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Research on the application of multigrid method in the past 60 years

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Abstract
The multigrid method has the property of fast convergence, computational simplicity and robustness, therefore, it has been applied to various problems. We analyze the research on the application of the multigrid method, and find that this method: (1) could be applied in many branches of physics; (2) mainly focuses on the problem of tomography in the field of image processing; (3) is generally applied in the problem of geological exploration in the field of geology; (4) has not been widely used yet in the field of finance.

Keywords: Multigrid method, physics, geology, image processing, finance

1. Introduction
Multigrid method is a one of the numerical methods for solving equations. It was first proposed by Fedorenko in 1961 to solve a poisson equation [1]. Because it has the characteristics of rapid convergence, low computational complexity and good stability, many scientists have applied this method to various problems in the field of physics, medicine, computer science and finance etc. [2-62].
In order to learn the application of multigrid method more clearly and provide direction for further study in the future, this paper intends analyze the relevant researches on the application of multigrid method in field of physics, image processing, geology and finance.

2. Basic principle of multigrid method
The problem to be solved by the multigrid method usually can be transformed into a minimization problem of cost functional $\min_u J(u)$, that is, a problem of finding the solution u which makes the cost functional minimum. Before the implementation of the multigrid method, the domain Ω of u is divided into a sequence of grid levels: $\Omega_0, \Omega_1, \cdots \Omega_K$, where Ω_K denotes the finest mesh hierarchy, Ω_0 denotes the coarsest one. The symbols in the following with the subscript *k* indicate that they are under the grid level *k*. Additionally, the relationship of the step sizes between adjacent grid levels is that the step size of the coarser is twice that of the finer grid. Next, the multigrid algorithm is stated in detail as follows:

1) *Initialization*. Let u_k^0 be an initial approximation at resolution *k*. If *k*=0, solve and return. Else, go to Step 2.

2) *Pre-optimization*. Apply γ_1 iterations of an optimization algorithm $\hbar_k(\cdot)$ to the problem at resolution *k*.

$$u_k^l = \hbar_k(u_k^{l-1}), l = 1, 2, \cdots \gamma_1$$

3) Coarse grid problem. Compute the fine-to-coarse gradient correction

$$\tau_{k-1} = \nabla J_{k-1}(\mathrm{I}_k^{k-1} u_k^{\gamma_1}) - \mathrm{I}_k^{k-1} \nabla J_k(u_k^{\gamma_1}).$$

Apply γ cycles of the coarse grid problem to obtain u_{k-1}.

4) *Coarse grid correction.* Minimizes J_k, update $u_k^{\gamma_1}$ by the following formulation

$$u_k^{\gamma_1+1} = u_k^{\gamma_1} + \mathrm{I}_{k-1}^k (u_{k-1} - \mathrm{I}_k^{k-1} u_k^{\gamma_1}).$$

5) *Post-optimization.* Apply γ_2 iterations of an optimization algorithm $\hbar_k(\cdot)$ to the problem at resolution *k*.

$$u_k^l = \hbar_k(u_k^{l-1}), l = \gamma_1 + 2, \cdots, \gamma_1 + \gamma_2 + 1$$

In the steps above, the symbols I_k^{k-1} and I_{k-1}^k are respectively restriction operator and prolongation operator, which need to be prescribed before the multigrid procedure. Moreover, $\gamma = 1$ and $\gamma = 2$ correspond to V-cycle and W-cycle multigrid method, which are two most common types of the multigrid method [2].

As the description of the multigrid method above, the sampling points are less on a coarser grid, resulting that the computational complexity is much simpler than that on the fine grid. Consequently, this multigrid strategy allows us to obtain an approximation efficiently comparing with single-grid methods.

3. The application of the multigrid method

We take "multigrid" as the keyword to search for literatures on the websites of China National Knowledge Infrastructure, Baidu Academic Search, Google Academic Search and the websites of web of SCI, and then selected the literatures on the application of multigrid method in recent 60 years, and got 62 articles. Based on the selected researches, it can be found that the applications of multigrid are mainly concentrated in the field of physics, geology, image processing and finance. Then these researches are analyzed in this section.

