibbs sampling technique of MCMC method under the OpenBUGS software. Some of the researchers used MCMC technique by OpenBUGS software (see, Kumar *et al.* (2012) [13], Adegoke *et al.* (2018) [1], Chaudhary *et al.* (2020) [4] and Srivastava *et al.* (2020) [26]. After that, from those samples, we estimate the parameters of GIL distribution and their reliability characteristics under classical and Bayesian inference. In classical, we estimate MLE along with their MSEs and also estimates the parameters under two different loss functions in the Bayesian set up. The reliability characteristics estimates and their MSEs are obtained in the both inference methods. In Bayesian, there are two choices for asymmetric loss functions (general entropy, $(q = -2, q = 2)$ and LINEX $(c = -2, c = 2)$). For the interval estimation, 5% level of significance we calculated, confidence interval in classical and credible and HPD interval under the Bayesian paradigm.

**Table 1:** MLE and Bayes estimates of $\alpha$ with their MSEs under type-II censoring scheme with $(\alpha, \theta)$~(1.5,1.5).

| n | r | MLE (MSE) | BS (MSE) | BG (MSE) | | BL (MSE) | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 25 | 10 | 1.7261 (0.2566) | 1.4745 (0.0012) | 1.4117 (0.0086) | 1.4936 (0.0057) | 1.4179 (0.0075) | 1.5311 (0.0015) |
| | 15 | 1.6391 (0.1287) | 1.0188 (0.2321) | 0.9637 (0.2882) | 1.0363 (0.2156) | 0.9839 (0.2669) | 1.0561 (0.1978) |
| | 25 | 1.5812 (0.0729) | 1.6682 (0.0287) | 1.6138 (0.0134) | 1.6859 (0.0349) | 1.6108 (0.0127) | 1.7295 (0.0532) |
| 50 | 10 | 1.7587 (0.2935) | 1.8992 (0.1597) | 1.8810 (0.1456) | 1.9052 (0.1645) | 1.8766 (0.1422) | 1.9221 (0.1785) |
| | 15 | 1.6701 (0.1582) | 2.0497 (0.3025) | 2.0351 (0.2866) | 2.0545 (0.3078) | 2.0299 (0.2812) | 2.0695 (0.3247) |
| | 25 | 1.5857 (0.0619) | 0.8604 (0.4117) | 0.8198 (0.4633) | 0.8748 (0.3955) | 0.8364 (0.4441) | 0.8888 (0.3831) |
| | 50 | 1.5422 (0.0307) | 1.5794 (0.0065) | 1.5516 (0.0028) | 1.5886 (0.0080) | 1.5509 (0.0027) | 1.6092 (0.0121) |
| 75 | 10 | 1.8032 (0.3512) | 2.0993 (0.3594) | 2.0863 (0.3440) | 2.1036 (0.3645) | 2.0814 (0.3381) | 2.1175 (0.3815) |
| | 15 | 1.6813 (0.1544) | 1.9041 (0.1634) | 1.8966 (0.1574) | 1.9065 (0.1654) | 1.8946 (0.1558) | 1.9134 (0.1711) |
| | 25 | 1.5967 (0.0699) | 2.0616 (0.3157) | 2.0549 (0.3082) | 2.0638 (0.3182) | 2.0525 (0.3055) | 2.0707 (0.3260) |
| | 50 | 1.5413 (0.0282) | 0.3199 (1.3937) | 0.3053 (1.4272) | 0.3298 (1.3746) | 0.3136 (1.4074) | 0.3368 (1.3665) |
| | 75 | 1.5271 (0.0202) | 1.7035 (0.0415) | 1.6867 (0.0350) | 1.70911 (0.0438) | 1.6847 (0.0342) | 1.7225 (0.0496) |

BS= Squared error; BG= General entropy; BL= LINEX; MSE=Mean square error.

