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Systems of difference equations

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Abstract
In mathematics, a system of differential equations is a finite set of differential equations. Such a system can be either linear or non-linear. In addition, such a system can be either a system of ordinary differential equations or a system of partial differential equations. Difference equation have played an important role in analysis of mathematical model of biology, physics. Economics and Engineering. Recently, there has been a great interest in studying properties of rational difference equation. Studying equilibrium properties requires studying the properties of a system of difference equation.

Keywords: Systems of difference equations: solution formula for some homogeneous, nonhomogeneous systems and autonomous systems

Introduction
In this section we are interested in finding solution of the following system of linear equations:

$$X_1(n+1)=a_{11}X_1(n)+a_{12}X_2(n)+\ldots+a_{1k}X_k(n)+b_1$$

$$X_2(n+1)=a_{21}X_1(n)+a_{22}X_2(n)+\ldots+a_{2k}X_k(n)+b_2$$

$$X_k(n+1)=a_{k1}X_1(n)+a_{k2}X_2(n)+\ldots+a_{kk}X_k(n)+b_k$$

Or. Equivalently

$$X(n+1)=\begin{pmatrix} X_1(n+1)\\ X_2(n+1)\\ \vdots\\ X_k(n+1)\end{pmatrix}=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k1} & a_{2k} & \cdots & a_{kk}\end{pmatrix}\begin{pmatrix} X_1(n)\\ X_2(n)\\ \vdots\\ X_k(n)\end{pmatrix}+\begin{pmatrix} b_1\\ b_2\\ \vdots\\ b_k\end{pmatrix}$$

This system may be written in the form

$$X(n+1)=AX(n)+B \tag{1}$$

Where A is a nonsingular constant matrix, and b_j is in the set of all integer including zero.

The matrix A is called *the companion matrix* of Equation (1).

Theorem1
Let $k\in\mathbb{N}, k\geq 2.$ A linear in inhomogeneous of k first order difference equations is given by Equation (1). Then the solution is given by

$$X(n)=A^nX(0)+\sum_{j=0}^{n-1}A^jB \tag{2}$$

Where $A^0 = I$.

Proof
From (2),

$$X(n+1) = A^{n+1}X(0) + \sum_{j=0}^{n} A^j B = A^{n+1}X(0) + A^n B + \cdots +$$

$$= A[A^n X(0) + A^{n-1}B + \cdots + IB] + B = A[A^n X(0) + \sum_{j=0}^{n-1} A^j B] + B$$

$$X(n+1) = AX(0) + B$$

Lemma 1
If $\lambda = 1$ is not an eigenvalue of a square matrix $\lambda = 1$, and be $n \in \mathrm{N}$, then

$$\sum_{j=0}^{n-1} A^j = (A^n - I)(A - I)^{-1}$$

Proof
Since $\lambda = 1$ is not an eigenvalue, then $|A - 1I| \neq 0, i.e (A - I)^{-1}$ exists. For $n = 1$

$$\sum_{j=0}^{1-1} A^j = A^0 = I = (A^1 - I)(A - I)^{-1}$$

Assume that the formula holds for $n = k, i.e$

$$\sum_{j=0}^{k-1} A^j = (A^k - I)(A - I)^{-1}$$

Then for $n = k + 1$

$$\sum_{j=0}^{k} A^j = A^k + \sum_{j=0}^{k-1} A^j = A^k + (A^k - I)(A - I)^{-1}$$

$$[A^k(A - I) + (A^k - I)](A - I)^{-1} = (A^{k+1} - A^k + A^k - I)$$

Hence

$$\sum_{j=0}^{k} A^j = (A^{k+1} - I)(A - I)^{-1}$$

Corollary
If $\lambda = 1$ is not an eigenvalue of a square matrix $\lambda = 1$, and be $n \in \mathrm{N}$,then the solution

$$X(n+1) = AX(n) + B$$

Is given by

$$X(n) = A^n X(0) + (A^n - I)(A - I)^{-1} B$$

Proof
From the theorem 1

$$A^n X(0)+\left(\sum_{j=0}^{n-1} A^j\right)B \quad X(n)=A^n X(0)+\sum_{j=0}^{n-1} A^j B=$$