3.1 Applications of multigrid method in the field of physics

Since the multigrid method was first proposed by Fedorenko, it has been immediately used to solve the poisson equation in a square region, that is, an equation describing the relationship between potential and charge density [1]. Then, the multigrid method has been widely applied in the branches of physics, such as optics, acoustics, hydrodynamics and thermodynamics [3-22].

Paola F. Antonietti *et al*. 2008 [4] presented a multigrid algorithm for the solution of the linear systems of equations stemming from the p-version of the Virtual Element discretization of a two-dimensional Poisson problem. Considered the time-dependent two-dimensional space-fractional diffusion equations which could model the anomalous diffusive systems with a V-cycle multigrid method [7]. Computed the ground state solution of Bose-Einstein condensations which belonged to the field of condensed matter physics by a multigrid method [8]. Besides, Sehar Iqbal and Paul Andries Zegeling solved a two-dimensional nonlinear boundary value problem which was strongly related to the well-known Gelfand-Bratu model emerged in nanotechnology, radioactive heat transfer, chemical reaction theory, elasticity theory, etc. [10] C. Nita and her group employed a multigrid scheme to DNS-based optimal control problem [16]. Due to 3D non-LTE radiative transfer problems are computationally demanding, Johan P. Bjørgen and Jorrit Leenaarts [18] implemented a non-linear multigrid and showed that multigrid can be employed in realistic problems with snapshots from 3D radiative magnetohydrodynamics simulation ns as input atmospheres. As a significant model measuring the flow positions and quantities in hydrodynamics, Navier-Stokes equations was tried to be solved by a structured overset grid method [20]. Wang Yajing *et al*. 2012 [22] attempted to estimate the particle size distribution by a multi-level Tikhonov regularization inversion method.

3.2 Applications of multigrid method in the field of image processing

Image processing refers to the technology of computer for images, including the aspects of image compression, enhancement and restoration, matching, description and recognition. In present, the multigrid method has been applied to image reconstruction, optical flow estimation, interpolation of missing image data, image segmentation, tomography, medical diagnosis etc. [23-45] For instance, Liliana Borcea proposed a nonlinear multigrid approach for imaging the electrical conductivity and permittivity of a heterogeneous and isotropic body with a Neumann-to-Dirichlet map at the boundary [30]. The constructed images by multigrid method not only had a good contrast and the correct geometrical features, but also was stable in the presence of noisy data. Additionally, Samy Wu Fung and Zichao Di 2020 [31] developed a multigrid-based optimization framework to reduce the computational costs of large-scale ptychographic phase retrieval problem. For a waveform imaging described by a hyperbolic wave equation, Mika Takala *et al*. 2018 [32] presented a rigorous multigrid-based forward approach, The numercal results suggested that it allowed recovering the robust results and the computational procedure can be speeded up.

In recent years, the application of multigrid method to the problem of tomography has attracted much attention, and there are many studies about it. Liisa-Ida Sorsa *et al*. 2020 [33] introduced and evaluates numerically a multigrid solver for non-linear tomographic radar imaging, and the results obtained suggested that the tomographic reconstruction quality could be improved. Cuiping Li *et al*. 2012 [34] studied the ultrasound tomography imaging in detecting breast cancer by the multigrid tomography technique. Seungseok Oh *et al*. 2006 [35] applied the a multigrid inversion method to iterative reconstruction for emission and transmission tomography.

3.3 Applications of multigrid method in the field of geology

Geology develops with the human demand for mineral resources such as oil, coal, metal and non-metal, as these are the fundamental source for human society. At the same time, as an important tool of resource exploration, seismic waves propagating underground can be measured or calculated by some means to reflect the underground structure, so that the mineral resources can be discovered. Therefore, the problem of geological exploration can often be transformed into a problem governed by wave equation. Due to the advantages of low computational complexity, multigrid method has been widely used in geological investigation and resource exploration [46-54].