In order to calculate MSEs under classical and with different loss functions under the Bayesian paradigm, we have replicated our results 30000 times. All the results of the MCMC method are reported in Tables (1)-(6). From Tables (1) to (6), we see the different choices of $(n, r)$ as (25,10), (25,15), (25,25), (50,10), (50,15), (50,25), (50,50), (75,10), (75,15), (75,25), (75,50) and (75,75). The parameters value of $(\alpha, \theta)$ as (1.5,1.5) show the estimates for both the parameters $(\alpha, \theta)$ along with their MSEs under classical and Bayesian paradigm (see Tables (1) and (2) respectively). In Tables (4) and (5) show the estimators of reliability characteristics $(R(t), H(t))$ at $t = 1$ along with their MSEs in classical and Bayesian paradigm respectively. In classical set up, the estimators of parameters $(\alpha, \theta)$ and reliability characteristics $(R(t), H(t))$ along with their MSEs are obtained under type-II censoring. With the different loss functions (symmetric (squared error) and asymmetric (LINEX, general entropy)), the estimators of parameters $(\alpha, \theta)$ and reliability characteristics $(R(t), H(t))$ along with their MSEs are obtained under type-II censoring scheme in the Bayesian paradigm. There are two choices for every asymmetric loss functions, for general entropy loss function $(q = -2, q = 2)$ and for LINEX loss function $(c = -2, c = 2)$. In the interval estimations, confidence intervals (lower limits and upper limits) also obtained in the classical setup. The credible and HPD intervals (lower limits and upper limits) also obtained in the Bayesian paradigm. Tables (5) and (6) show the interval estimation for both the parameters $\alpha$ and $\theta$ respectively.

**Table 2:** MLE and Bayes estimates of $\theta$ with their MSEs under type-II censoring scheme with $(\alpha, \theta)$~(1.5,1.5).

| n | r | MLE(MSE) | BS (MSE) | BG (MSE) | | BL (MSE) | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 25 | 10 | 1.4727 (0.1008) | 1.9606 (0.2122) | 1.9595 (0.2112) | 1.9610 (0.2125) | 1.9592 (0.2109) | 1.9619 (0.2134) |
| | 15 | 1.5125 (0.0792) | 1.9706 (0.2215) | 1.9701 (0.2209) | 1.9708 (0.2217) | 1.9699 (0.2208) | 1.9713 (0.2222) |
| | 25 | 1.5318 (0.0737) | 1.5890 (0.0081) | 1.5457 (0.0023) | 1.6025 (0.0107) | 1.5454 (0.0023) | 1.6305 (0.0172) |
| 50 | 10 | 1.4046 (0.1017) | 1.9509 (0.2033) | 1.9492 (0.2018) | 1.9514 (0.2038) | 1.9487 (0.2014) | 1.9528 (0.2051) |
| | 15 | 1.4561 (0.0605) | 1.9763 (0.2268) | 1.9758 (0.2264) | 1.9764 (0.2269) | 1.9757 (0.2263) | 1.9768 (0.2273) |
| | 25 | 1.5018 (0.0387) | 1.9937 (0.2437) | 1.9937 (0.2437) | 1.9938 (0.2438) | 1.9937 (0.2437) | 1.9938 (0.2438) |
| | 50 | 1.5197 (0.0335) | 1.3468 (0.0236) | 1.3216 (0.0139) | 1.3554 (0.0211) | 1.3245 (0.0309) | 1.3707 (0.0169) |
| 75 | 10 | 1.3459 (0.1259) | 1.9541 (0.2061) | 1.9525 (0.2048) | 1.9545 (0.2066) | 1.9521 (0.2044) | 1.9558 (0.2077) |
| | 15 | 1.4223 (0.0620) | 1.9783 (0.2288) | 1.9780 (0.2285) | 1.9784 (0.2289) | 1.9779 (0.2284) | 1.9787 (0.2292) |
| | 25 | 1.4769 (0.0324) | 1.9934 (0.2434) | 1.9933 (0.2434) | 1.9934 (0.2434) | 1.9933 (0.2434) | 1.9934 (0.2434) |
| | 50 | 1.5071 (0.0206) | 1.9969 (0.2469) | 1.9969 (0.2469) | 1.9969 (0.2470) | 1.9969 (0.2469) | 1.9970 (0.2470) |
| | 75 | 1.5121 (0.0199) | 1.1937 (0.0938) | 1.1807 (0.1019) | 1.1982 (0.0911) | 1.1833 (0.1003) | 1.2048 (0.0871) |

