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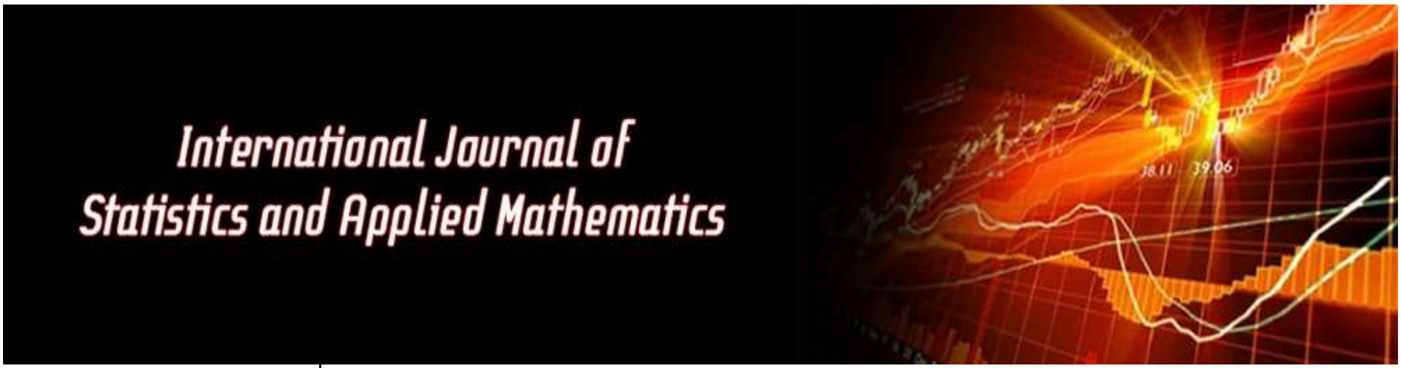


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# Localization with flat envelopes and flat covers of complexes

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**Abstract**
The aim of this article is mainly to check :
1. if the lacalized complex $S_C^{-1}(C)$ has a flat cover when $C$ does.
2. If the class of complexes of left $S^{-1}A$ is flat enveloping when the class of complex of left $A$-modules does.
3. If the class of complexes of left $S^{-1}A$ is flat covering when the class of complex of left $A$-modules does.
And finally we generalize the fact that "if $A$ is an integral noetherian commutative ring with finite Krull dimension then the class of complexes of $A$-modules is covering" to duo-rings.

**Keywords:** Saturated multiplicative subset, left Ore conditions, localization, category of complexes, flat envelopes, flat covers

## Introduction

In this article $Comp(A - Mod)$ is the category of left $A$-modules.
A complex of left $A$-modules (or complex ) $C$ is denoted by

$$\ldots \longrightarrow C^{n-1} \xrightarrow{d_C^{n-1}} C^n \xrightarrow{d_C^n} C^{n+1} \longrightarrow \cdots.$$

If $M$ is a $A$-module, we denote by $\underline{M}$ the complex with $M$ in the zeroth place and 0 elsewhere. If $C$ is a complex of left $A$-modules, we denote by $C[i]$ the complex with $C[i]^n = C^{n+i}$ and $d^n_{C[i]} = (-1)^{n+i} d^n_C$. And finally if $C$ and $D$ are complexes of left $A$-modules, $Hom^\bullet(C, D)$ is the complex of $\mathbb{Z}$-modules such that:

$$Hom^\bullet(C, D)^n = \prod_{t \in \mathbb{Z}} Hom(C^t, D^{n+t})$$

and such that if $f \in Hom^\bullet(C, D)^n$ then:

$$(d^n_{Hom^\bullet(C,D)} f)^m = d_D^{n+m} f^m + (-1)^{n+1} f^{m+1} d_C^m$$

We organize this paper as following:
We give in the first section reminders and preliminary results.
In the second one we prove those results for existence of flat envelopes:

1. If $C$ is a complex of left $A$-modules and if $f: C \longrightarrow F$ is a flat pre-envelope of $C$ then $S_C^{-1}(f): S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat pre-envelope of $S_C^{-1}(C)$;
2. If $C$ is a complex of left $S^{-1}A$-modules and if $f: C \longrightarrow F$ is a flat envelope of $C$ then $S_C^{-1}(f): S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat envelope of $S_C^{-1}(C)$;

