International Journal of Statistics and Applied Mathematics 2021; 6(6): 148-157

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2021; 6(6): 148-157

www.mathsjournal.com
Received: 07-09-2021
Accepted: 11-10-2021

Dr. Anshu Sharma
Assistant Professor, Department of Mathematics, Govt. College for Women Gandhi Nagar Jammu, Jammu and Kashmir, India

Corresponding Author:
Dr. Anshu Sharma
Assistant Professor, Department of Mathematics, Govt. College for Women Gandhi Nagar Jammu, Jammu and Kashmir, India

Integration operators from H^∞, bloch and bmoa spaces to Bloch type spaces

Dr. Anshu Sharma

Abstract
In this paper, we characterized boundedness and compactness of generalized integration operator from H^∞, Bloch and BMOA spaces to Bloch type spaces. Moreover, the operator norm is estimated.

Keywords: Integration, operators, bmoa, spaces, Bloch

1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}, H(\mathbb{D})$ the space of all functions holomorphic on $\mathbb{D}, dA(z) = \frac{1}{\pi}dxdy = \frac{1}{\pi}rdrd\theta$ be the normalized area measure on $\mathbb{D}$ and H^∞ the space of all bounded holomorphic functions with the norm $\| f \|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$.
Let $\alpha > 0$. The α-Bloch space B^α on $\mathbb{D}$ is the space of all holomorphic functions f on $\mathbb{D}$ such that $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty$.
The little α-Bloch space B_0^α consists of all $f \in B^\alpha$ such that $\lim_{|z|\to 1}(1-|z|^2)^\alpha|f'(z)| = 0$.=
Both spaces $\mathcal{B}^\alpha$ and $\mathcal{B}_0^\alpha$ are Banach spaces with the norm $\| f \|_{\mathcal{B}\alpha} = |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)|$, and $\mathcal{B}_0^\alpha$ is a closed subspace of $\mathcal{B}^\alpha$. If $\alpha = 1$, they become the classical Bloch space B and little Bloch space B_0 respectively. Throughout this paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. When $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$.
It is well known (see Theorem 5.4 of [21]) that for each $n \in \mathbb{N}$ and $f \in \mathcal{B}$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^n\left|f^{(n)}(z)\right| \lesssim \| f \|_{\mathcal{B}}.$$

Since $H^\infty \subset B$ and $\| f \|_{\mathcal{B}} \leq 2 \| f \|_\infty$ for all $f \in H^\infty$ (see Proposition 5.1 of [21]). Then for all functions $f \in H^\infty$, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^n\left|f^{(n)}(z)\right| \lesssim \| f \|_\infty. \tag{1.1}$$

For more information about Bloch spaces we refer to [21].
For $0 < p < \infty$, the Hardy space H^p containing all holomorphic functions $f: \mathbb{D} \to \mathbb{C}$ such that

$$\| f \|_{H_p} = \sup_{0<r<1}\left(\int_0^{2\pi}\left|f\left(re^{i\theta}\right)\right|^p\frac{d\theta}{2\pi}\right)^{1/p} < \infty.$$

If $0 < p < 1, H^p$ is a complete metric space. For $1 \leq p < \infty, H^p$ is a Banach space under the above norm (see [6]).
For $a \in \mathbb{D}$, let $\sigma_a(z) = (a-z)/(1-\bar{a}z)$ be the automorphism of $\mathbb{D}$ that interchanges 0 and a. Let the Green function in $\mathbb{D}$ with logarithmic singularity at a is given by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right| = \log\frac{1}{|\sigma_a(z)|}$$

The space BMOA consists of all $f \in H^2$ such that

$$\sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^2} < \infty.$$

BMOA is a Banach space under following norm (see, for example, [7])

$$\| f \|_{\mathrm{BMOA}} = |f(0)| + \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^2}$$

BMOA can be described in terms of Carleson measures. Recall that a positive measure μ on $\mathbb{D}$ is a Carleson measure if

$$\sup_{I\in\partial\mathbb{D}} \frac{\mu(S(I))}{|I|} < \infty,$$

Where $S(I) = \{z: 1-|I| \le |z| < 1, z/|z| \in I\}$ is a Carleson box based on the arc $I \subset \partial\mathbb{D}$ of length $|I|$. Following is the well known Carleson measure characterization of BMOA space.

