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A combination of dividend and jump diffusion process on Heston model in deriving black Scholes equation

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Abstract
The reality that exists in a stock market situation is that assets do pay dividend to owners of assets or derivative securities. Dupire, Derman and Kani built an option pricing process with a dividend yielding diffusion process but lacked the jump diffusion component. Jump diffusion process that mostly been captured in modern stock market do exhibit discontinuous behavior on pricing of assets. Introduction of Black Scholes concept equation that assumes volatility is constant led to several studies that have proposed models that address the shortcomings of Black – Scholes model. Heston's models stands out amongst most volatility models because the process of volatility is greater than zero and exhibit mean reversion process and that is what is observed from the stock market. One of the shortcomings of Heston's model is that it doesn't incorporate the dividend yield jump diffusion process. Black Scholes equation revolves around Geometric Brownian motion and its extensions. We, therefore, incorporate dividend yield and jump diffusion process on Heston's model and use it to formulate a new Black Scholes equation using the knowledge of partial differential equations.

Keywords: Dividend yield, Jump diffusion, Poisson distribution, volatility, geometric Brownian motion, black – Scholes formula, Heston's model

1. Introduction

The main idea why we use the jump diffusion process in describing the stock price movement is that streaming of information is discontinuous with no short term delivery of information [12]. Jump diffusion is a model categorized into two parts with first diffusion component built on a Brownian motion with the property of unpredictable return on the asset as a result of vibrations caused by the price of underlying portfolio. The second part is the jump component with poisson distribution property that occurs as a result of vibrations caused by abnormal price of the underlying portfolio. The prices of assets exhibiting the jumps processes are assumed to be independently and identically distributed. The standard Black Scholes equation [2] is one of the models that formulate the prices of options and it is derived under the following assumptions;

- Volatility is constant
- Dividend payments do not occur during the lifetime of a portfolio.
- Payment of interest rates are known and constant
- The returns are log-normally distributed.
- No commission and transaction cost.
- The market is perfectly liquid

Oduor [11] reviewed one extension of the assumptions of the original Black Scholes equation where the initial Black Scholes equation was modified to reflect the reality in the market. It represented Black – Scholes model with non – linear Brownian motion on a continuous dividend – paying asset. The pricing of stock derived from a function of time, S_t was initially done from geometric Brownian motion using a standard model given by;

$$dS_t = \mu S_t dt + \sigma S_t dZ_t , \tag{1}$$

Where, μ is the percentage growth of the asset, S_t is the price of the asset or portfolio, σ is the volatility describing the uncertainty in price of the asset and dZ_t is the standard Brownian motion. One of the short – comings of equation (1) is that it involves a non – dividend paying asset which is vital in financial investments. Many extensions have been done by researchers to address the short – coming of equation [1]. Among them, studies by Hull and White [6], Stein and Stein [13] and Heston [4] built models on stochastic volatilities that are two factor models where one of the major factors is responsible for the changes in coefficient of volatility dynamics. Heston's model [4] stands out among these models because the processes for volatility which is greater than zero and has mean reversion which is in contrary to Black – Scholes model that assumes constant volatility. Heston's model [4] also has existence solution of vanilla options which is of closed form. It assumes that the spot price has a diffusion process of the form;

$$dS_t = \mu S_t dt + S_t \sqrt{\upsilon_t} dZ_t , \tag{2}$$

Where, μ (linear drift rate) is a constant, υ_t is a non – constant instantaneous volatility and Z_t is a Brownian motion (Wiener process). This is a process resembling Geometric Brownian motion with a proposed volatility having a component which is mean reverting. Equation (2) also does not involve a dividend – paying asset but in reality assets do pay dividends to security holders. It is from this Heston's volatility model that we superimpose a diffusion process with a dividend - paying and jump diffusion component that is used to derive a new Black – Scholes equation.