3. If all complex of left $A$-modules has a flat envelope then all complex of left $S^{-1}A$-modules has a flat envelope;
4. if $A$ is right coherent then each complex of left $S^{-1}A$-modules admits a flat pre-envelope;

And finally, we show those result for existence of flat cover:

1. If all complex of left $A$-modules admit a flat precover then all complex of left $S^{-1}A$-modules admit a flat precover;
2. Let $A$ be an integral and noetherian duo-ring of finite Krull dimension $d$. Then every complex of left $A$-modules admit a flat cover.

**1. Définitions et résultats préliminaires**

**Définition 1.1** Let A be a ring. A is said to be duo-ring if every left ideal is two sided and if every right ideal is two sided.

**Définition 1.2** Let A a ring and S a multiplicative subset of A. It is said that S is a saturated multiplicative subset of A verifying the left Ore conditions if:

- S is a saturated multiplicative subset of A if S is multiplicative subset of A and if for all $s, s' \in A$, if $ss' \in S$ then $s \in S$ and $s' \in S$.
- S is a saturated multiplicative subset of A verifying the left Ore conditions if S is a saturated multiplicative subset of A and then:

(a) $\forall a \in A, \forall s \in S, \exists (b,t) \in A \times S: ta = bs$,

(b) $\forall a \in A, \forall s \in S$, if $as = 0$ then there exist $t \in S$ such as $ta = 0$.

**Remark**: Through the paper A is considered to be a ring and S a saturated subset of A with left Ore conditions.

**Proposition 1.1** The binary relation defined on $S \times M$ by:

$$(s,m)\mathcal{R}(s',m') \Leftrightarrow \exists\, x,y \in S: \begin{cases} xm & = & ym' \\ xs & = & ys' \end{cases}$$

is an equivalence relation.

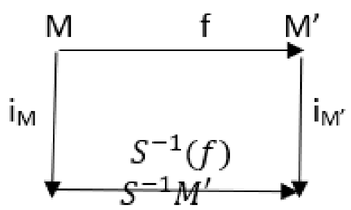
we denote by $S^{-1}M$ the set of classes and $(s,m)$ by $\frac{m}{s}$. $S^{-1}A$ can be equipped to a ring structure and $S^{-1}M$ can be equipped to a $S^{-1}A$-module structure.

**Proposition 1.2** *Let $f: M \longrightarrow M'$ a morphism of left $A$-modules. Then*

$$S^{-1}f: S^{-1}M \longrightarrow S^{-1}M'$$

$$\frac{m}{s} \longmapsto \frac{f(m)}{s}$$

is a morphism of left $S^{-1}A$-modules and we have the following commutative diagram:




$S^{-1}M$

Where $i_M(m) = \frac{m}{1}$.

**Proof** see [1]

**Proposition 1.3** *The following relation:*

$S_C^{-1}(): Comp(A - Mod) \longrightarrow Comp(S^{-1}A - Mod)$ such as

1. if $C := \cdots \longrightarrow C^n \xrightarrow{\delta_C^n} C^{n+1} \longrightarrow \cdots$ is an objet of $Comp(A - Mod)$ then :

$$S_C^{-1}(C) := \cdots \longrightarrow S^{-1}C^n \xrightarrow{S^{-1}\delta_C^n} S^{-1}C^{n-1} \longrightarrow \cdots$$

is an objet of $Comp(S^{-1}A - Mod)$

2. if $f: C \longrightarrow D$ is a morphism of $Comp(A - Mod)$ then

$S_C^{-1}(f): S_C^{-1}(C) \longrightarrow S_C^{-1}(D)$ is a morphism of $Comp(S^{-1}A - Mod)$

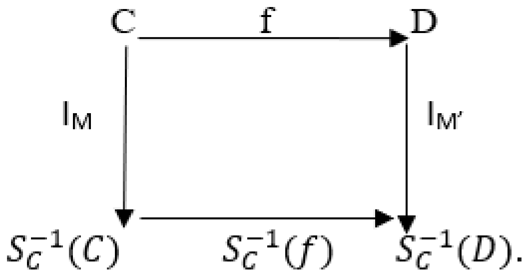
Then $S_C^{-1}()$ is a covariant functor that is exact.

