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Power size biased exponential distribution with applications in bio-science

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Abstract
In this article, a modified version of size biased exponential distribution has been introduced and named as Power Size Biased Exponential Distribution (PSBED). Various statistical properties of the said distribution have been derived. The pdf, CDF and HRF of the distribution has been represented through the graphs. The method of maximum likelihood estimation MLE has been used to estimate the parameters of the distribution. Finally the performance of the distribution is checked through the data set where it shows the better results than the parallel distributions.

Keywords: Bio-science, power size biased exponential distribution (PSBED), pdf, CDF and HRF

1. Introduction

In recent developments, researchers focused on generating more flexible, tractable and meaningful distributions and modeled various types of lifetime data with monotone failure rates. In spite of their simplicity in solving many problems of lifetime data and reliability studies, such existing distributions are not useful to model bathtub and multimodal shaped failure rates and also fail to provide sound parametric fit to some practical application. In recent past, new families of probability distributions have been defined that are extension of well-known families of distributions. These newly developed families/classes of distributions provide greater flexibility in modeling complex data. Weighted in general and size biased in particular distributions arise in practice when observations from a sample are recorded with unequal probability and provide unifying approach for the problems when the observations fall in the non-experimental, non replicated and non-random categories, first introduced by to model ascertainment bias these are later formulized in uniform theory by. Discussed that weighted distributions have various statistical applications, especially in analysis of data relating to ecology and human populations. For the first, time applied weighted distribution to connect with sampling wood cells. Gove (2003) [9] studied some of the latest results on size-biased distributions especially conceder the Weibull family relating to application and parameter estimation with method of moments and maximum likelihood in forestry. A weighted version of exponential distribution is discussed by Mir *et al*. (2013) [5]. They derive some mathematical properties and estimate the parameter with method of moments, maximum likelihood and Bayesian method.

The probability density function of size biased exponential distribution is given by;

$$f(x,\theta) = \theta^2 x e^{-\theta x}, x > 0, \theta > 0 \tag{1.1}$$

And the corresponding cdf is given as;

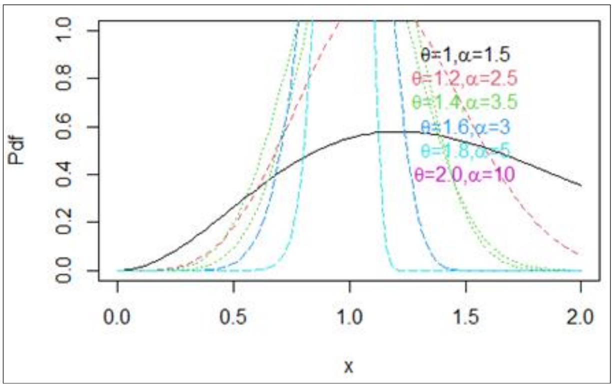
$$F(x;\theta) = 1 - (1 - \theta x)e^{-\theta x}, x > 0, \theta > 0 \tag{1.2}$$

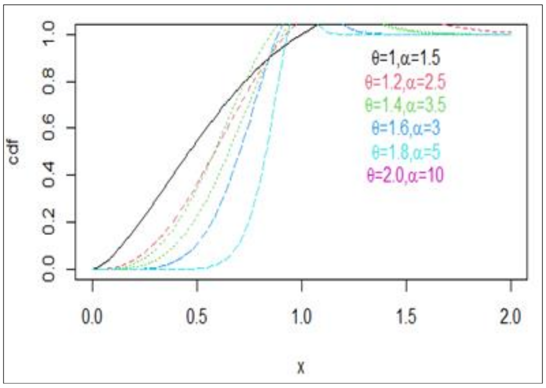
2. The Power Size Biased Exponential Distribution

If Y be a random variable that follows Size Biased Exponential distribution, then the random variable $X = Y^{\frac{1}{\beta}}$ is said to follow Power Size Biased Exponential distribution (PSBED) if its probability density function pdf is given as;

$$f(x,\alpha,\theta) = \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} , x > 0 , \alpha,\theta > 0 \quad (2.1)$$

$$F(x;\alpha,\theta) = 1 - (1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}}, x > 0 , \alpha,\theta > 0 \quad (2.2)$$





3. Structural Properties of Power Size Biased Exponential Distribution

a) Survival Analysis

The reliability function of a random variable x denoted by $S(x,\alpha,\theta)$, can be obtained as ;