Theorem A. The following statements are equivalent:

1. $f \in BMOA$.
2. $d\mu_f$ is a Carleson measure, where $d\mu_f(z) = |f'(z)|^2 \log 1/|z| dA(z)$.
3. $\sup_{a\in D} \int_{\mathbb{D}} |f'(z)|^2 g(z,a) dA(z) < \infty$.

Any $g \in H(\mathbb{D})$ induces integral operators as following

$$I_g f(z) = \int_0^z f'(\zeta) g(\zeta) d\zeta,\ T_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$$

Where$f \in H(\mathbb{D})$. Let φ be a holomorphic self-mapping of $\mathbb{D}$ and $h \in H(\mathbb{D})$. For a non-negative integer n, we define a generalized integral operator as following

$$I_{h,\varphi}^{(n)} f(z) = \int_0^z f^{(n)}(\varphi(\zeta)) h(\zeta) d\zeta,\ f \in H(\mathbb{D}).$$

It is easy to see that $I_{h,\varphi}^{(n)}$ induces many well know linear operators, like I_g, T_g. If $n = 1$, the integral operator $I_{h,\varphi}^{(n)}$ is the generalized composition operator defined in [8]. For $h(z) = \varphi'(z)$, then the $I_{h,\varphi}^{(n)}$ is related with the products of differentiation and composition operators $C_\varphi D^m$ defined in [14].
The boundedness and compactness of integral operators I_g and T_g between spaces of holomorphic functions have been studied in several papers. For example, Aleman, Cima and Siskakis investigated integral operators T_g on Hardy spaces and Bergman spaces in [1, 2, 3]; Siskakis and Zhao [11] studied T_g on BMOA; Yoneda [17, 18] studied I_g and T_g on Bloch type spaces. Recently, the second author and Anshu Sharma studied the generalized integral operator $I_{h,\varphi}^{(n)}$ on Bergman space. In this paper, we study the integral operator $I_{h,\varphi}^{(n)}$ between some classical holomorphic function spaces.

2. Some lemmas

It is well known that $H^\infty \subset BMOA \subset B$. From the definition of the norm, we know

$$\| f \|_{\mathrm{BMOA}} \lesssim \| f \|_\infty, f \in H^\infty.$$

Indeed, Girela proved that

$$f \|_{\mathcal{B}} \le \| f \|_{\mathrm{BMOA}_1} \qquad (2.1)$$

in Corollary 5.2 of [7]. The $\|\cdot\|_{\mathrm{BMOA}_1}$ is an equivalent norm of $\|\cdot\|_{\mathrm{BMOA}}$ (See [7]). we will prove that (2.1) also holds for the norm $\|\cdot\|_{BMOA}$. The following lemma is from Lemma 6 in [15].

Lemma 2.1. If $f \in H(\mathbb{D})$, then

$$|f(0)|^2 \le 2\int_{\mathbb{D}} |f(z)|^2 \log\frac{1}{|z|} dA(z).$$

The proof of Lemma 2.1 is similar with Lemma 4.12 of [21].

Lemma 2.2. Let $f \in H(\mathbb{D})$. Then

$$\| f \|_B \le \| f \|_{BMOA}. \tag{2.2}$$

Proof. Applying Littlewood-Paley identity

$$\| f \|_{H^2}^2 = |f(0)|^2 + 2\int_{\mathbb{D}} |f'(z)|^2 \log\frac{1}{|z|} dA(z)$$

and Lemma 2.1, We have

$$\begin{aligned}\sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{H^2} &= \sup_{a\in\mathbb{D}}\left(2\int_{\mathbb{D}} \left|f'(\sigma_a(z))\sigma_a'(z)\right|^2 \log\frac{1}{|z|} dA(z)\right)^{\frac{1}{2}} \\ &\ge \sup_{a\in\mathbb{D}} (1-|a|^2)|f'(a)| \\ &= \sup_{z\in\mathbb{D}} (1-|z|^2)|f'(z)|.\end{aligned}$$

It follows from the definitions of Bloch space and BMOA space that

$$\| f \|_B \le \| f \|_{\mathrm{BMOA}}.$$

By Theorem 6.2 of [7] and the proof of Theorem 1 of [4], we have following lemma 2.3.

Lemma 2.3. Let n be a fixed positive integer and $f \in B$ with $(0) = f'(0) = \cdots = f^{(n-1)}(0) = 0$. If

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^2 (1-|z|^2)^{2n-2}(1-|\sigma_a(z)|^2) dA(z) \lesssim 1,$$

Then $\| f \|_{BMOA} \lesssim 1$.