**From Tables (1)-(6), we conclude that**

1. Tables (1) and (2) show the MSEs of both the parameters $(\alpha, \theta)$ decreases when the different choices of $(n, r)$ increases respectively for both classical and Bayesian inferences.
2. The Bayes estimator for LINEX loss function exhibits lower MSEs among other Bayes estimators for the parameters α and θ respectively [Tables (1) and (2)].
3. For the reliability characteristics, the MSEs for both reliability function $R(t)$ and hazard rate function $H(t)$ decreases in the increment of $(n, r)$ with given different choices in Tables (3) and (4) respectively under both estimation methods (classical and Bayesian).
4. Tables (3) and (4) show the Bayes estimators for general entropy loss function exhibits lower MSEs for $R(t)$ and $H(t)$ among other Bayes estimators respectively.
5. Tables (5) and (6) show that the HPD interval length is smaller than other intervals length (confidence interval and credible interval) of both the parameters $(\alpha, \theta)$.

**Table 3:** MLEs and Bayes estimates of Reliability function $R(t)$ with their MSEs under type-II censoring scheme and $R(t = 1)$ = 0.6429.

| n | r | MLE (MSE) | BS (MSE) | BG (MSE) | | BL (MSE) | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 25 | 10 | 0.6215 (0.0111) | 0.7658 (0.0151) | 0.7657 (0.1501) | 0.7659 (0.0151) | 0.7658 (0.0151) | 0.7659 (0.0151) |
| | 15 | 0.6372 (0.0075) | 0.7681 (0.1565) | 0.7680 (0.0156) | 0.7681 (0.0156) | 0.7680 (0.0156) | 0.7681 (00156) |
| | 25 | 0.6442 (0.0066) | 0.6653 (0.0005) | 0.6552 (0.0001) | 0.6683 (0.0006) | 0.6611 (0.0003) | 0.6692 (0.0007) |
| 50 | 10 | 0.5989 (0.0139) | 0.7636 (0.0145) | 0.7634 (0.0145) | 0.7637 (0.0145) | 0.7635 (0.0145) | 0.7637 (0.0145) |
| | 15 | 0.6210 (0.0072) | 0.7693 (0.0159) | 0.7693 (0.0159) | 0.7694 (0.0159) | 0.7693 (0.0159) | 0.7694 (0.0159) |
| | 25 | 0.6386 (0.0039) | 0.7731 (0.0169) | 0.7731 (0.0169) | 0.7731 (0.0169) | 0.7731 (0.2919) | 0.7731 (0.0169) |
| | 50 | 0.6451 (0.0031) | 0.5875 (0.0030) | 0.5798 (0.0040) | 0.5901 (0.0028) | 0.5846 (0.0034) | 0.5905 (0.0027) |
| 75 | 10 | 0.5768 (0.0191) | 0.7642 (0.0146) | 0.7642 (0.0146) | 0.7644 (0.0147) | 0.7643 (0.0147) | 0.7644 (0.0147) |
| | 15 | 0.6098 (0.0081) | 0.7698 (0.0161) | 0.7697 (0.0161) | 0.7698 (0.0161) | 0.7697 (0.0161) | 0.7698 (0.0161) |
| | 25 | 0.6313 (0.0036) | 0.7730 (0.0169) | 0.7730 (0.0169) | 0.7730 (0.0169) | 0.7730 (0.0169) | 0.7730 (0.0169) |
| | 50 | 0.6426 (0.0021) | 0.7738 (0.0171) | 0.7737 (0.0171) | 0.7738 (0.0171) | 0.7737 (0.0171) | 0.7738 (0.0171) |
| | 75 | 0.6443 (0.0019) | 0.5304 (0.0126) | 0.5257 (0.0137) | 0.5321 (0.0123) | 0.5287 (0.0131) | 0.5322 (0.0122) |

**Table 4:** MLEs and Bayes estimates of hazard rate function $H(t)$ with their MSEs under type-II censoring scheme and $H(t = 1)$ = 0.9369.