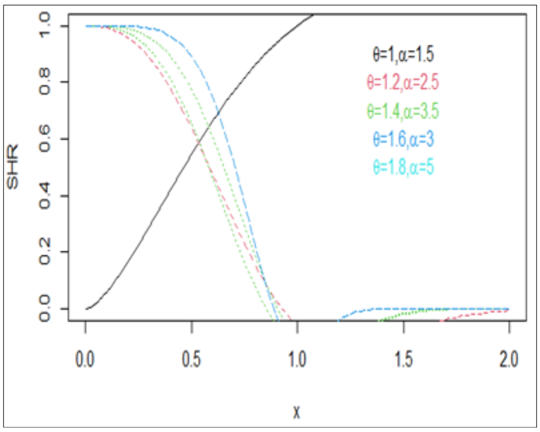
$$S(x,\alpha,\theta) = 1 - F(x;\alpha,\theta)$$
$$= (1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}} \quad (3.1)$$

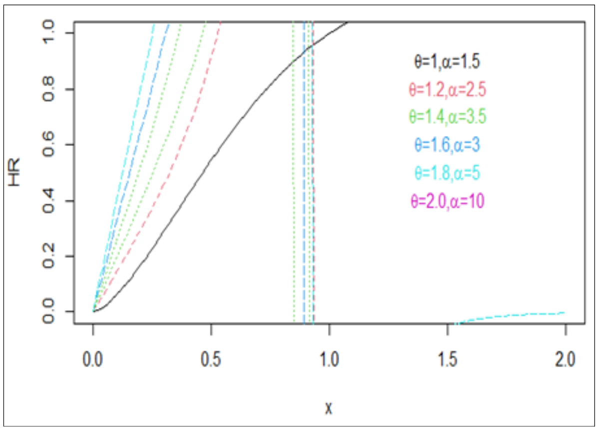
The hazard rate function denoted as $h(x,\alpha,\theta)$ of a random variable x can be obtained as;

$$h(x,\alpha,\beta) = \frac{f(x,\alpha,\theta)}{S(x,\alpha,\theta)}$$

$$h(x,\alpha,\beta) = \frac{\theta^2 x e^{-\theta x}}{(1-\theta x^{\alpha}) e^{-\theta x^{\alpha}}}$$

$$= \frac{\theta^2 x}{(1-\theta x^{\alpha})} \quad (3.2)$$





b) Moments

Suppose x be a random variable follows PSBED. Then the r moment denoted by $\mu_r'$is given as;

$$\mu_r' = E(x^r) = \int_0^{\infty} x^r f(x,\alpha,\theta)\, dx = \int_0^{\infty} x^r \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} dx$$

Substituting $\theta x^{\alpha} = z$,then solving the integral we get;

$$\mu_r' = E(x^r) = \theta^{r/\alpha} \Gamma\left(\frac{r}{\alpha} + 2\right) \quad (3.3)$$

By substituting$r = 1, 2, 3$ and 4 we obtain the first four moments of the distribution about origin

$$\mu_1' = \theta^{1/\alpha} \Gamma\left(\frac{1}{\alpha} + 2\right) \quad (3.3)$$

$$\mu_2' = \theta^{2/\alpha} \Gamma\left(\frac{2}{\alpha} + 2\right) \quad (3.4)$$

$$\mu_3' = \theta^{3/\alpha} \Gamma\left(\frac{3}{\alpha} + 2\right) \quad (3.5)$$

$$\mu_4' = \theta^{4/\alpha} \Gamma\left(\frac{4}{\alpha} + 2\right) \quad (3.6)$$

Mean of the distribution is $\mu = \theta^{1/\alpha} \Gamma\left(\frac{1}{\alpha} + 2\right)$