The following criterion for compactness is a useful tool to us and it follows from standard arguments, for example, Proposition 3.11 of [5] and Lemma 2.10 of [12].

Lemma 2.4. Let $\alpha > 0$ and $n \in \mathbb{N} \cup \{0\}$. Suppose that h and φ are in $H(\mathbb{D})$ such that $\varphi(\mathbb{D}) \subset \mathbb{D}$. Then $I_{h,\varphi}^{(n)}: X \to B^\alpha$, where $X = B, BMOA$ or B_0, is compact if and only if for any sequence $\{f_m\}$ in X with $\sup \|f_m\|_X = M < \infty$ and which converges to zero locally uniformly on $\mathbb{D}$, we have $\lim_{m\to\infty} \left\|I_{h,\varphi}^{(n)} f_m\right\|_{\mathcal{B}^\alpha} = 0$.

3. Main results

In this section, we characterize boundedness and compactness of $I_{h,\varphi}^{(n)}$ from the Bloch space, H^∞, BMOA and B_0 space to weighted Bloch spaces.

Theorem 3.1. Let $\alpha > 0, h \in H(\mathbb{D}), n \in \mathbb{N}$ and φ be a holomorphic self-map of D.Then following statements are equivalent:

1. $I_{h,\varphi}^{(n)}: BMOA \to B^\alpha$ is bounded.
2. $I_{h,\varphi}^{(n)}: B_0 \to B^\alpha$ is bounded.
3. $I_{h,\varphi}^{(n)}: B \to B^\alpha$ is bounded.
4. $\sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n} |h(z)| < \infty$.

Moreover, the following asymptotic relation holds

$$\|I_{h,\varphi}^{(n)}\| \approx \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)|. \tag{3.1}$$

Proof. Suppose (4) holds. Since

$$\begin{aligned}\|I_{h,\varphi}^{n} f\|_{\mathcal{B}^\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^{\alpha}\left|f^{(n)}(\varphi(z))\right||h(z)| \\ &= \sup_{z\in\mathbb{D}}(1-|\varphi(z)|^2)^{n}\left|f^{(n)}(\varphi(z))\right|\frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)| \\ &\lesssim \| f \|_{\mathcal{B}}\cdot \sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)|.\end{aligned}$$

Note that $\| f \|_{\mathcal{B}} \leq \| f \|_{\mathrm{BMOA}}$. Thus, we have $(4) \Rightarrow (3), (4) \Rightarrow (2)$ and $(4) \Rightarrow (1)$. Moreover, we have

$$\|I_{h,\varphi}^{(n)}\| \lesssim \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)|.$$

On the other hands, set

$$\lambda = \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)|.$$

Assume that $\lambda < \infty$. For any $\varepsilon > 0$, then we can find $b \in \mathbb{D}$ such that

$$\frac{(1-|b|^2)^{\alpha}}{(1-|\varphi(b)|^2)^n}|h(b)| > \lambda - \varepsilon.$$

If $\varphi(b) = 0$, then choose the text function $g(z) = z^n$. It is clearly that $g \in \mathcal{B}_0$ and $\| g \|_{\mathcal{B}} \approx 1$. In the meantime $\| g \|_{BMOA} \approx 1$. So,

$$\|I_{h,\varphi}^{(n)}\| \gtrsim \|I_{h,\varphi}^{(n)} g(z)\|_{\mathcal{B}^\alpha} > \lambda - \varepsilon.$$

If $\varphi(b) \neq 0$, consider the function

$$g(z) = \frac{1}{\bar{a}^n}\frac{\left(1-|a|^2\right)^n}{\left(1-\bar{a}z\right)^n} \triangleq \sum_{j=1}^{\infty} c_j z^j,$$

Where
$a = \varphi(b)$. Let $F(z) = \sum_{j=n}^{\infty} c_j z^j$. Then $F(0) = F'(0) = \cdots = F^{(n-1)}(0) = 0$ and

$$F^{(n)}(z) = \left(\frac{1-|a|^2}{(1-\bar{a}z)^2}\right)^n$$

It is easy to see that

$$(1-|z|^2)^n\left|F^{(n)}\right| = (1-|\sigma_a(z)|^2)^n \leq 1.$$

So, by Theorem 5.4 of [21], we have $\| F \|_{\mathcal{B}} \approx 1$. Simultaneously, $F \in \mathcal{B}_0$. By Lemma 1 of [19] and Lemma 2.3, we get $\| F \|_{BMOA} \approx 1$. Therefore we have

$$\|I_{h,\varphi}^{(n)}\| \gtrsim \|I_{h,\varphi}^{(n)} F(z)\|_{\mathcal{B}^\alpha} > \lambda - \varepsilon.$$

Thus, we get

$$\|I_{h,\varphi}^{(n)}\| \approx \sup_{z\in\mathbb{D}} \frac{(1-|z|^2)^{\alpha}}{(1-|\varphi(z)|^2)^n}|h(z)|.$$

This completes the proof.