| n | r | MLE (MSE) | BS (MSE) | BG (MSE) | | BL (MSE) | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 25 | 10 | 1.1316 (0.2409) | 0.7041 (0.0543) | 0.6728 (0.0699) | 0.7137 (0.0499) | 0.6905 (0.0608) | 0.7177 (0.0481) |
| | 15 | 1.0353 (0.1012) | 0.4833 (0.2058) | 0.4569 (0.2305) | 0.4918 (0.1983) | 0.4752 (0.2132) | 0.4917 (0.1983) |
| | 25 | 0.9832 (0.0604) | 0.9990 (0.0041) | 0.9382 (0.0002) | 1.0202 (0.0072) | 0.9594 (0.0007) | 1.0456 (0.0121) |
| 50 | 10 | 1.2120 (0.3545) | 0.9125 (0.0007) | 0.9011 (0.0014) | 0.9165 (0.0005) | 0.9055 (0.0011) | 0.9199 (0.0004) |
| | 15 | 1.0958 (0.1571) | 0.9692 (0.0011) | 0.9615 (0.0006) | 0.9717 (0.0012) | 0.9642 (0.0008) | 0.9742 (0.0014) |
| | 25 | 0.9999 (0.0503) | 0.4024 (0.2862) | 0.3834 (0.3064) | 0.4092 (0.2795) | 0.3969 (0.2919) | 0.4286 (0.2803) |
| | 50 | 0.9583 (0.0250) | 1.0811 (0.0210) | 1.0411 (0.0111) | 1.0943 (0.0250) | 1.0532 (0.0137) | 1.1111 (0.0306) |
| 75 | 10 | 1.2964 (0.5162) | 1.0067 (0.0049) | 0.9977 (0.0037) | 1.0098 (0.0053) | 1.0006 (0.0014) | 1.0131 (0.0058) |
| | 15 | 1.1278 (0.1771) | 0.8991 (0.0014) | 0.8951 (0.0017) | 0.9005 (0.0013) | 0.8967 (0.0016) | 0.9016 (0.0012) |
| | 25 | 1.0235 (0.0641) | 0.9646 (0.0008) | 0.9614 (0.0006) | 0.9656 (0.0008) | 0.9625 (0.0007) | 0.9666 (0.0009) |
| | 50 | 0.9637 (0.0220) | 0.1494 (0.6205) | 0.1425 (0.6311) | 0.1541 (0.6141) | 0.1476 (0.6231) | 0.1525 (0.6162) |
| | 75 | 0.9513 (0.0167) | 1.2645 (0.1074) | 1.2393 (0.0915) | 1.2728 (0.1129) | 1.2438 (0.0943) | 1.2856 (0.1217) |

**Table 5:** Classical and Bayesian Interval estimation for α under type-II censoring scheme.

| n | r | Confidence interval | | Credible interval | | HPD interval | |
|---|---|---|---|---|---|---|---|
| | | LL | UL | LL | UL | LL | UL |
| 25 | 10 | 0.9423 | 2.5101 | 1.0130 | 1.9510 | 1.1080 | 1.9090 |
| | 15 | 1.0252 | 2.2531 | 0.6718 | 1.4220 | 0.6650 | 1.4100 |
| | 25 | 1.0956 | 2.0667 | 1.2120 | 2.1750 | 1.1930 | 2.1500 |
| 50 | 10 | 0.9563 | 2.5612 | 1.6020 | 2.2010 | 1.5900 | 2.1870 |
| | 15 | 1.0511 | 2.2891 | 1.7700 | 2.3270 | 1.7630 | 2.3170 |
| | 25 | 1.1287 | 2.0432 | 0.5938 | 1.1870 | 0.5816 | 1.1580 |
| | 50 | 1.2082 | 1.8762 | 1.2570 | 1.9320 | 1.2410 | 1.9150 |
| 75 | 10 | 0.9745 | 2.6320 | 1.8380 | 2.3670 | 1.8380 | 2.3660 |
| | 15 | 1.0535 | 2.3091 | 1.7090 | 2.0910 | 1.7130 | 2.0940 |
| | 25 | 1.1374 | 2.0560 | 1.8690 | 2.2550 | 1.8630 | 2.2460 |
| | 50 | 1.2234 | 1.8592 | 0.2343 | 0.4065 | 2.3190 | 0.3995 |
| | 75 | 1.2573 | 1.7971 | 1.4310 | 1.9730 | 1.4350 | 1.9760 |

LL= Lower limit; UL= Upper limit.