$$A^n X(0)+(A^n-I)(A-I)^{-1}B$$

First Order System of Difference Equations
Consider the first order system of difference equation

$$X(n+1)=AX(n)+B(n) \qquad A\neq 0$$

Theorem 2[3]
Let

$$X(n)=\begin{pmatrix} X_1(n)\\ X_2(n)\\ \vdots \\ X_k(n)\end{pmatrix}, \quad A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k}\\ a_{21} & a_{22} & \cdots & a_{2k}\\ \vdots & \vdots & \ddots & \vdots\\ a_{k1} & a_{2k} & \cdots & a_{kk}\end{pmatrix}, \quad B(n)=\begin{pmatrix} b_1(n)\\ b_2(n)\\ \vdots \\ b_k(n)\end{pmatrix}$$

Then, the system of first order of difference equations

$$X(n+1)=AX(n)+B(n)$$

Has the solution

$$X(n)=A^n X(0)+\sum_{j=0}^{n-1} A^{n-1-j}B(j) \tag{4}$$

Proof

From (4), replace $"n" \quad by \quad "n+1"$,

$$X(n+1)=A^{n+1}X(0)+\sum_{j=0}^{n} A^{n-j}B(j)=A^{n+1}X(0)+A^nB(0)+A^{n-1}B(1)+\cdots+AB(n-1)+B(n)$$

$$A[A^n X(0)+A^{n-1}B(0)+A^{n-2}B(1)+\cdots+B(n-1)]+B(n)=$$

$$A[A^n X(0)+\sum_{j=0}^{n-1} A^{n-1-j}B(j)]+B(n).$$

Thus, from (4)

$$X(n+1)=AX(n)+B(n)$$

First Order System of Difference Equations
Homogenous
We start the analysis with the discussion of the homogenous equation

$$X(n+1)=AX(n) \tag{5}$$

Where $n\geq 0$.
The general solution of the equation $X(n+1)=AX(n)$ is

$$X(n+1)=A^n X(0) \tag{6}$$

Where

$$A^j : R^j \to R, \qquad j = 0,1,\cdots,n$$

Let n=0, then

$$X(n) = A^0 X(0)$$

Where

$$A^0 = I, \quad X(0) = (X_1(0), X_2(0),\cdots, X_k(0)) \quad \textit{are given}$$

Let $X(0) = c$, then the general solution can be written as

$$X(n) = A^n c, \quad n \in N \tag{7}$$

Where c is an arbitrary constant vector. To find the general solution of Equation (5), we need to find A^n.

Definition 1 [3]
The solution $X(n) = (X_1(n), X_2(n),\cdots, X_k(n))$ are linearly independent for $n \geq 0$ if and if the matrix A is nonsingular $(\det A \neq 0)$, for all $n \geq 0$.

Let $\Phi(n)$ a $k \times k$ matrix whose columns of $X(n) = A^n X(0),$ we write

$$\Phi(n) = (X_1(n), X_2(n),\cdots, X_k(n))$$

Implies

$$\Phi(n+1) = (A(n)X_1(n), A(n)X_2(n),\cdots, A(n)X_k(n))$$

$$= A(n)(X_1(n), X_2(n),\cdots, X_k(n)) = A(n)\Phi(n)$$

Then

$$\Phi(n+1) = A(n)\Phi(n), \tag{9}$$

Definition 2 [4]
If $\Phi(n)$ is a matrix that is nonsingular for all $n \geq 0$ and satisfies the homogenous matrix system equation $\Phi(n+1) = A(n) \quad \Phi(n)$, then it is said to be *a fundamental matrix* for system of equation $X(n) = A^n X(0).$

Remark
If $\Phi(n)$ is a fundamental matrix and C is any nonsingular matrix, then $\Phi(n)C$ is also a fundamental matrix.
The Casarotian matrix $W(n)\, of \quad X^{(1)}, X^{(2)}, \cdots, X^{(k)} \quad with \quad k \geq 1$ is defined as

$$W(n) = \det\begin{pmatrix} X_1^{(1)}(n) & X_1^{(2)}(n) & \cdots & X_1^{(k)}(n) \\ X_2^{(1)}(n) & X_2^{(2)}(n) & \cdots & X_2^{(k)}(n) \\ \vdots & \vdots & \ddots & \vdots \\ X_k^{(1)}(n) & X_k^{(2)}(n) & \cdots & X_k^{(k)}(n) \end{pmatrix}$$

If $n = 0$, this implies that $\Phi(n) = I_k$, then $\Phi(n)$ is called *a principal fundamental matrix.*
Let $\Phi(n) = A^n$, then $X(n) = \Phi(n)C = A^n C$.