If $\varphi(z) = z$ in Theorem 3.1, then we can obtain a result for $I_h^{(n)}$ quickly, where

$$I_h^{(n)} f(z) = \int_0^z f^{(n)}(\zeta)h(\zeta)d\zeta,\ f \in H(\mathbb{D}).$$

Corollary 3.2. Let $\alpha > 0, h \in H(\mathbb{D})$ and $n \in \mathbb{N}$. Then following statements are equivalent:

(1) $I_h^{(n)}: BMOA \to B^\alpha$ is bounded.

(2) $I_h^{(n)}: B_0 \to B^\alpha$ is bounded.

(3) $I_h^{(n)}: B \to B^\alpha$ is bounded.

(4) $\sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha-n}|h(z)| < \infty$.

Moreover, the following asymptotic relation holds

$$\|I_h^{(n)}\| \approx \sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha-n}|h(z)|. \tag{3.2}$$

In the formulation of our next result, we write $\mathcal{A}_\infty^\alpha$ for the space of holomorphic functions f on $\mathbb{D}$ for which

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha|f(z)| < \infty$$

Corollary 3.3. Let $\alpha > 0, h \in H(\mathbb{D})$ and $n \in \mathbb{N}$. Then $I_h^{(n)}: X \to B^\alpha$ is bounded, where $X = B, BMOA$ or B_0 if and only if $h \in Y$, where

$$Y = \begin{cases} \mathcal{A}_\infty^{\alpha-n} & \text{if } n < \alpha \\ H^\infty & \text{if } n = \alpha \\ \{0\} & \text{if } n > \alpha \end{cases}$$

Proof. First suppose that $I_h^{(n)}: X \to B^\alpha$ is bounded. Then by Corollary 3.2, we have

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha-n}|h(z)| < \infty.$$

Thus for $n < \alpha, h \in \mathcal{A}_\infty^{\alpha-n}$ and for $n = \alpha, h \in H^\infty$. If $n > \alpha$, the condition

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)^{\alpha-n}|h(z)| < \infty$$

Implies that there is a positive constant C such that

$$|h(z)| \le C(1 - |z|^2)^{n-\alpha}$$

For all $z \in \mathbb{D}$. It follows that $|h(z)| \to 0$ as $|z| \to 1^-$. So by the Maximum modulus Theorem, we have $h \equiv 0$. Converse is obvious. We omit the details.

Corollary 3.4. Let $\alpha > 0, h \in H(\mathbb{D}), n \in \mathbb{N}$ and φ be a holomorphic self-map of D such that $|\varphi(z)| \le K < 1$. Then following statements are equivalent:

1. $I_{h,\varphi}^{(n)}: BMOA \to B^\alpha$ is bounded.
2. $I_{h,\varphi}^{(n)}: B_0 \to B^\alpha$ is bounded.
3. $I_{h,\varphi}^{(n)}: \mathcal{B} \to \mathcal{B}^\alpha$ is bounded.
4. $\sup_{z\in\mathbb{D}}(1 - |z|^2)^\alpha|h(z)| < \infty$.

Moreover, the following asymptotic relation holds

$$\|I_{h,\varphi}^{(n)}\| \approx \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)|. \quad (3.3)$$

Proof. It suffices to prove that if $|\varphi(z)| \le K < 1$, then

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| < \infty \quad (3.4)$$

if and only if

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| < \infty \quad (3.5)$$

Since

$$(1-|z|^2)^\alpha|h(z)| \le \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)|,$$

so (3.5) implies (3.4). Again if $|\varphi(z)| \le K$, then

$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| \le \frac{(1-|z|^2)^\alpha}{(1-K^2)^n}|h(z)|$, so (3.4) implies (3.5).