**Table 6** Classical and Bayesian Interval estimation for $\theta$ under type-II censoring scheme.

| n | r | Confidence interval | | Credible interval | | HPD interval | |
|---|---|---|---|---|---|---|---|
| | | LL | UL | LL | UL | LL | UL |
| 25 | 10 | 0.9215 | 2.0239 | 1.8610 | 1.9980 | 1.8840 | 2.0000 |
| | 15 | 1.0079 | 2.0171 | 1.8960 | 1.9990 | 1.9150 | 2.0000 |
| | 25 | 1.0436 | 2.0201 | 1.1710 | 1.9560 | 1.2190 | 1.9850 |
| 50 | 10 | 0.8628 | 1.9464 | 1.8310 | 1.9990 | 1.8590 | 2.0000 |
| | 15 | 1.0204 | 1.8917 | 1.9150 | 1.9990 | 1.9300 | 2.0000 |
| | 25 | 1.1360 | 1.8676 | 1.9780 | 2.0000 | 1.9810 | 2.0000 |
| | 50 | 1.1789 | 1.8605 | 1.0740 | 1.6700 | 1.0570 | 1.6480 |
| 75 | 10 | 0.7721 | 1.9197 | 1.8380 | 1.9990 | 1.8660 | 2.0000 |
| | 15 | 0.9775 | 1.8672 | 1.9250 | 1.9990 | 1.9380 | 2.0000 |
| | 25 | 1.1371 | 1.8168 | 1.9760 | 2.0000 | 1.9800 | 2.0000 |
| | 50 | 1.2239 | 1.7904 | 1.9890 | 2.0000 | 1.9910 | 2.0000 |
| | 75 | 1.2356 | 1.7886 | 1.0200 | 1.4180 | 1.0010 | 1.3820 |

**Real data analysis**

In this section, we have considered a real data set from Wingo (1983) [29] which represents a clinical trial describe a relief time (in hours) for 50 arithmetic patients. The data set is-

0.70, 0.84, 0.58, 0.50, 0.55, 0.82, 0.59, 0.71, 0.72, 0.61, 0.62, 0.49, 0.54, 0.36, 0.36, 0.71,
0.35, 0.64, 0.84, 0.55, 0.59, 0.29, 0.75, 0.46, 0.46, 0.60, 0.60, 0.36, 0.52, 0.68, 0.80, 0.55,
0.84, 0.34, 0.34, 0.70, 0.49, 0.56, 0.71, 0.61, 0.57, 0.73, 0.75, 0.44, 0.44, 0.81, 0.80, 0.87,
0.29, 0.50.

We have considered the dataset to show the applicability of the GIL distribution with other different models belongs to the same family of distributions which are Inverse Lindley (IL) distribution, Generalized Lindley (GL) distribution, gamma distribution and Weibull distribution (see, Table 7). To test the goodness of fit of the considered distribution, we have used estimated negative log likelihood function (-Log L) and Kolmogorov-Smirnov (K-S) test statistic for the distribution selection criterion. We also used Akaike Information Criterion (AIC) and Bayesian Information Criterion (BIC). The AIC and BIC are expressed as,

$$AIC = 2 \times k - 2 \times log\hat{L}$$

$$BIC = k \times log(n) - 2 \times log\hat{L}$$

**Where**

$k$ = Number of parameters,
$n$ = Sample size, and,
$\hat{L}$ = value of the maximum likelihood for the considered distribution.

The K-S test statistic $D$ is defined as

$$D = \underset{x}{Sup}|F_n(x) - F(x)|,$$

Where

$F_n(x)$ = Empirical distribution function.