Where $C \in R^k$ is a constant vector. Then any linearly independent set of solutions of $X(n+1) = AX(n).$

The general solution of this equation is defined to be

$$X(n)=\sum_{i=1}^{k}C_i X_i(n), \qquad C_i \neq 0$$

Where

$C_i=(C_1,C_2,\cdots,C_k)^T \in R, and \quad X_i(n)=(X_1(n),X_2(n),\cdots,X_k(n))^T$ is a fundamental matrix.

Solution Formula Homogenous Systems

To find solution of the homogenous system $X(n)=AX(n-1).$, we need to find the eigenvalues of $A\{\lambda_1,\lambda_2,\cdots,\lambda_k\}$, and eigenvectors $\{v_i, i=1,2,\ldots,k\}$ that is satisfies

$$Av_i=\lambda_i v_i \rightarrow (A-\lambda I)v_i=0$$

Let $p=v_i, i=1,2,\ldots,k, \quad where \quad v_i=(v_{i1},v_{i2},\cdots,v_{ik})^T$.

Put

$$D=diag[\lambda_1,\lambda_2,\cdots,\lambda_k]$$

$$D=\begin{pmatrix}\lambda_1 & 0 & \cdots & 0\\ 0 & \lambda_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_k\end{pmatrix}$$

Then $P^{-1}AP=D=diag[\lambda_i]_{i=1,2,\ldots,k}$, equivalently $A=PDP^{-1}$ note that for all λ_i there exists unique eigenvector v, satisfies

$$(A-\lambda_i I)v_i=0, \qquad forall \quad i=1,2,\ldots,k.$$

To find the solution of system $X(n)=A^n X(0).$, we need to find A^n is follows as

$$A^n=PD^nP^{-1}$$

$$D^n=\begin{pmatrix}\lambda_1^n & 0 & \cdots & 0\\ 0 & \lambda_2^n & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_k^n\end{pmatrix}$$

Where

Put

$$X_i(n)=v_i\lambda_i^n, \qquad i=1,2,\ldots,k.$$

We obtain

$$X_i(n)=\sum C_i X_i(n)=\sum C_i v_i \lambda_i^n, \qquad n\geq 0 \tag{9}$$

For some constants $C_1,C_2,\ldots,C_k$.

Definition 3 [9]

Let $v_1, v_2, \ldots, v_k$ be a set of k nonzero vectors (all the same dimension), these vectors are called linearly independent if the vector equation

$$C_1 v_1 + C_2 v_2 + \ldots + C_k v_k = 0.$$

Has only the trivial solution.

Example

To find the general solution of

$$X(n+1) = AX(n),$$

With

$$A = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

We solve the equation

$$(A - \lambda_i I)v_i = 0,$$

$$\begin{vmatrix} -\lambda & 0 & -1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0 \rightarrow \lambda_1 = \lambda_2 = 1, \lambda_3 = -1$$

at $\lambda = 1$

$$\begin{pmatrix} -1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} v_{11} \\ v_{21} \\ v_{31} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \quad and \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

At $\lambda = -1$

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_{13} \\ v_{23} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$P = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

This implies that

$$P^{-1} = \begin{pmatrix} \frac{-1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Where

$$PAP^{-1} = D = diag(\lambda_i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Note that

$$A^n = diag(\lambda_i^n) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (-1)^n \end{pmatrix}$$

The general solution of equation

$X(n) = A^n X(0)$ is

$$X(n) = \sum_{i=1}^{3} C_i v_i \lambda_i^n = C_1 v_1 \lambda_1^n + C_2 v_2 \lambda_2^n + C_3 v_3 \lambda_3^n = C_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + C_3 (-1)^n \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

$$X(n) = \begin{pmatrix} -C_1 + C_3(-1)^n \\ C_2 \\ C_1 + C_3(-1)^n \end{pmatrix}.$$

Suppose that

$$X(0) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

To find the solution $X(0)$ with initial value, then

$$-C_1 + C_3 = 2$$
$$C_2 = 0$$
$$C_1 + C_3 = 0$$

Solving this system gives $C_1 = -1, C_2 = 0, \quad and \quad C_3 = 1,$ leading us to the solution.