2. Preliminaries

2.1 Itô Process

Given the constants a and b that exhibits some functions of the variable X and t, then Itô process which is also a generalized Brownian motion (Wiener process) is mathematically represented by;

$$dG = a(G,t)dt + b(G,t)dZ_t \tag{3}$$

The expected growth rate and variance rate both are reliable to some change over given small interval say from t to $t+\Delta t$. These changes are therefore expressed as;

$$\Delta G = a(G,t)\Delta t + b(G,t)\varepsilon\sqrt{\Delta t} , \tag{4}$$

Where, the drift rate is given as $a(G,t)$ and variance rate is given as $b^2(G,t)$ from equation (4). Therefore $\Delta t \sim N\left(a(G,t), b(G,t)\sqrt{\Delta t}\right)$

2.2 Itô Lemma

Stochastic differential equations are best solved by Itô Lemma, where wiener - like differential process are put into mathematical formulation of partial differential equations to obtain solutions of stochastic differential equations. In deriving Itô lemma we consider value of a variable P that follows an Itô process from equation (3), where P is said to have a linear drift rate of a and a percentage variance of b^2 such that from Itô lemma, it is stated that a function $G(P,t)$ that can be differentiable twice in P and once in t forms an Itô process of the form;

$$dG = \left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)dt + \frac{\partial G}{\partial P}bdZ , \tag{5}$$

Where, $\left(\frac{\partial G}{\partial P}a + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial P^2}b^2\right)$ is the percentage drift and a percentage variance derived by $\left(\frac{\partial G}{\partial P}\right)^2 b^2 dt$

2.3 Geometric Brownian Motion

Geometric Brownian motion is a specific Itô Process following a diffusion process given by

$$dF = aFdt + bFdZ \tag{6}$$

Where, $a(F,t) = aF$, $b(F,t) = bF$ and Z is the standard wiener process. A geometric Brownian motion used in the application of stock pricing is given by;

$$dS = \mu Sdt + \sigma SdZ , \tag{7}$$

Where, S is the price of the underlying asset, μ is the expected growth rate or the rate of return of the underlying asset and σ is the percentage volatility of price of the underlying asset. Re - written equation (7) results to the following;

$$dS = \mu Sdt + \sigma SdZ \tag{8}$$

A discrete time model over a given time frame t to Δt that is normally used in pricing of the underlying asset is given by;

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}\ , \tag{9}$$

Where, ΔS is the rate of change in price of the asset S within a time interval Δt and $Z = \varepsilon\sqrt{\Delta t}$ such that ε is a random variable derived by a standardized normal distribution with mean zero and standard deviation of one [5].

2.4 Stochastic Process

Stochastic processes are activities or events whose occurrences are random over a given period of time; this makes them obey the law of probability. Mathematically it can be defined as a process represented by X_t which is a collection of random variables $[X_t : t \in T]$ found in a probability space that varies over a given set T. We define various types of stochastic processes below.

2.4.1 Markov Process

It is a stochastic process where factors from the past history of the asset does not influence the behavior of the current asset price since it is believed that the current price already contain relevant information from the past history that could affect the new price of the underlying stock.

2.4.2 Wiener Process

It is a Brownian motion with mean rate of change zero and variance rate of one. It can also be defined as a random variable of value Z that follows a Wiener process with the following properties;

Property 1: Over a small period of time ΔZ is defined by;

$$\Delta Z = \varepsilon\sqrt{\Delta t} \tag{10}$$

Where, ε is normally distributed by mean of zero and variance of 1. That is $\varepsilon \sim N(0,1)$

Property 2: Over a two varied short time periods, ΔZ are independent. That is $\operatorname{cov}(\Delta Z_i, \Delta Z_j) = 0$, $i \neq j$ [5].

Dividend – Paying Security

These are distributions of profits to shareholders in form of returns of securities invested in a company or corporation. Derman and Kani [3].

2.5 Jump Process

This occurs when the assets deviate abruptly or suddenly from its normal path due to non – systematic risk. This happens due to new information which causes negative or positive effects in asset price. The information may be firm or sector oriented. Such information represents non – systematic risk meaning that the jumps are uncorrelated with the market.