Now, we will give an example of φ and h such that (3.4) and (3.5) holds, but $\|\varphi\|_\infty = 1$

Example 3.5. Let $\alpha > 0, \beta > 0$ and $n \in \mathbb{N}$ be such that $\alpha = n + \beta$. Let $\varphi(z) = \frac{1-z}{2}$ and $h(z) = \frac{1}{(1-z)^\beta}$. Then by simple computation, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| < \infty$$

And

$$\sup_{z\in\mathbb{D}}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| < \infty.$$

Theorem 3.6. Let φ be a holomorphic self-map of $\mathbb{D}$ and $h \in H(\mathbb{D})$. Then following statements are equivalent:

1. $I_{h,\varphi}^{(0)}: \mathcal{B} \to B^\alpha$ is bounded.
2. $I_{h,\varphi}^{(0)}: BMOA \to B^\alpha$ is bounded.
3. $I_{h,\varphi}^{(0)}: B_0 \to B^\alpha$ is bounded.
4. $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| < \infty.$

Moreover, the following asymptotic relation holds

$$\|I_{h,\varphi}^{(0)}\| \approx \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\log\frac{2}{(1-|\varphi(z)|^2)}|h(z)|. \quad (3.6)$$

Proof. (1) ⇒ (4). Suppose that (1) holds. For $a \in \mathbb{D}$, let $f_a(z) = \log\frac{2}{1-\varphi(a)z}$. Then $f_a'(z) = \frac{\overline{\varphi(a)}}{1-\overline{\varphi(a)z}'}$, so that

$$|f_a(z)| \le \frac{1}{(1-|z|)}.$$

Thus $f_a \in B$. Moreover, $\|f_a\|_B \lesssim 1$. Thus we have

$$\begin{aligned}\left\|I_{h,\varphi}^{(0)}\right\| &\gtrsim \left\|I_{h,\varphi}^{(0)} f_a\right\|_{\mathcal{B}^\alpha} \\ &\geq (1-|a|^2)^\alpha |f(\varphi(a))h(a)| \\ &= (1-|a|^2)^\alpha \log\frac{2}{(1-|\varphi(a)|^2)}|h(a)|.\end{aligned}$$

Since $a \in \mathbb{D}$ is arbitrary, we have

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| < \infty.$$

The proof of (2) ⇒ (4) and (3) ⇒ (4) is similarly with (1) ⇒ (4). Since the text function $f_a(z)$ belongs to B_0 and BMOA, and by Lemma 1 of [16],

$$\|f_a(z)\|_{\mathrm{BMOA}} \leq \left\|\log\frac{2}{1-z}\right\|_{\mathrm{BMOA}} \lesssim 1.$$

(4) ⇒ (1). Suppose (2) holds. Let $f \in \mathcal{B}$, then for $z \in \mathbb{D}$, we have

$$(1-|z|^2)^\alpha |f(\varphi(z))||h(z)| \leq (1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| \parallel f \parallel_{\mathcal{B}}$$

hence $I_{h,\varphi}^{(0)}: B \to B^\alpha$ is bounded. Since $B_0 \subset B$, BMOA $\subset B$ and Lemma 2.2. Similarly with the proof of (4) ⇒ (1), we get (4) ⇒ (2) and (4) ⇒ (3).

Remark 3.7. For a holomorphic self-mapping φ of $\mathbb{D}$ and $n \in \mathbb{N} \cup \{0\}$, we define

$\mathcal{M}_n(\varphi, X, Y) = \left\{h \in H(\mathbb{D}): I_{h,\varphi}^{(n)}\right\}: X \to Y$ is bounded.

If $X = BMOA, B$ or B_0 and $Y = B^\alpha$ and $n_1 \geq n_2$, then we have

$\{0\} \subset \mathcal{M}_{n_1}(\varphi, X, Y) \subset \mathcal{M}_{n_2}(\varphi, X, Y) \subset \mathcal{A}_\infty^\alpha$.

If $|\varphi(z)| \leq K < 1$, then

$\mathcal{M}_n(\varphi, X, Y) = \mathcal{A}_\infty^\alpha$ for arbitrary n.

Theorem 3.8. Let $\alpha > 0, h \in H(\mathbb{D})$ and φ be a holomorphic self-map of $\mathbb{D}$. Then following statements are equivalent:
(1) $I_{h,\varphi}^{(0)}: H^\infty \to \mathcal{B}^\alpha$ is bounded.

(2) $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)| < \infty$.