Let $X(n) = \Phi(n)\Phi^{-1}(0)X(0), \qquad where \qquad \Phi(n) = (v_i\,\lambda_i^n, i = 1,2,3)$,

This given

$$\Phi(n) = \begin{pmatrix} -1 & 0 & (-1)^n \\ 0 & 1 & 0 \\ 1 & 0 & (-1)^n \end{pmatrix} \Rightarrow \Phi(0) = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix},$$

Thus

$$\Phi^{-1}(n)=\begin{pmatrix} \frac{-1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix},$$

This given

$$X(n)=\begin{pmatrix} -1 & 0 & (-1)^n \\ 0 & 1 & 0 \\ 1 & 0 & (-1)^n \end{pmatrix}$$

$$\begin{pmatrix} \frac{-1}{2} & 0 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix},$$

Then

$$X(n)=\begin{pmatrix} 1+(-1)^n \\ 0 \\ -1+(-1)^n \end{pmatrix}.$$

Example

To find the general solution of equation 8

$$X(n)=A^n X(0)$$

Where

$$A=\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Implies

$$|A-\lambda I|=\begin{vmatrix} \cos\theta-\lambda & \sin\theta \\ -\sin\theta & \cos\theta-\lambda \end{vmatrix}=0, \quad or \quad \lambda^2-2\lambda\cos\theta+1=0,$$

$$\lambda_{1,2}=\frac{2\cos\theta\pm\sqrt{4\cos^2\theta-4}}{2}=\cos\theta\pm i\sin\theta,$$

At

$$\lambda=\cos\theta+i\sin\theta, \quad implies \quad \begin{pmatrix} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{pmatrix}\begin{pmatrix} v_{11} \\ v_{21} \end{pmatrix}= \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

This given

$$v_{11} = -iv_{21} \quad or \quad v_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix},$$

At

$$\lambda = \cos\theta - i\sin\theta, \quad implies \quad \begin{pmatrix} i\sin\theta & \sin\theta \\ -\sin\theta & i\sin\theta \end{pmatrix}\begin{pmatrix} v_{12} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

This given

$$v_{12} = -\frac{1}{i}v_{22} \quad or \quad v_2 = \begin{pmatrix} i \\ 1 \end{pmatrix},$$

Hence

$$P = \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix} \quad \Rightarrow \quad p^{-1} = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{-i}{2} & \frac{1}{2} \end{pmatrix},$$

Where

$$PAP^{-1} = diag[\lambda_1, \lambda_2].$$

The general solution of this system is

$$X(n) = \sum_{i=1}^{2} C_i v_i \lambda_i^n = C_1 v_1 \lambda_1^n + C_2 v_2 \lambda_2^n = C_1(\cos\theta + i\sin\theta)^n \begin{pmatrix} -i \\ 1 \end{pmatrix} + C_2(\cos\theta - i\sin\theta)^n \begin{pmatrix} i \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -iC_1(\cos n\theta + i\sin n\theta) + iC_2(\cos n\theta - i\sin n\theta) \\ C_1(\cos n\theta + i\sin n\theta) + C_2(\cos n\theta - i\sin n\theta) \end{pmatrix}.$$

Let

$$X(0) = \begin{pmatrix} i \\ 3 \end{pmatrix}, \quad then \quad \begin{pmatrix} -iC_1 + iC_2 \\ C_1 + C_2 \end{pmatrix} = \begin{pmatrix} i \\ 3 \end{pmatrix}.$$

$$-iC_1 + iC_2 = i, \quad C_1 + C_2 = 3,$$
$$\Rightarrow C_1 = 1, \quad C_2 = 2$$

Implies

$$X(n) = \begin{pmatrix} -i(\cos n\theta + i\sin n\theta) + 2i(\cos n\theta - i\sin n\theta) \\ (\cos n\theta + i\sin n\theta) + 2(\cos n\theta - i\sin n\theta) \end{pmatrix}. = \begin{pmatrix} 3\sin n\theta + i\sin n\theta \\ 3\cos n\theta - i\sin n\theta \end{pmatrix}.$$

If $\lambda = \alpha + i\beta$ is an eigenvalue of A , then $\lambda = \alpha - i\beta$ is also an eigenvalue of A .