Variance of the distribution is given by;

$$\sigma^2 = \mu_2' - (\mu_1')^2$$

Substituting the values of μ_2' and μ_1' from the above equations, we get;

$$\sigma^2 = \theta^{2/\alpha} \Gamma\left(\frac{2}{\alpha} + 2\right) - \left\{\theta^{1/\alpha} \Gamma\left(\frac{1}{\alpha} + 2\right)\right\}^2$$

$$\sigma^2 = \theta^{2/\alpha} \left[\Gamma\left(\frac{2}{\alpha} + 2\right) - \left(\Gamma\left(\frac{1}{\alpha} + 2\right)\right)^2\right] \quad (3.7)$$

c) Moment generating function

Let x be a random variable follows power biased exponential distribution then the moment generating function denoted by $M_x(t)$ is obtained as;

$$M_x(t) = \int_0^\infty e^{tx} f(x, \alpha, \theta) dx$$

Using Taylor's theorem, we get

$$= \int_0^\infty \left\{1 + tx + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \cdots\right\} f(x, \alpha, \theta) dx$$

$$= \int_0^\infty \sum_{r=0}^\infty \frac{t^r}{r!} x^r f(x, \alpha, \theta) dx$$

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$$= \sum_{r=0}^\infty \frac{t^r}{r!} \mu_r'$$

$$M_x(t) = \sum_{r=0}^\infty \frac{t^r}{r!} \theta^{r/\alpha} \Gamma\left(\frac{r}{\alpha} + 2\right) \quad (3.8)$$

d) Characteristics function of power size biased exponential distribution

Let x be a random variable follows power size biased exponential distribution then the moment generating function denoted by $\phi_x(t)$ is obtained as.

$$\phi_x(t) = \int_0^\infty e^{itx} f(x, \alpha, \theta) dx$$

Using Taylor's theorem, we get

$$= \int_0^\infty \left\{1 + itx + \frac{(itx)^2}{2!} + \frac{(itx)^3}{3!} + \cdots\right\} f(x, \alpha, \theta) dx$$

$$= \int_0^\infty \sum_{r=0}^\infty \frac{(it)^r}{r!} x^r f(x, \alpha, \theta) dx$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \int_0^{\infty} x^r f(x, \alpha, \theta) dx$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \mu_r'$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \theta^{r/\alpha} \Gamma\left(\frac{r}{\alpha} + 2\right) \qquad (3.9)$$

e) Harmonic mean of power size biased exponential distribution

The harmonic mean (H) is given as.

$$H = E\left(\frac{1}{x}\right) = \int_0^{\infty} \frac{1}{x} f(x, \alpha, \theta)\, dx$$

$$= \int_0^{\infty} \frac{1}{x} \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} dx$$

Substituting $\theta x^{\alpha} = m$,then solving the integral we get;

$$H = \theta^{1/\alpha} \Gamma\left(2 - \frac{1}{\alpha}\right) \qquad (3.10)$$

4. Order Statistics of Power size biased exponential Distribution

Let us suppose $x_1, x_2, x_3 \dots, x_n$ be random samples of size n from power size biased exponential distribution with p.d.f $f(x)$ and c.d.f F(x). Then the probability density function of k^{th} order statistics is given as.

$$f_{x(k)}(x) = \frac{n!}{(k-1)!\,(n-k)!} [F(x)]^{k-1} [1 - F(x)]^{n-k} f(x)$$

Now using the values of $f(x)$ and F(x) in the above equation.Then the probability of k^{th} order statistics of Power size biased exponential distribution is given as;

$$f_{x(k)}(x, \alpha, \theta) = \frac{n!}{(k-1)!(n-k)!} \left[1 - (1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}}\right]^{k-1} \left[(1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}}\right]^{n-k} \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} \quad k = 1,2,3, \dots, n \qquad (4.1)$$

Then, the p.d.f of first order x_1 Power size biased exponential distribution is given as;

$$f_{x(1)}(x, \alpha, \theta) = n\left[(1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}}\right]^{n-1} \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} \qquad (4.2)$$

Then, the p.d.f of first order x_n Power size biased exponential distribution is given as

$$f_{x(n)}(x, \alpha, \theta) = n\left[1 - (1 - \theta x^{\alpha}) e^{-\theta x^{\alpha}}\right]^{n-1} \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}} \qquad (4.3)$$

5. Method of Maximum Likelihood Estimation

The estimation of parameters of Power size biased exponential distribution is doing by using the method of maximum likelihood estimation. Suppose $x_1, , x_3 \dots x_n$ be random samples of size n from Power size biased exponential distribution. Then the likelihood function of Power size biased exponential distribution is given as;

$$l = \prod_{i=1}^{n} f(x_i, \alpha, \theta)\, dx$$

$$= \prod_{i=1}^{n} \alpha\theta^2 x^{2\alpha-1} e^{-\theta x^{\alpha}}\, dx$$

$$= (\alpha\theta^2)^n \prod_{i=1}^{n} x_i^{2\alpha-1} e^{-\theta \sum_i^n x_i^{\alpha}}$$