Moreover, the following asymptotic relation holds

$$\left\|I_{h,\varphi}^{(0)}\right\| \approx \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)|. \quad (3.7)$$

Proof. (1) ⇒ (2). Suppose that (1) holds. Consider the text function $f(z) = 1$. Then we have

$$\left\|I_{h,\varphi}^{(0)}\right\| \geq \left\|I_{h,\varphi}^{(0)} f(z)\right\|_{B^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)|.$$

(2) ⇒ (1). Since

$$\left\|I_{h,\varphi}^{(0)} f(z)\right\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha f(\varphi(z))|h(z)| \leq \parallel f \parallel_\infty \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)|.$$

Theorem 3.9. Let $\alpha > 0, h \in H(\mathbb{D}), n \in \mathbb{N}$ and φ be a holomorphic self-map of D.
Then following statements are equivalent:

1. $I_{h,\varphi}^{(n)}: BMOA \to B^\alpha$ is compact.
2. $I_{h,\varphi}^{(n)}: B \to B^\alpha$ is compact.

3. $I_{h,\varphi}^{(n)}: \mathcal{B}_0 \to \mathcal{B}^\alpha$ is compact.

4. $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| < \infty$ and $\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| = 0$.

Remark 3.10. Noted that

$$\limsup_{|\varphi(z)|\to 1}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| = \lim_{s\to 1^-}\sup_{|\varphi(z)|>s}\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)|.$$

It is understood that if $\{z: |\varphi(z)| > s\}$ is an empty set for some $0 < s < 1$ the supremum equals 0. This happens when $\varphi(\mathbb{D})$ is a relatively compact subset of $\mathbb{D}$. So, if $\| \varphi \|_\infty < 1$, then $I_{h,\varphi}^n: \mathcal{B} \to B^\alpha$ is compact if and only if $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| < \infty$.
Proof. It is clear that (2) ⇒ (1) and (2) ⇒ (3). Thus to complete the proof, we only need to prove that (4) ⇒ (2), (2) ⇒ (4), (1) ⇒ (4) and (3) ⇒ (4).
(4) ⇒ (2). Let $\{f_m\}$ be a norm bounded sequence in $\mathcal{B}$ that converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_m\|f_m\|_{\mathcal{B}} < \infty$. Let $\varepsilon > 0$. Then there exists an r such that for $|\varphi(z)| > r$, we have

$$\frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^n}|h(z)| < \varepsilon.$$

Thus for $z \in \mathbb{D}$, we have

$$\begin{aligned}\left\|I_{h,\varphi}^{(n)}f_m\right\|_{\mathcal{B}\alpha} &= \sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha\left|f_m^{(n)}(\varphi(z))\right||h(z)| \\ &\lesssim \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha\left|f_m^{(n)}(\varphi(z))\right||h(z)| + \sup_{|\varphi(z)|>r}\frac{(1-|z|^2)^\alpha|h(z)|\|f_m\|_{\mathcal{B}}}{(1-|\varphi(z)|^2)^n} \\ &\le K\sup_{|z|\le r}\left|f_m^{(n)}(z)\right| + \varepsilon M,\end{aligned}$$

where $K = \sup_{z\in\mathcal{D}}(1-|z|^2)^\alpha|h(z)|$. Since $\varepsilon > 0$ is arbitrary and $\{f_m\} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus, we have $\left\|I_{h,\varphi}^{(n)}f_m\right\|_{\mathcal{B}\alpha} \to 0$ as $m \to \infty$. It follows from Lemma 2.4 that $I_{h,\varphi}^{(n)}: \mathcal{B} \to \mathcal{B}^\alpha$ is compact. (2) ⇒ (4). By taking $f(z) = \frac{z^n}{n!} \in \mathcal{B}$, we have that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|h(z)| < \infty.$$

Suppose that the second term in (4) does not hold. Then there is a sequence $\{z_m\}$ in $\mathbb{D}$ and $\delta > 0$ such that $|\varphi(z_m)| \to 1$ as $m \to \infty$ and

$$\frac{(1-|z_m|^2)^\alpha}{(1-|\varphi(z_m)|^2)^n}|h(z_m)| > \delta, \quad \text{as } m \to \infty.$$

Choose the subsequence of $\{z_m\}$ if necessary, suppose that $\inf_m|\varphi(z_m)| > 1/2$. Let

$$f_m(z) = \frac{1}{\overline{\varphi(z_m)^n}}\frac{(1-|\varphi(z_m)|^2)^n}{(1-\overline{\varphi(z_m)}z)^n}, z \in \mathbb{D}.$$