Some theorems pertaining to the eigenvalue problem

- The eigenvalues of a symmetric matrix are real.
- The eigenvalues of an anti-symmetric matrix are purely imaginary or zero.
- The eigenvalues of a unitary matrix have absolute value equals to one.

- If the eigenvalues of a square matrix are not distinct, the corresponding eigenvalue form a basis (i.e. they from a linearly independent set.)

If the system

$$\sum_{i=1}^{k} C_i v_i = b$$

consistent, we say that b is a linear combination of $v_1, v_2, \ldots, v_k$. [2, 9].

Non-homogeneous First Order System of Difference Equation

Consider now the first order system of difference equation

$$X(n) = AX(n-1) + B(n) \tag{10}$$

Where

$$X(n) = (X_i(n), i = 1,2,\ldots,k)^T; \qquad B(0) = (B_i(n), i = 1,2,\ldots,k)^T$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{2k} & \cdots & a_{kk} \end{pmatrix}.$$

Theorem 3 [7]

Every solution $X(n)$ to the first order nonhomogeneous system (10) can be represented as the sum of the general solution $X_g(n)$ to the homogeneous system (5), and a particular solution to the nonhomogeneous system (10), $X_p(n)$:

$$X(n) = X_g(n) + X_p(n) = A^n C + X_p(n).$$

A particular solution can be found by iterating the difference equation backwards

$$X(n) = AX(n-1) + B(n)$$

$$X(n) = A(AX(n-2) + B(n-1)) + B(n) = A^2 X(n-2) + AB(n-1) + B(n)$$

$$\vdots$$

$$X(n) = A^n X(0) + A^{n-1}B(1) + A^{n-2}B(2) + \ldots + AB(n-1) + B(n)$$

$$= A^n X(0) + \sum_{j=0}^{n-1} A^j B(n-j)$$

This suggests to take:

$$X_p(n) = \sum_{j=0}^{n-1} A^j B(n-j)$$

As particular solution. Then, we can write the general solution to the non-homogeneous system in the form

$$X(n) = A^n C + \sum_{j=0}^{n-1} A^j B(n-j)$$

For the initial value problem $X(0) = C$.

Example

Non-homogeneous First Order Matrix Difference Equations and the Steady State

An example of a non-homogeneous first order matrix difference equation is

$$X(n) = AX(n-1) + B$$

With additive constant vector B ,the steady state of this system is a value X^* of the vector X which. If reached, would not be deviated form in terms of deviations from the steady state

$$X(n) = X(n-1) = X^*$$

In the difference equation and solving for X^* to obtain

$$X^* = [I - A]^{-1} B$$

Where I is the $k \times k$ identity matrix, and where it is assumed that $[I - A]$ is invertible. Then, the non-homogeneous equation can be rewritten in homogeneous form in terms of deviations from the steady state

$$[X(n) - X^*] = A[X(n-1) - X^*].$$

Stability of the First Order case
Definition 5

The first-order matrix difference equation $[X(n) - X^*] = A[X(n-1) - X^*]$. is stable, that is, $X(n)$ converges asymptotically to the steady state X^*.

Theorem 4 [8]

The first-order matrix difference equation $[X(n) - X^*] = A[X(n-1) - X^*]$. is stable if and only if all eigenvalues of the transition matrix A (whether real or complex) have an absolute value which is less than 1.

Stability of the Linear System

The eigenvalue can be real or complex value. If $\lambda = \alpha + i\beta$ is an eigenvalue

- Every solution is stable if all the eigenvalues of A have negative real part.
- Every solution is unstable if at Least one eigenvalues of A has positive real part.
- Suppose that the eigenvalues of A are $\lambda_j = \alpha_j \pm i\beta_j$

If $\alpha \le 0$, then every solution is stable if A has linearly independent eigenvectors for each λ_j , otherwise, every solution is unstable.