In, Bhogeswara, R 1991 [47] Realistic had developed a variant of multigrid for the simulation of multiphase and multicomponent fluid flow in oil reservoirs which was computationally intensive, and the results obtained showed that the performance was comparable to current mainframe supercomputers [46]. Klaus Stüben 2003 [48] and his team introduced an algebraic multigrid for ground water flow and oil reservoir. M. Adil Sbai and A. Larabi 2021 [49] focused on a single groundwater flow solve and a pure advective transport solve with algebraic multigrid preconditioning. R.P.Hammersley *et al*. integrated an algebraic multigrid technique into a parallel reservoir simulation code and found that it performed well when solving for scalar solution to a system of ellipic equations. More recently, Steen Hørsholt *et al*. 2019 [50] presented a hierarchical multigrid method for oil production optimization of industry-scale the reservoir models. For tracking seismic reflections to identify horizons, Xinming Wu and Sergey Fomel 2018 [51] utilized the multigrid method that globally fit both local slopes and multigrid correlations of seismic traces [70]. Takumi Veda and Michael S. Zhdanov applied a multigrid approach for modeling of Multitransmitter electromagnetic data to speed up the process. Sun Q and Hu Y 2021 [53] employed the geometric multigrid method to borehole electromagnetic sensing and validated its performance as a potential fast solver for practical application.

3.4 Applications of multigrid method in the field of finance

The application of multigrid method in the field of finance, unlike the situation in mechanics, geophysics and image processing mentioned above, emerged relatively late. In the 1990s, multigrid techniques started to be used to the related problem of pricing and hedging financial derivatives. For example, Nigel Clarke and Kevin Parrott in 1999 [56] had developed multigrid solvers for pricing American equity options in a stochastic volatility framework [55].

To date, the application of multigrid method in the field of finance is not yet widespread. Only some researches on the typical financial problems were performed by the multigrid method, such as the problems about the option pricing. For instance, Markus Holtz provided a multigrid approach for the fast and accurate computation of American style option prices in the Black-Scholes framework and of their derivatives with respect to the underlying. Presented a multigrid iterative algorithm for solving a system of coupled free boundary problems for pricing American put options with regime-switching [57]. Dong Han and Justin W. L. Wan proposed multigrid methods for solving the discrete algebraic equations arising from the discretization of the second order Hamilton-Jacobi-Bellman and Hamilton-Jacobi-Bellman-Isaacs equations which could discribe financial problems [58]. Fractional diffusion equations, as a model in computational finance, were solved by two computationally favourable variants of the proposed multigrid method [59]. Aicha Driouch and Hassan Al Moatassime dealt with the task of pricing European basket options in the presence of transaction costs by a nonlinear multigrid method [60, 61]. Justin W. L. Wan 2019 [63] tried to solve the discrete partial integro-differential equations arising from pricing European options by using a fast multigrid method [62]. In order to improve the efficiency of the option pricing, Darae Jeong *et al*. 2013 [2] had approximately solved the two-dimensional Black-Scholes PDE by a multigrid method.

4. Conclusions

This paper has presented a brief overview of multigrid techniques and their applications to problems in several fundamental fleids including physics, image processing, geology and finance. By analyzing the researches in each fleid comprehensively, we now arrive at the following conclusions:

1. The application of multigrid method in the field of physics originated earliest, resulting that this method was not only applied in many branches of physics, but also deeply investigated and improved for specific physical problems in the process of implementation;
2. In the field of image processing, tomography is a very important problem for its non-invasive characteristics. The research on the application of multigrid to tomography is a subject of great tremendous interest, and many related research have been made;
3. In the field of geology, multigrid method is mainly applied in the problem of geological exploration;
4. At present, the application of multigrid method in the field of finance is not widely applied yet, and it mainly focuses on several typical problems. Therefore, the multigrid method may have a wider applicability to the problems in the field of finance in the future.

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