Applying log on both sides, we get the log likelihood function

$$Log\ l = Log\left[(\alpha\theta^2)^n \prod_{i=1}^{n} x_i^{2\alpha-1} e^{-\theta \sum_i^n x_i^\alpha}\right]$$

$$= Log(\alpha\theta^2)^n + \sum_{i=1}^{n} Log\ x_i^{2\alpha-1} + Log e^{-\theta \sum_i^n x_i^\alpha}$$

$$Log\ l = nLog\alpha + 2n\ Log\theta + (2\alpha - 1)\sum_{i=1}^{n} Logx_i - \theta \sum_i^n x_i^\alpha \quad (5.1)$$

Differentiating the log likelihood function w.r.t α, we get;

$$\frac{\partial logl}{\partial \alpha} = \frac{n}{\alpha} + 2\sum_{i=1}^{n} Logx_i - \theta \sum_{i}^{n} x_i^\alpha Logx_i$$

Equating $\frac{\partial logl}{\partial \alpha} = 0$, we get;

$$\alpha\{\theta \sum_i^n x_i^\alpha Logx_i - 2\sum_{i=1}^{n} Logx_i\} = n \quad (5.2)$$

Differentiating the log likelihood function w.r.t θ we get;

$$\frac{\partial logl}{\partial \theta} = \frac{2n}{\theta} - \sum_{i}^{n} x_i^\alpha$$

Equating $\frac{\partial logl}{\partial \theta} = 0$ we get;

$$\hat{\theta} = \frac{2n}{\sum_i^n x_i^\alpha} \quad (5.3)$$

Which gives the estimate of θ.

6. Data Analysis

The usefulness of the formulated distribution has been explained through two bioscience related data sets. The data sets are as follows

Data set 1: The data set represents the remission times (in months) of a random sample of 128 bladder cancer patients. The observations are follows

0.08, 2.09, 2.73, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.22, 3.52, 4.98, 6.99, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 15.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.93, 8.65, 12.63, 22.69

In order to compare the efficiency of the formulated distribution, we consider various criterions for goodness of fit such as $-\log l$, AIC, CAIC and BIC. The established model is compared with Weighted Ailamujia distribution (WAD), Area Biased Ailamujia Distribution (ABAD), Length Biased Ailamujia Distribution (LBAD), Ailamujia Distribution (AD). The distribution having lower value of AIC, CAIC and BIC is considered to be best fit. The descriptive statistics of data set 1 and data set 2 are presented in table 1 and 3. The estimates of the parameters, log-likelihood, Akaike information criteria (AIC) for the data sets are generated and presented in Table 2 and 4.

Table 1: Data summary for the first data set

Min.	1stQu.	Median	Mean	3rdQu.	Max.	Skew.	Kurt.
0.080	3.29	6.05	9.31	11.67	79.05	3.31	18.54