**Proof** see [6], proposition 1

**Proposition 1.4** *Let $f: C \longrightarrow D$ a chain map of complexes of left $A$-modules. Then:*

1. The following diagram is commutative

Where $I_C$ is the chain map of component $i_{C^n}$.



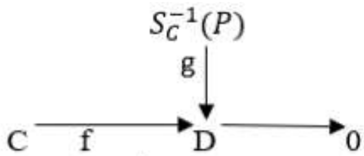

2. If $C$ is a complex of left $S^{-1}A$-modules then $C \cong S_C^{-1}(C)$.

**Proof**

1. It follows from proposition 1.2.
2. By corralary 2.2.6 of [4], for all $n$, $C^n \cong S^{-1}C^n$. By considering $i_{C^n}$ that isomorphism we see that $I_C$ is an isomorphism of complexes of left $S^{-1}A$-modules. Hence $C \cong S_C^{-1}(C)$.

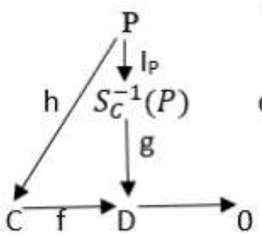
**Proposition 1.5** *If $P$ is a projective object of $Comp(A - Mod)$ then $S_C^{-1}(P)$ is a projective object of $Comp(S^{-1}A - Mod)$. Otherwise the functor $S_C^{-1}()$ keeps projectivity of complexes .*

**Proof** Consider $P$ to be a projective object of $Comp(A - Mod)$ and let be the diagram:

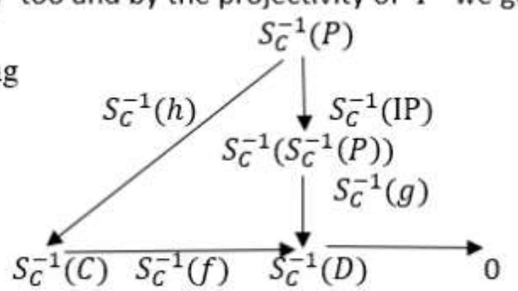



$f$ is an epimorphism of $Comp(S^{-1}A - Mod)$ and $g$ a morphism of $Comp(S^{-1}A - Mod)$.

Then $f$ and $g$ are morphism of $Comp(A - Mod)$ too and by the projectivity of $P$ we get:

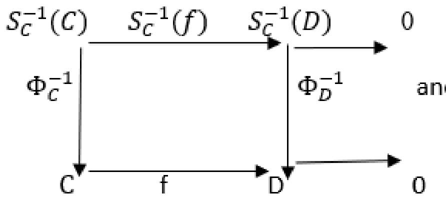



That implies the following commutative diagram

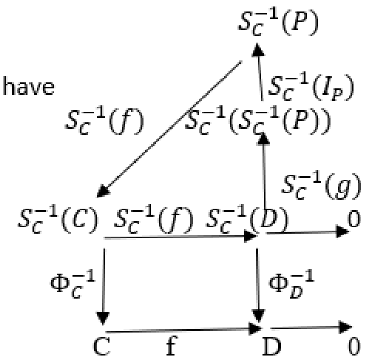



Besides, $C \simeq S_C^{-1}(C)$ and $D \simeq S_C^{-1}(D)$. Let then be $\Phi_C: C \longrightarrow S_C^{-1}(C)$ and $\Phi_D: D \longrightarrow S_C^{-1}(D)$ the corresponding isomorphisms.