Lemma 2 [3]

The zero solution of the homogeneous system $X(n) = AX(n-1)$ is stable if and only if there exists $M > 0$ such that

$$\|A^n\| \le M \qquad forall \quad n \ge 0$$

Theorem 5 [7]

For the homogeneous system $X(n) = AX(n-1)$ the following statements are true

- The zero solution is stable if and only if $\rho(A) \le 1$ and the eigenvalues on the unit circle are semisimple.
- The zero solution is asymptotically stable if and only if $\rho(A) < 1$, where $\rho(A) = \max\{|\lambda| : \lambda$ is an eigenvalue of $A\}$.

Theorem 6 [3]

A hyperbolic matrix is matrix with no eigenvalues on the unit circle. If A is a hyperbolic matrix, then the corresponding linear homogeneous difference equation $X(n) = AX(n-1)$ is also called *hyperbolic*.

Theorem 7 [3]

The zero solution of the hyper linear difference equation $X(n) = AX(n-1)$ is called *asaddle point* if there exist at least two eigenvalues of A, λ_u *and* λ_s , such that $|\lambda_u| > 1 and \quad |\lambda_s| < 1$.

Solution and Stability of Higher-Order Cases

Matrix difference equations of higher order-that is, with a time lag longer than one period-can be solved, and their stability analyzed. By converting them into first –order form using a block matrix.

Example

Suppose we have the second-order equation

$$X(n) = AX(n-1) + BX(n-2)$$

With the variable vector X being $k \times 1$ matrix, and A *and* B being $k \times k$ matrix. This can be stacked in the form

$$\begin{pmatrix} X(n) \\ X(n-1) \end{pmatrix} = \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} X(n-1) \\ X(n-2) \end{pmatrix}$$

Where I is the $k \times k$ identity matrix, and 0 is the zero matrix. Then denoting the $2k \times 1$ stacked vector of current and once-lagged variables as $Y(n)$ and $2k \times 2k$ block matrix as A, we have as before the solution

$$Y(n) = A^n Y(0)$$

Also as before. This stacked equation and thus the original second-order equations arestable if and only if all eigenvalues of the matrix A are smaller than unity in absolute value.

Autonomous Systems

We say that the two $k \times k$ matrices A *and* B are similar if there exists a nonsingular matrix p such that

$$P^{-1}AP = B$$

This implies that A *and* B have the some eigenvalues.

Diagonalizable Matrices

Theorem 6 [3]

A $k \times k$ square matrix is diagonalizable if and only if it has k linearly independent eigenvectors.

Proof

Let $\lambda_1, \lambda_2, \cdots, \lambda_k$ be the eigenvalues of D and let $v_1, v_2, \ldots, v_k$. be the corresponding linearly Independent eigenvectors of D .then form formula

$$\Phi(n) = PD^n = P\begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix}$$

We have

$$\Phi(n) = [v_1, v_2, \ldots, v_k]\begin{pmatrix} \lambda_1^n & 0 & \cdots & 0 \\ 0 & \lambda_2^n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k^n \end{pmatrix} = [v_1\lambda_1^n, v_2\lambda_2^n, \ldots, v_k\lambda_k^n].$$

Notice that since colummns of $\Phi(n)$ are solution of the system $X(n) = AX(n-1)$, then $X(n) = v_i\lambda_i^n$, $i = 1,2,\ldots,n$,a solution of this system.

Hence, the general solution of $X(n) = A^n X(0)$ my be given by

$$X(n)=\sum_{i=1}^{k} C_i v_i \lambda_i^n .$$

The Jordan Form:

If the matrix A is not diagonalizable, then

A has repeated eigenvalues, and one is not stable to generate k linearly independent eigenvectors.

If a $k \times k$ matrix A is not diagonalizable, then it is akin to the so-called Jordan form, i.e. $P^{-1}AP = J$, where

$$J = diag(J_1, J_2, \ldots, J_r) \quad {}_{1 \le r \le n} \tag{11}$$

$$J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots & 0 \\ 0 & \lambda_i & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_i \end{pmatrix} \tag{12}$$

The matrix J_i is called a *Jordan block* .this matrix has only one eigenvector, $v_1 = (1,0,0,\ldots,0)^T$.