**Table 7:** ML estimates of the parameters, -Log L, K-S distance, AIC and BIC for the fitted models.

| Models | Estimates | | -Log L | K-S | AIC | BIC |
|---|---|---|---|---|---|---|
| | α | θ | | | | |
| IL | - | 0.7041 | 43.0361 | 0.3234 | 90.0722 | 93.8963 |
| GL | 0.3277 | 0.2947 | 94.7228 | 0.5560 | 193.4456 | 197.2697 |
| GILD | 2.2881 | 0.3707 | 4.1835 | 0.1559 | 12.367 | 16.1911 |
| Gamma | 0.4942 | 4.5608 | 79.3829 | 0.5119 | 162.7657 | 166.5898 |
| Weibull | 0.4371 | 2.2538 | 67.6233 | 0.8247 | 139.2466 | 143.0707 |

From the Table 7, we can say that the GIL distribution indicated by the lowest values of the respective -Log L, AIC and BIC as compared to the other distributions values. Hence, GIL distribution is found to fit most suitably to chosen data set. In addition, we notice that the real data set of 50 arithmetic patients support GIL distribution with the K-S distance as 0.1559 for $\alpha = 2.2881$ and $\theta = 0.3707$. The empirical cdf, Q-Q plot and P-P plot shows that GIL distribution fits best for the considered real data set (see, figure 1).

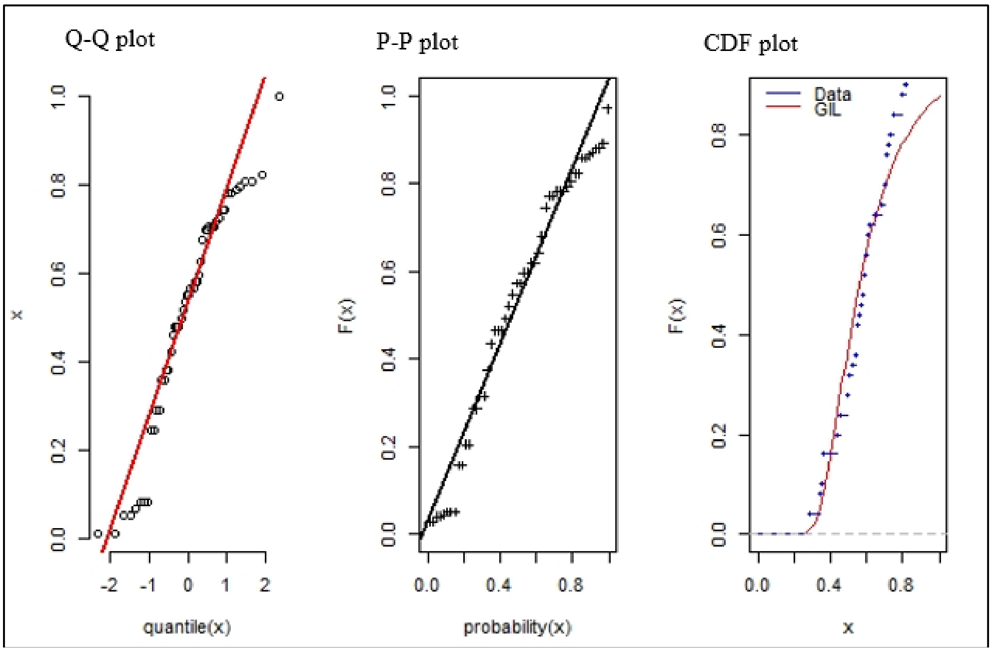


**Fig 1:** QQ-plot, PP-plot and empirical cdf plot of the GIL distribution for real data set.

Now, for the considered real data sets of 50 arithmetic patients, we generate the samples from type-II censoring scheme as done in the simulation study. For $n = 50$, we consider various combination of number of failures $r = 10, 25$ and 45 under type -II censoring scheme. For the real data set, we calculate the ML estimators and the Bayes estimators under symmetric and asymmetric loss functions. For asymmetric loss function, we consider $c = (-2, 2)$ in LINEX loss function and $q = (-2, 2)$ in general entropy loss functions. The calculated estimators of α and $\theta$ under type-II censoring for the real data set in Table 8 and Table 9 respectively. The reliability function $R(t)$ and hazard rate function $H(t)$ as $t = 1$ are given in Table 10 and Table 11 respectively for the real data set under type-II censoring.

Table 12 and Table 13 show the confidence intervals, credible intervals and HPD intervals for the parameter α and θ respectively at 5% level of significance.