And since the following diagram is commutaive

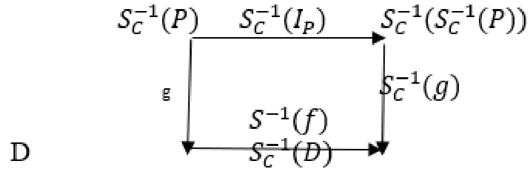



and then we have




Hence $\Phi_D^{-1} \circ S_C^{-1}(g) \circ S_C^{-1}(I_P) = f \circ \Phi_C^{-1} \circ S_C^{-1}(h)$ (1)

On the other hand, we have:




That is $S_C^{-1}(g) \circ S_C^{-1}(I_P) = \Phi_D \circ g$ and then

$$g = \Phi_D^{-1} \circ S_C^{-1}(g) \circ S_C^{-1}(I_P) \ (2)$$

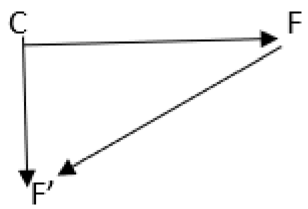
Hence (1) and (2) imply:

$$g = f \circ \Phi_C^{-1} \circ S_C^{-1}(h)$$

Thus $S_C^{-1}(P)$ is projective .

**2. Localization and flat envelopes**

**Définition 2.1** Let A be a ring and C be a complex of left A-modules. A flat pre-envelope of C is a morphism $\mathrm{f}: \mathrm{C} \longrightarrow \mathrm{F}$ where F is a flat object of $\mathrm{Comp(A-Mod)}$ such as for all morphism $\mathrm{g}: \mathrm{C} \longrightarrow \mathrm{F'}$ where $\mathrm{F'}$ is a flat object of $\mathrm{Comp(A-Mod)}$, there exists a morphism $\mathrm{h}: \mathrm{F} \longrightarrow \mathrm{F'}$ such as the diagram below is commutative:




that is $g = h \circ f$.

If in addition, we can only complete the diagram below by automorphism $h: F \longrightarrow F$ then we say that $f$ is a flat enveloppe of $C$.

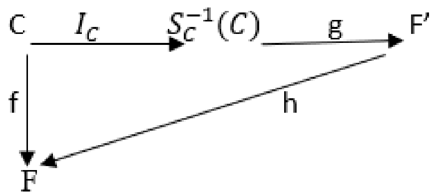

**Théorème 2.1** *The functor $S_C^{-1}()$ keeps flat pre-envelope. Otherwise, if $f: C \longrightarrow F$ is a flat pre-envelope of $C$ then $S_C^{-1}(f): S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat pre-envelope of $S_C^{-1}(C)$.*

**Proof**

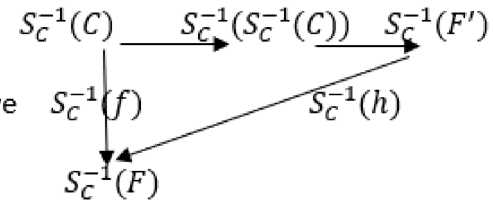
$S_C^{-1}(F)$ is flat because $F$ is flat (see [6], theorem 2)

Let $g: S_C^{-1}(C) \longrightarrow F'$ be a map of complexes of left $A$-modules. Then consider

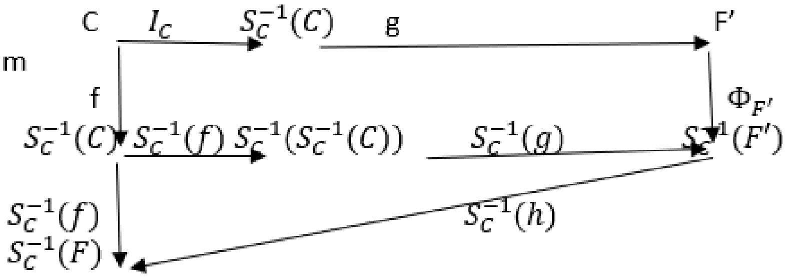
$g \circ I_C: C \longrightarrow S_C^{-1}(C) \longrightarrow F'$, there exists a map of complexes $h: F \longrightarrow F'$ such as the following diagram is commutative:




and then we have




That gives us the following: commutative diagram




So, we have:

$$\Phi_{F'} \circ g \circ I_C = S_C^{-1}(h) \circ S_C^{-1}(f) \circ I_C$$

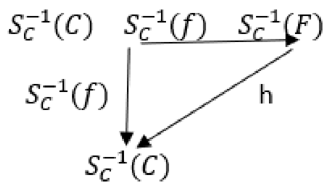
$$\Longrightarrow \Phi_{F'} \circ g = S_C^{-1}(h) \circ S_C^{-1}(f)$$