2.6 Heston's Model with Jump diffusion model

We superimpose dividend rates and jump diffusion process onto Heston's model of closed – form solution of vanilla options. It will therefore follow a diffusion process given by;

$$dS_t = S_t\left((\mu - y_t - \lambda k)dt + \sqrt{\upsilon_t}\, dZ_t\right) + (g-1)dN\ , \tag{11}$$

Where, μ (linear drift rate) is a constant, y_t is the dividend rate, υ_t is a non – constant instantaneous volatility, λ is the rate at which jumps occurs per unit time, k is the proportional increase in price of the asset measuring the jump size, N is Poisson process that generates the jumps diffusion process and Z_t is the standard Wiener process. Andanje [1], Opondo [8] and Oduor [11].

3. Main results

3.1 Formulation of Black – Scholes equation using a superimposed dividends and jump diffusion on Heston model

We take a Heston's jump model of a diffusion process of the form;

$$dS_t = S_t((\mu - y_t - \lambda k)dt + \sqrt{\upsilon_t}\, dZ_t) + (g-1)S_t dN \tag{12}$$

Where, μ is the percentage rate of growth, y_t is a dividend paying rate of asset at given time t, υ_t is the non – constant volatility, λ is the percentage at which jumps occurs per unit time, k is the proportional increase in price of the asset measuring the jump size and N is Poisson process that generates the jump diffusion process. We therefore have that dZ_t and dN are independently and identically distributed.

Assuming that over a small change of time interval dt the price of the underlying option moves from S to gS where g is absolute price jump size such that the relative jump of price of the asset is given by;

$$\frac{dS}{S} = \frac{gS - S}{S} = g - 1 \tag{13}$$

Thus the relative price jump size of S_t and $g-1$ are log – normally distributed with mean of;

$$E(g-1)=\exp\left(\mu+\frac{1}{2}\sigma^2\right)-1\equiv k$$

and rate of variance given by;

$$E\left(g-1-E(g-1)^2\right)^2=\exp\left(2\mu+\sigma^2\right)\exp(\left(\sigma^2\right)-1)$$

Merton [7]

The assumption that the absolute jump price of size $(q-1)$ is lognormal random variable such that;

$$(g-1)\sim i.i.d\log normal\left[k=e^{\mu+\frac{1}{2}\sigma^2}-1,e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)\right]$$

Equivalently, the log – return jump size $\ln\left(\frac{qS}{S}\right)$ is a normal random variable such that;

$$\ln\left(\frac{gS}{S}\right)=\ln(g)\sim i.i.dN\left(\mu,\sigma^2\right)$$

We therefore note that;

$$\ln E[(g-1)]=E[\ln(g)]\neq E[\ln(g-1)]$$

Using the jump diffusion component dN, the expected relative price change $E\left[\frac{dS_t}{S_t}\right]$ over the time interval dt is λkdt derived as;

$$E[(g-1)dN]-E[(g-1)]E[dN]=k\lambda dt \tag{14}$$

The resulting solution is a predictable function of the jump diffusion process. In order to have the jump diffusion component non -predictable, the return expected from the underlying asset μdt is computed by $-\lambda kdt$ on the adjusted drift term on the jump diffusion. This is given by;

$$E\left[\frac{dS_t}{S_t}\right]=E[(\mu-y_t-\lambda k)]dt+E\left[\sqrt{\upsilon_t}dZ_t\right]+E[(g-1)dN] \tag{15}$$

Simplifying equation (15) gives

$$E\left[\frac{dS_t}{S_t}\right]=(\mu-y_t)dt \tag{16}$$

We take a variable G that is a function of S and t which form a price of any call option that is differentiable twice in S and once in t. Then using Itồ lemma we have;

$$\partial G=\left(\frac{\partial G}{\partial S}(\mu-y_t-\lambda k)S_t+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_2 S_t^2\right)dt+\frac{\partial G}{\partial S}\sqrt{\upsilon_t}S_t dZ_t+(GgS_t-GS_t)dN \tag{17}$$