Notice that J may be written as $J = \lambda I + N$, where

$$N = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Example

If A a 3×3 square matrix with an eigenvalue λ , then has one of the following four forms

$$A = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad C = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \quad D = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix},$$

The matrix A is diagonalizable, then, it is has three eigenvalues

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The matrices B *and* C have two eigenvectors

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

The matrix D has one eigenvector $v = (1,0,0)^T$.

Lemma 3 [3]

For any $\alpha > 0,$ then A is similar to a matrix

$$B = \lambda I + \alpha N = \begin{pmatrix} \lambda & \alpha & 0 & \cdots & 0 \\ 0 & \lambda & \alpha & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha \\ 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Example

Two-Dimensional systems

We start by deriving a particular solution for the autonomous system, with two-dimensional square matrix A.

Particular solution

The particular solution is the steady state, if it exists. Let $X(n) = X^*$ for all n. Then,

$$X^* = AX^* + B \Rightarrow X^*(I_2 - A) = B$$

If $(I_2 - A)$ is invertible,

$$X_p(n) = X^* = (I - A)^{-1} B \Rightarrow (I_2 - A)^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 1 - a_{22} & a_{12} \\ a_{21} & 1 - a_{11} \end{pmatrix}$$

Which is well defined as long as $\det(A) \neq 0,$

Or equivalently if and only if

$$1 - tr(A) + \det(A) \neq 0.$$

Complementary and general solutions

Consider for this purpose the homogeneous first order system $X(n) = A^n X(0)$ written as a two equation system

$$X(n) = AX(n-1)$$

Where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Whose characteristic equation is given by

$$\lambda^2 - tr(A)\lambda + \det(A) = 0, \quad where \quad tr(A) = \sum_{i=1}^{k} a_{ii}.$$

Let the matrix of eigenvalues and eigenvectors of A be

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad P = [v_1, v_2] = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}.$$

Case 1: Distinct eigenvalues

Observe that M is of full rank (hence invertible) provided eigenvalues are distinct, $\lambda_1 \neq \lambda_2$.

$$AP = PA \quad \Leftrightarrow \quad A = PAP^{-1} \quad \Leftrightarrow \quad P^{-1}AP = D$$

Distinct eigenvalues allow for a diagonalization of A (i.e.to transform A into a diagonal matrix A). Substituting into the homogeneous system

$$X(n) = PDP^{-1} X(n-1) \quad \Rightarrow \quad P^{-1}X(n) = DP^{-1}X(n-1)$$

Put $Y(n) = P^{-1} X(n)$

Explicitly, we obtain

$$Y(n) = DP^{-1} X(n-1) = DY(n-1).$$

The strategy is to derive a solution for $Y(n)$ to later recover, the solution of the original variable $X(n)$. The solution is determined since the matrix D is diagonal, given initial conditions,

$$Y(0) = P^{-1} X(0)$$

$$Y(n) = \begin{pmatrix} Y_1(0)\lambda_1^n \\ Y_2(0)\lambda_2^n \end{pmatrix}$$

$$X_g(n) = PY(n) = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \quad \begin{pmatrix} Y_1(0)\lambda_1^n \\ Y_2(0)\lambda_2^n \end{pmatrix} \tag{13}$$

More generally

$$X_i(n) = X_i + \sum_{j=1}^{k} v_{ij} Y_i(0)\lambda_i^n, \quad for \quad all \quad i = 1,2,\ldots,k$$

$$X_i(n) = X_i^* + p\sum_{j=1}^{k} Y_i(0)\lambda_i^n$$

In matrix from, the general solution

$$X_i(n) = X_i^* + pD^n Y(0), \quad Y(0) = p^{-1} X(0).$$

Case 2: Repeated eigenvalue

Suppose Φ is not already a diagonal matrix, then, it is not diagonalizable whenever it has repeated eigenvectors are linearly independent.

i. If λ_i is an eigenvalue with multiplicity $r_i \; for\,all \quad i, \quad then$

$$(A - \lambda_i I_m)^{r_i} v_i = 0$$

ii. If λ_i distinct for some eigenvalue, then

$$(A - \lambda_i I_m)^{r_i} v_i = v_{i-1} \Leftrightarrow \begin{cases} (A - \lambda_i I_m)v_1 = v_1 \\ (A - \lambda_i I_m)v_2 = v_1 \\ \vdots \\ (A - \lambda_i I_m)^{r-1} v_i = v_1 \end{cases}$$