Thus $g = \Phi_{F'}^{-1} \circ S_C^{-1}(h) \circ S_C^{-1}(f)$

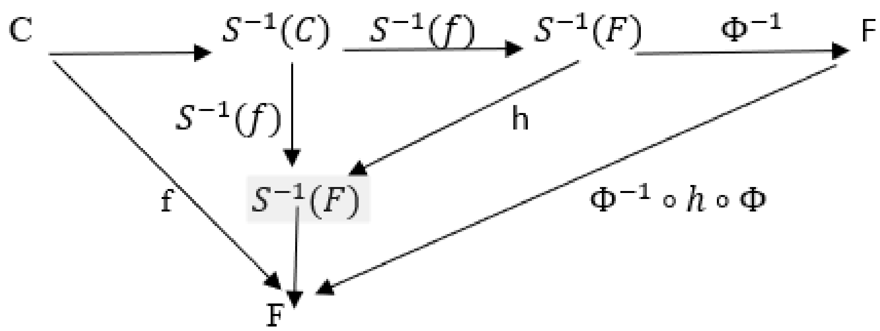
**Théorème 2.2** *Let $C$ be a complex of left $S^{-1}A$-modules. If $f: C \longrightarrow F$ is a flat envelope of $C$ then $S^{-1}(f): S^{-1}(C) \longrightarrow S^{-1}(F)$ is a flat envelope of $S^{-1}(C)$.*

**Proof**

By the previous theorem $S^{-1}(f): S^{-1}(C) \longrightarrow S^{-1}(F)$ is a flat preenvelope. Consider now the following diagram:




Show now that $h$ is an automorphism. Consider the f commutative diagram below which is composed with three commutative diagrams.
Since $f$ is flat envelope then $\Phi^{-1} \circ h \circ \Phi$ is an automorphism and so is $h$.
Hence $S^{-1}(f): S^{-1}(C) \longrightarrow S^{-1}(F)$ is a flat envelope of $S^{-1}(C)$.




**Théorème 2.3** *If all complex of left A-modules has a flat envelope then all complex of left $S^{-1}A$-modules has a flat envelope.*

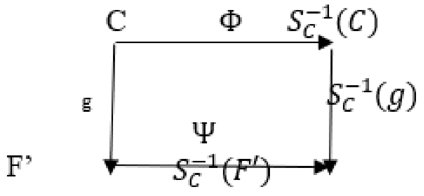
**Proof**

Let $C$ be a complex of left $S^{-1}A$-modules. Since $C$ is also a complex of left $A$-modules then $C$ has a flat pre-envelope $f: C \longrightarrow F$ and then we have:
$S_C^{-1}(f): S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$. Since $C \cong S_C^{-1}(C)$ looked as complexes of left $S^{-1}A$-modules, let be $\Phi: C \longrightarrow S_C^{-1}(C)$ that isomorphism.
Let us Show now that $S_C^{-1}(f) \circ \Phi: C \longrightarrow S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat pre-envelope of $C$ looked as a complex of left $S^{-1}A$-modules.
Since $F$ is flat then $S_C^{-1}(F)$ is flat.

Let $g: C \longrightarrow F'$ be a map of complexes of left $S^{-1}A$-modules. Then since $g$ is also a map of complexes of left $A$-modules, there exists a morphism $h: F \longrightarrow F'$ such as

$g = h \circ f$, then $S_C^{-1}(g) = S_C^{-1}(h) \circ S_C^{-1}(f)$.

Since $F' \cong S_C^{-1}(F')$ then let be $\Psi: F' \longrightarrow S_C^{-1}(F')$ that isomorphism.
We then have this commutative diagram:

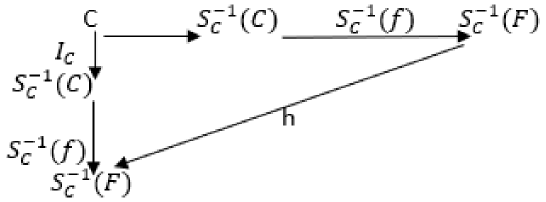



We get :$S_C^{-1}(g) \circ \Phi = \Psi \circ g$ and then $S_C^{-1}(g) = \Psi \circ g \circ \Phi^{-1}$. By replacing $S_C^{-1}(g)$ by its expression in $S_C^{-1}(g) = S_C^{-1}(h) \circ S_C^{-1}(f)$ we get:

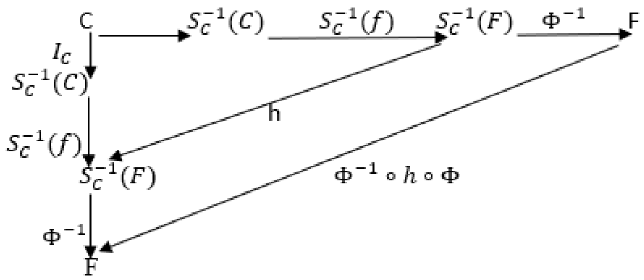
$\Psi \circ g \circ \Phi^{-1} = S_C^{-1}(h) \circ S_C^{-1}(f)$

$\Longrightarrow g = \Psi^{-1} \circ S_C^{-1}(h) \circ S_C^{-1}(f) \circ \Phi$

And then $S_C^{-1}(f) \circ \Phi: C \longrightarrow S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat pre-envelope of $C$ looked as a complex of left $S^{-1}A$-modules. For now, show that $S_C^{-1}(f) \circ \Phi: C \longrightarrow S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat envelope of $C$.Consider the following commutative diagram




Let prove that $h$ is an automorphism. The following diagram is commutative:




Indeed, it is composed with the previous commutative diagram and the other square diagram is commutative by construction.
Remark also that $\Phi^{-1} \circ S_C^{-1}(f) \circ I_C = f$.
So since $f$ is a flat envelope then $\Phi^{-1} \circ h \circ \Phi$ is an automorphism and so is $h$. Hence $S_C^{-1}(f) \circ \Phi: C \longrightarrow S_C^{-1}(C) \longrightarrow S_C^{-1}(F)$ is a flat envelope of $C$.

**Corollaire 2.1** *Let A be a right coherent ring. Then all complex of left $S^{-1}A$-modules has a flat preenvelope.*

**Proof**
All complex of left $A$-modules has a flat preenvelope by [[2], theorem 5.2.2] and by the last proposition we get the result.

### 3. Localization and flat covers
Cover is the dual notion of envelope.

**Théorème 3.1** *If each complex of left A-modules admits a flat precover then each complex of $S^{-1}A$-modules admits a flat precover.*

**Proof**

Let $C$ be a complex of $S^{-1}A$-modules. Since $C$ is also a complex of left $A$-modules then it has a flat precover $f: F \longrightarrow C$ and then we have: $S_C^{-1}(f): S_C^{-1}(F) \longrightarrow S_C^{-1}(C)$. Since $C \cong S_C^{-1}(C)$ looked as complexes of left $S^{-1}A$-modules, let $\Phi: S_C^{-1}(C) \longrightarrow C$ that isomorphism.

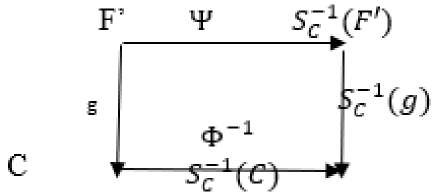
So, let us show that $\Phi \circ S_C^{-1}(f): S_C^{-1}(F) \longrightarrow S_C^{-1}(C) \longrightarrow C$ is a flat precover of $C$ looked as a complex of left $S^{-1}A$-modules.

Since $F$ is flat then $S_C^{-1}(F)$ is flat too.

Let $g: F' \longrightarrow C$ be a map of complexes of left $S^{-1}A$-modules. Then since $g$ is also a map of complexes of left $A$-modules then there exists a morphism $h: F' \longrightarrow F$ such as $g = f \circ h$. That imply $S_C^{-1}(g) = S_C^{-1}(f) \circ S_C^{-1}(h)$.

Since $F' \cong S_C^{-1}(F')$ then let $\Psi: F' \longrightarrow S_C^{-1}(F')$ be that isomorphism.