Then $f_m \in B$ and $\|f_m\|_B \lesssim 1$. Moreover,

$$f_m^{(n)}(z) = \frac{(1-|\varphi(z_m)|^2)^n}{\left(1-\overline{\Phi(Z_m)}Z\right)^{2n}}.$$

Thus

$$\left\|I_{h,\varphi}^{(n)}f_m\right\|_{\mathcal{B}^\alpha} \gtrsim \frac{(1-|z_m|^2)^\alpha}{(1-|\varphi(z_m)|^2)^n}|h(z_m)| > \delta,$$

as $m \to \infty$. This contradict with Lemma 2.4. Since the test function used in (2) ⇒ (4) is in BMOA and $\mathcal{B}_0$, so the proof of (1) ⇒ (4) and (3) ⇒ (4) on the same lines as the proof of (2) ⇒ (4). We omit the details.

Corollary 3.11. Let $\alpha > 0, h \in H(\mathbb{D})$ and $n \in \mathbb{N}$. Then following statements are equivalent:

1. $I_h^{(n)}: BMOA \to B^\alpha$ is compact.
2. $I_h^{(n)}: B \to B^\alpha$ is compact.
3. $I_h^{(n)}: B_0 \to B^\alpha$ is compact.
4. $\lim_{|z|\to 1}(1-|z|^2)^{\alpha-n}|h(z)| = 0$.

Theorem 3.12. Let φ be a holomorphic self-map of $\mathbb{D}$ and $h \in H(\mathbb{D})$. Then following statements are equivalent:

1. $I_{h,\varphi}^{(0)}: B \to B^\alpha$ is compact.
2. $I_{h,\varphi}^{(0)}: BMOA \to B^\alpha$ is compact.
3. $I_{h,\varphi}^{(0)}: \mathcal{B}_0 \to \mathcal{B}^\alpha$ is compact.
4. $\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)| < \infty$ and

$$\limsup_{|\varphi(z)|\to 1^-}(1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| = 0.$$

Proof. It is clear that $(1) \Rightarrow (2)$ and $(1) \Rightarrow (3)$. Thus to complete the proof, we only need to prove that $(4) \Rightarrow (1), (1) \Rightarrow (4), (2) \Rightarrow (4)$ and $(3) \Rightarrow (4)$.

$(4) \Rightarrow (1)$. Let $\{f_m\}$ be a norm bounded sequence in B that converges to zero uniformly on compact subsets of $\mathbb{D}$. Let $M = \sup_m \|f_m\|_{\mathcal{B}} < \infty$. Let $\varepsilon > 0$. Then there exists an r such that for $|\varphi(z)| > r$, we have

$$(1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| < \varepsilon.$$

So,

$$\begin{aligned}\left\|I_{h,\varphi}^{(0)} f_m\right\|_{\mathcal{B}\alpha} &= \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f_m(\varphi(z))||h(z)| \\ &\le \sup_{|\varphi(z)|\le r}(1-|z|^2)^\alpha |f_m(\varphi(z))||h(z)| + \sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha |f_m(\varphi(z))||h(z)| \\ &\le K\sup_{|z|\le r}|f_m(z)| + \|f_m\|_{\mathcal{B}} \sup_{|\varphi(z)|>r}(1-|z|^2)^\alpha \log\frac{2}{(1-|\varphi(z)|^2)}|h(z)| \\ &\le K\sup_{|z|\le r}|f_m(z)| + M\varepsilon,\end{aligned}$$

where $K = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)|$. Since $\varepsilon > 0$ is arbitrary and $\{f_m\} \to 0$ uniformly on compact subsets of $\mathbb{D}$. Thus,

$$\left\|I_{h,\varphi}^{(0)} f_m\right\|_{\mathcal{B}^\alpha} \to 0$$

as $m \to \infty$. Hence, $I_{h,\varphi}^{(0)}: \mathcal{B} \to B^\alpha$ is compact. $(1) \Rightarrow (4)$. Choose $f(z) = 1 \in \mathcal{B}$, we have

$$\left\|I_{h,\varphi}^{(0)} f\right\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |h(z)| < \infty.$$

Assume that the second term in (4) does not hold. Then there exist a sequence $\{a_n\} \in \mathbb{D}$ and $\delta > 0$ such that $|\varphi(a_n)| \to 1^-$as $n \to \infty$ and

$$(1-|a_n|^2)^\alpha \log\frac{2}{(1-|\varphi(a_n)|^2)}|h(a_n)| > \delta, \text{ as } n \to \infty.$$