Example

Let A be the 3×3 square matrix, with $\lambda_1 = 2, \quad \lambda_2 = \lambda_3 = 3,$ then

$$\lambda_1 = 2, \Rightarrow \quad (A - 2I_3)v_1 = 0$$

$$\lambda_2 = \lambda_3 = 3, \Rightarrow \quad (A - 3I_3)v_3 = v_2 \Rightarrow \quad (A - 3I_3)^2 v_3 = 0.$$

$$p = (v_1, v_2, v_3) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

Case 3: Complex eigenvalues
If the conjugate complex eigenvalues are

$$\lambda_{1,2} = \alpha \pm i\beta,$$

Then A is similar to the matrix

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} = |\lambda| \quad \begin{pmatrix} \cos\omega & \sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix}$$

Where $|\lambda| = \sqrt{\alpha^2 + \beta^2}$, with $\omega = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$.

If A has the Jordan form $A = PJP^{-1}$, then we make the variable transformation

$$Y(n) = P^{-1}X(n).$$

This implies that

$$Y(n+1) = AP^{-1}X(n+1) = P^{-1}APP^{-1}X(n) = JY(n) \tag{14}$$

Where has one of the following three forms

$$J = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

Distinct or repeated semi simple real eigenvalues

$$J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Repeated eigenvalues with one independent eigenvector, or

$$J = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

Complex eigenvalue: $\lambda_{1,2} = \alpha \pm i\beta.$

Theorem 7 [7]
The homogeneous two-dimensional system has an asymptotically stable solution if and only if

Proof
The characteristic polynomial, $\rho(\lambda)$, *of a* 2×2 matrix is

$\rho(\lambda) = \lambda^2 - tr(A) + \det A$

This implies that

$$\lambda_{1,2} = \frac{tr(A) \pm \sqrt{tr^2(A) - \det A}}{2}$$

Suppose that the zero point is asymptotically stable, then $|\lambda_{1,2}| < 1$. But, in the case of real root, this equivalent to the following two inequalities:

$$-2 - tr(A) < \sqrt{tr^2(A) - 4\det A} < 2 - tr(A)$$

$$-2 - tr(A) < -\sqrt{tr^2(A) - 4\det A} < 2 - tr(A) \cdots$$

Equivalently

$$tr(A) < 1 + \det(A)$$

$$tr(A) < -1 - \det(A)$$

Combining both results gives, the first part of the stability condition

$$|trA| < 1 + \det(A) < 2$$

The second part follows from the observation that $\det(A) = \lambda_1 \lambda_2$ and the assumption that $|\lambda_1 \lambda_2| < 1$.

If the roots are complex, the conjugate are complex, so that the second part of the stability $|trA| < 1 + \det(A) < 2$ results from $\det A = \lambda_1 \lambda_2 = |\lambda_1||\lambda_2| < 1,$

This implies that

$$tr^2(A) - 4\det A < 0 \quad \Rightarrow \quad 0 < tr(A) < 4\det A$$

This can be used to show that

$$4(1 + \det A - tr(A)) > 4 + tr^2(A) - 4\det A = (2 - tr(A)) > 0.$$

Which is the required inequality.
Conversely; if the stability condition $|trA| < 1 + \det(A) < 2$ satisfied and if the roots are real, we have

$$-1 < \frac{-2 \pm \sqrt{tr^2(A) - 4\det A}}{2} < \lambda_1 = \frac{tr(A) + \sqrt{tr^2(A) - 4\det A}}{2}$$

$$< \frac{tr(A) + \sqrt{tr^2(A) + 4 - 4tr(A)}}{2} = \frac{tr(A) + \sqrt{(2 - tr(A))^2}}{2} < 1.$$

Similarly, for λ_2. If the roots are complex, they are conjugate complex and we have

$$|\lambda_1|^2 = |\lambda_2|^2 = \lambda_1 \lambda_2 = \frac{tr^2(A) - tr^2(A) + 4\det A}{4} = \det A < 1.$$ [1, 2-5-6-9].

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