We have the following cummutative diagram:




Hence :$S_C^{-1}(g) \circ \Psi = \Phi^{-1} \circ g$ and we have $S_C^{-1}(g) = \Phi^{-1} \circ g \circ \Psi^{-1}$. By replacing $S_C^{-1}(g)$ by its expression in $S_C^{-1}(g) = S_C^{-1}(f) \circ S_C^{-1}(h)$ we obtain:

$$\Phi^{-1} \circ g \circ \Psi^{-1} = S_C^{-1}(f) \circ S_C^{-1}(h)$$

$$\Longrightarrow g = \Phi \circ S_C^{-1}(f) \circ S_C^{-1}(h) \circ \Psi$$

And then $\Phi \circ S_C^{-1}(f): S_C^{-1}(F) \longrightarrow C$ is a flat pre-cover of $C$ looked as a complex of left $S^{-1}A$-modules.

**Définition 3.1** $A$ is considered to be a ring.

1. An $A$-module $M$ is cotorsion if $Ext_A^1(F, M) = 0$ for all flat $A$-module $F$.
2. A complex $C$ is $DG$-cotorsion if $EXT^1_{Comp(A-Mod)}(F, C) = 0$ for all flat complex $F$. $EXT^n_{Comp(A-Mod)}$ is the $n-th$ derived functor of $Hom_{Comp(A)}$.
3. A complex $C$ is cotorsion if it is exact and $Ker(d_C^i)$ is cotorsion.

**Proposition 3.1** *Let $A$ be a ring and $C$ be a complex of left $A$-modules.*

1. $C$ is $DG$-cotorsion if and only if $C^n$ is cotorsion and $Hom^\bullet(F, C)$ is exact for all complex of left $A$-modules $F$.
2. If $C$ is bounded then $C$ is $DG$-cotorsion if and only if $C^n$ is cotorsion for all $n \in \mathbb{Z}$.

**Théorème 3.2** *Let $A$ be an integral noetherian duo-ring of finite Krull dimension $d. dim(A) = d$. Then all complex of left A-modules admit a flat cover.*

**Proof**

By lemma 4.4.2 in [2], it is enough to show only the result for exact complexes of left $A$-modules .

Consider $C$ to be an exact complex of left $A$-modules and

$$0 \longrightarrow C \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_d \longrightarrow D \longrightarrow 0$$

be an injective partial resolution of $C$. Show that $D$ is $DG$- cotorsion. We know that $D$ is exact since $C$ is exact. we have also for all complex of left $A$-modules $E$, $Hom(\underline{R}[-i], E) \cong Ker(d_E^i)$. Then by applying the $Hom(\underline{R}[-i], -)$ functor to the injective partial resolution we get:

$$0 \longrightarrow Hom(\underline{R}[-i], C) \longrightarrow Hom(\underline{R}[-i], I_0) \longrightarrow \cdots Hom(\underline{R}[-i], I_d) \longrightarrow Hom(\underline{R}[-i], D) \longrightarrow 0$$

And then we have:

$$0 \longrightarrow Ker(d_{I_0}^i) \longrightarrow Ker(d_{I_1}^i) \longrightarrow \cdots \longrightarrow Ker(d_{I_d}^i) \longrightarrow Ker(d_D^i) \longrightarrow 0$$

that is exact and such as $Ker(d_{I_j}^i)$ is injective. By [[8], theorem 4.4.13] we have:

$$0 = Ext^{d+2}(F, Ker(d_C^i)) = Ext^1(F, Ker(d_D^i))$$

Hence $Ker(d_D^i)$ is cotorsion for all $i \in \mathbb{Z}$. And then $D$ is $DG$-cotorsion. Elsewhere, since $I_0, I_1, \dots, I_d$ are injectives then they are $DG$-cotorsion.
By splitting:

$$0 \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_d \longrightarrow D \longrightarrow 0$$

into two sequences:

$$0 \longrightarrow C \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_{d-1} \longrightarrow N \longrightarrow 0$$

and

$$0 \longrightarrow N \longrightarrow I_d \longrightarrow D \longrightarrow 0$$

And by applying lemma 4.4.5 in [2] to the exact sequence

$$0 \longrightarrow N \longrightarrow I_d \longrightarrow D \longrightarrow 0$$

we get that $D$ admit a flat pre-cover of $DG$-cotorsion kernel. By repeating the same thing we get that $C$ admit a flat pre-cover of $DG$-cotorsion kernel.

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