Choose the subsequence of $\{a_n\}$ if necessary, suppose that $\inf_n |\varphi(a_n)| > 1/2$. Let

$$f_n(z) = \left(\log\frac{2}{1-\overline{\Phi(a_n)}Z}\right)^2 \Big/ \log\frac{2}{1-|\varphi(a_n)|^2}.$$

Then $f_n \in \mathcal{B}$ and $\|f_n\|_{\mathcal{B}} \lesssim 1$. So, we have

$$\left\|I_{h,\varphi}^{(0)} f_n\right\|_{\mathcal{B}^\alpha} \gtrsim (1-|a_n|^2)^\alpha \log\frac{2}{(1-|\varphi(a_n)|^2)}|h(a_n)| > \delta$$

as $n \to \infty$. This contradict with Lemma 2.4. Since the test function used in (1) ⇒ (4) is also in $BMOA$ and B_0. So the proof of (2) ⇒ (4) and (3) ⇒ (4) is similarly with (1) ⇒ (4). We omit them for details.

References

1. Aleman A, Cima JA. An integral operator on H^p and Hardy's inequality, J Anal. Math. 2001;85:157-176.
2. Aleman A, Siskakis AG. An integral operator on H^p, Complex Variables Theory Appl. 1995;28(2):149-158.
3. Aleman A, Siskakis AG. Integration operators on Bergman spaces, Indiana Univ. Math. J. 1997;46:337-356.
4. Aulaskari R, Nowak M, Zhao R. The nth derivative characterisation of Möbius invariant Dirichlet space, Bull. Austral. Math. Soc. 1998;58(1):43-56.
5. Cowen CC, MacCluer BD. Composition operators on spaces of analytic functions, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL. 1995.
6. Duren PL. Theory of H^p spaces, Pure and Applied Mathematics, 38. Academic Press, New York-London 1970.
7. Girela D. Analytic functions of bounded mean oscillation, Complex Function Spaces (Mekrijärvi, 1999), 61C170, Univ. Joensuu Dept. Math. Rep. Ser, 4, Univ. Joensuu, Joensuu, 2001.
8. Li S, Stević S. Generalized composition operators on Zygmund spaces and Bloch type spaces, J Math. Anal. Appl. 2008;338(2):12821295.
9. Manhas J, Zhao R. New estimates of essential norms of weighted composition operators between Bloch type spaces, J Math. Anal. Appl. 2012;389(1):32-47.
10. Sharma A, Sharma AK. Carleson measures and a class of generalued integration operators on the Bergman space, Rocky Mountain J Math. 2011;41(5):1711-1724.
11. Siskakis, A. G. and Zhao, R., A Volterra type operator on spaces of analytic functions, Function Spaces (Edwardsville, IL, 1998), 299-311, Contemp. Math, 232, Amer. Math. Soc, Providence, RI. 1999.
12. Tjani M. Compact composition operators on some Möbus invariant Banach spaces, the sis (Ph.D.) Michigan State University, 1996.
13. Wulan H, Zheng D, Zhu K. Compact composition operators on BMOA and the Bloch space, Proc. Amer. Math. Soc. 2009;137(11):3861-3868.
14. Wu Y, Wulan H. Products of differentiation and composition operators on the Bloch space, Collectanea Mathematica. Doi:10.1007/s13348-010-0030-8.
15. Xiong C, Liu J. Norm-attaining integral operators on analytic function spaces, Manuscript.
16. Ye S, Lou Z. Cyclic vectors and cellular indecomposable operators on Q_p spaces, Acta Math Sci. Ser. B Engl. Ed. 2011;31(2):434440.
17. Yoneda R. Integration operators on weighted Bloch spaces, Nihonkai Math. J. 2001;12(2):123-133.
18. Yoneda R. Multiplication operators, integration operators and companion operators on weighted Bloch space, Hokkaido Math. J. 2005;34(1):135-147.
19. Zhao R. Distances from Bloch functions to some Möbus invariant spaces, Ann Acad. Sci. Fenn. Math. 2008;33(1):303-313.
20. Zhao R. Essential norms of composition operators between Bloch type spaces, Proc. Amer. Math. Soc. 2010;138(7):2537-2546.
21. Zhu K. Operator theory in function spaces, Second edition. Mathematical surveys and Mono graphs, 138. Amer. math. soc, Providence, 2007.