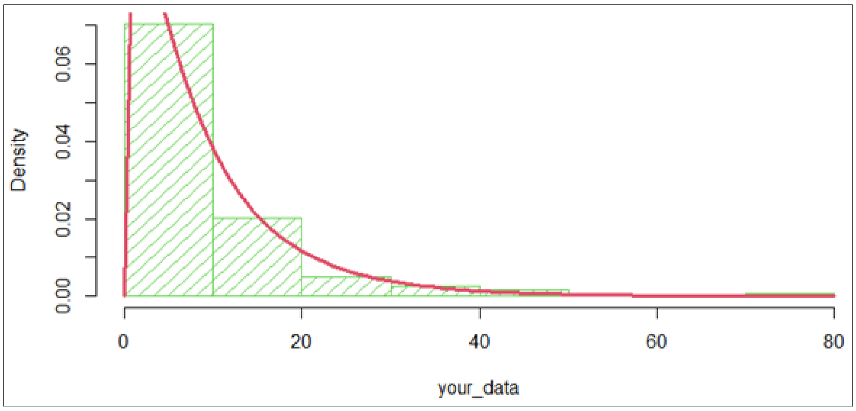

Fig 1.1: PSBE Fitting

Table 2: Performance of distributions for first Data set

Model	Parameter Estimation	Standard Error	$-\log l$	AIC	CAIC	BIC
PSBE	$\hat{\alpha} = 0.72031$ $\hat{\beta} = 0.43880$	$\hat{\alpha} = 0.0287$ $\hat{\beta} = 0.0470$	397.79	799.59	799.69	805.23
WAD	$\hat{\alpha} = 0.1074$ $\hat{\beta} = 0.0010$	$\hat{\alpha} = 0.0143$ $\hat{\beta} = 0.2359$	413.49	830.99	831.095	836.63
ABAD	$\hat{\alpha} = 0.2148$	$\hat{\alpha} = 0.0096$	489.35	980.713	980.74	983.53
LBAD	$\hat{\alpha} = 0.1074$	$\hat{\alpha} = 0.0068$	413.470	828.94	828.97	831.76
AD	$\hat{\alpha} = 0.1074$	$\hat{\alpha} = 0.0068$	413.47	828.94	828.97	831.762

Data Set 2: This data represents the survival times, in weeks of 33 patients suffering from acute myelogenous leukemia. The observations are as follows;
65, 156, 100, 134, 16, 108, 121, 4, 39, 143, 56, 26, 22, 1, 1, 5, 65, 56, 65, 17, 7, 16, 22, 3, 4, 2, 3, 8, 4, 3, 30, 4, 43

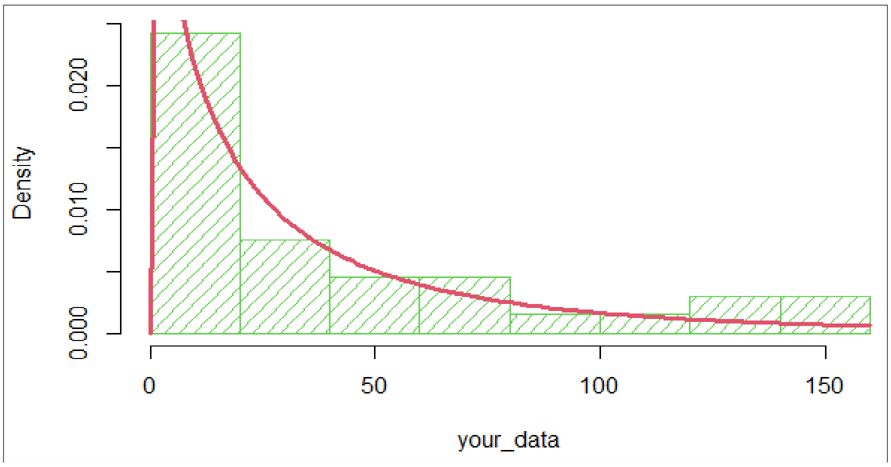

Fig 1.2: PSBE Fitting

Table 3: Data summary for the first data set

Min.	1stQu.	Median	Mean	3rdQu.	Max.	Skew.	Kurt.
1.00	4.00	22.00	40.88	65.00	156.0	1.164	3.122

Table 4: Performance of distributions for second data set

Model	Parameter Estimation	Standard Error	$-\log l$	AIC	CAIC	BIC
PSBE	$\hat{\alpha} = 0.5122$ $\hat{\beta} = 0.3583$	$\hat{\alpha} = 0.0510$ $\hat{\beta} = 0.067$	153.50	311.01	311.41	314.006
WAD	$\hat{\alpha} = 0.0244$ $\hat{\beta} = 0.0010$	$\hat{\alpha} = 0.0063$ $\hat{\beta} = 0.4557$	171.79	347.58	347.98	347.58
ABAD	$\hat{\alpha} = 0.0489$	$\hat{\alpha} = 0.0042$	217.79	437.58	437.71	439.08
LBAD	$\hat{\alpha} = 0.0244$	$\hat{\alpha} = 0.0030$	171.77	345.54	345.67	347.03
AD	$\hat{\alpha} = 0.0244$	$\hat{\alpha} = 0.0030$	171.77	345.54	345.67	347.03

It is evident from table 2 and table 4 that the formulated distribution has lower value of AIC, CAIC, BIC than compared ones. Hence power size biased exponential distribution leads better fit.

7. Conclusion

In this article, a new version of size biased exponential distribution has been introduced and named as Power Size Biased Exponential Distribution (PSBED). Various statistical properties of the said distribution have been derived. The PDF, CDF and HRF of the distribution have been represented through the graphs. The method of maximum likelihood estimation MLE has been used to estimate the parameters of the distribution. Finally, the performance of the distribution is checked through the data set where it shows better results than the parallel distributions.

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