Where, (GqS_t-GS_t) shows that there is occurrence in jumps. We take the discrete version of equation (12) and (17) resulting to;

$$\Delta S_t=S_t((\mu-y_t-\lambda k)\Delta t+\sqrt{\upsilon_t}\Delta Z_t)+(g-1)S_t\Delta N \tag{18}$$

and

$$\Delta G = \left(\frac{\partial G}{\partial S}(\mu - y_t - \lambda k)S_t + \frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_2 S_t^2 \right)\Delta t + \frac{\partial G}{\partial S}\sqrt{\upsilon_t}\, S_t \Delta Z_t + \left(GqS_t - GS_t\right)\Delta N \tag{19}$$

Where, ΔS and ΔG gives the rate of changes in S and G over a small rate of change of time, Δt. Using Itồ lemma mentioned in section 2.2 both G and S in equation (18) and (19) have the same effect of uncertainty on ΔZ. We need to eliminate the Wiener process by choosing a portfolio of an asset and derivative. We consider a portfolio that is short of one derivative and takes $+\frac{\partial G}{\partial S}$: Shares. We also define Π as the value of the portfolio such that the portfolio holder will have both short and long option position in acquiring quantity of shares. By definition.

$$\Pi = G - \frac{\partial G}{\partial S}S \tag{20}$$

The discrete value of equation (20) in the interval Δt is given by;

$$\Delta\Pi = \Delta G - \frac{\partial G}{\partial S}\Delta S \tag{21}$$

When we substitute equation (18) and equation (19) onto equation (21) and removing the discrete aspect, we obtain

$$d\Pi = \left(\frac{\partial G}{\partial S} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)dt + \left(GqS_t - GS_t - \frac{\partial G}{\partial S}S_t(g-1) \right)dN \tag{22}$$

Equation (22) shows the elimination growth rate of asset, μ. This shows that the value of the option is not affected by the growth rate of the price of the asset. There is no arbitrage principle asserting that the risk free portfolio return is equivalent to the risk free rate. Applying the concept of Derman and Kani [3] of a dividend yielding asset, r is denoted by $(\mu - y_t)$, where μ is the growth rate of the asset and y_t is the dividend – paying rate of the asset that reduces its growth. This implies that the time interval dt is equivalent to r for the percentage rate of return on the underlying portfolio. This is given by;

$$E(d\Pi) = r\Pi dt, \tag{23}$$

Where, r the interest rate that is risk free and is denoted by $(\mu - y_t)$.

When we substitute equation (20) and equation (22) onto equation (23) gives;

$$E\left[\left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)dt + \left[GgS_t - GS_t - \frac{\partial G}{\partial S}S_t(g-1) \right]dN \right] = (\mu - y_t)\left(G - \frac{\partial G}{\partial S}S \right)dt \tag{24}$$

This simplifies to;

$$\left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)dt + \left[(g-1)GS_t - \frac{\partial G}{\partial S}S_t(g-1) \right]EdN = (\mu - y_t)\left(G - \frac{\partial G}{\partial S}S \right)dt \tag{25}$$

The equivalence of equation (27) is expressed as;

$$\left(\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 \right)dt + \left[(g-1)GS_t - \frac{\partial G}{\partial S}S_t(g-1) \right]\lambda dt = (\mu - y_t)\left(G - \frac{\partial G}{\partial S}S \right)dt \tag{26}$$

Simplifying equation (26) further gives;

$$\frac{\partial G}{\partial t} + \frac{1}{2}\frac{\partial^2 G}{\partial S^2}\upsilon_t S_t^2 + (g-1)\lambda GS_t - \frac{\partial G}{\partial S}S_t\lambda(g-1) = (\mu - y_t)G \tag{27}$$

Equation (27) is the Black Scholes equation with Heston’s Dividend – paying asset and jump diffusion process model.

4. Conclusion

In this paper, we have derived a Black - Scholes equation using a superimposed dividend paying asset and jump-diffusion process on the Heston model. Historical data can be applied in this model to compare the results with other existing extended Black – Scholes – Merton models which can help investors on analyzing their investment strategies and make viable decisions.

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