**Table 8:** MLEs and Bayes estimates of $\alpha$ under type-II censoring scheme for the real data set.

| n | r | MLE | BS | BG | | BL | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 50 | 10 | 1.7587 | 1.8367 | 1.8185 | 1.8427 | 1.8148 | 1.8591 |
| | 25 | 1.5859 | 1.8603 | 1.8418 | 1.8664 | 1.8378 | 1.8828 |
| | 45 | 1.5449 | 1.7173 | 1.7091 | 1.7200 | 1.7080 | 1.7262 |

**Table 9:** MLEs and Bayes estimates of $\theta$ under type-II censoring scheme for the real data set.

| n | r | MLE | BS | BG | | BL | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 50 | 10 | 1.4046 | 1.4695 | 1.4468 | 1.4770 | 1.4477 | 1.4919 |
| | 25 | 1.5018 | 1.3231 | 1.3018 | 1.3303 | 1.3045 | 1.3426 |
| | 45 | 1.5190 | 1.0623 | 1.0583 | 1.0638 | 1.0593 | 1.0657 |

**Table 10:** MLEs and Bayes estimates of reliability function $R(t)$ under type-II censoring scheme for the real data set.

| n | r | MLE | BS | BG | | BL | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 50 | 10 | 0.5989 | 0.6301 | 0.6243 | 0.6320 | 0.6277 | 0.6325 |
| | 25 | 0.6386 | 0.5793 | 0.5727 | 0.5815 | 0.5768 | 0.5819 |
| | 45 | 0.6449 | 0.4758 | 0.4741 | 0.4764 | 0.4753 | 0.4764 |

**Table 11:** MLEs and Bayes estimates of hazard function $H(t)$ under type-II censoring scheme for the real data set.

| n | r | MLE | BS | BG | | BL | |
|---|---|---|---|---|---|---|---|
| | | | | $q = -2$ | $q = 2$ | $c = -2$ | $c = 2$ |
| 50 | 10 | 1.2120 | 1.1779 | 1.1315 | 1.1932 | 1.1429 | 1.2156 |
| | 25 | 0.9999 | 1.2930 | 1.2464 | 1.3080 | 1.2549 | 1.3326 |
| | 45 | 0.9605 | 1.3658 | 1.3526 | 1.3698 | 1.3543 | 1.3762 |

**Table 12:** Classical and Bayesian Interval estimation for $\alpha$ under type-II censoring scheme for the real data set.

| n | r | Confidence interval | | Credible interval | | HPD interval | |
|---|---|---|---|---|---|---|---|
| | | LL | UL | LL | UL | LL | UL |
| 50 | 10 | 0.9563 | 2.5612 | 1.5530 | 2.1360 | 1.5680 | 2.1460 |
| | 25 | 1.1287 | 2.0432 | 1.5710 | 2.1530 | 1.5790 | 2.1570 |
| | 45 | 1.1983 | 1.8914 | 1.4850 | 1.8860 | 1.5080 | 1.9000 |

**Table 13:** Classical and Bayesian Interval estimation for $\theta$ under type-II censoring scheme for the real data set.

| n | r | Confidence interval | | Credible interval | | HPD interval | |
|---|---|---|---|---|---|---|---|
| | | LL | UL | LL | UL | LL | UL |
| 50 | 10 | 0.8628 | 1.9464 | 1.1870 | 1.7800 | 1.1700 | 1.7510 |
| | 25 | 1.1360 | 1.8676 | 1.0670 | 1.6090 | 1.0570 | 1.5930 |
| | 45 | 1.1773 | 1.8608 | 1.0020 | 1.2300 | 1.0000 | 1.1870 |

## Conclusion

In this paper, we have considered the statistical inference of the unknown parameters, reliability characteristics (reliability and hazard rate function) of the GIL distribution when the data are type-II censored. The maximum likelihood estimators (MLEs) and the Bayes estimators of the parameters and reliability characteristics are obtained. The MLEs are not in closed forms. therefore, numerical approximation techniques have been implemented to evaluate them. To compute the Bayesian estimates and associated MSEs, we employed symmetric loss function (SELF) and asymmetric loss functions (LINEX, general entropy). To measure the performance level of the developed estimators, simulation technique (MCMC) was used for the different compositions of *n* & *r*., a real data set has been considered to illustrate the computation of various estimators.

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