International Journal of Statistics and Applied Mathematics 2022; 7(1): 94-97

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2022; 7(1): 94-97

www.mathsjournal.com
Received: 16-10-2021
Accepted: 02-12-2021

Wanjala Victor
Department of Mathematics, Kibabii University, Bungoma, Kenya

AM .Nyongesa
Department of Mathematics, Kibabii University, Bungoma, Kenya

John Wanyonyi Matuya
Department of Mathematics and Physical Sciences, Maasai Mara University, Narok, Kenya

Corresponding Author:
Wanjala Victor
Department of Mathematics, Kibabii University, Bungoma, Kenya

On metric equivalence of operators of order N

Wanjala Victor, AM Nyongesa and John Wanyonyi Matuya

DOI: https://doi.org/10.22271/maths.2022.v7.i1b.777

Abstract
Class of operators satisfying $V^{*n}V^n = Q^{*n}Q^n$ is introduced which is an equivalence relation, that is Metric equivalence of order n. We examine some properties that this class enjoys. On the same note, we study the relation of this class to other classes of operators like K^* quasinormal operators of order n and normal of order n. We give a condition under which $V^{*n}V^n = Q^{*n}Q^n$ is same as $V^*V^n = Q^*Q^n$ and $V^{*n}V^m = Q^{*n}Q^m$, that is n-power and (n, m)-power-Metric equivalences respectively.

Keywords: N-metric equivalence, metric equivalence, metric equivalence of order n operators, (n, m)-metric equivalence

1. Introduction
We take into consideration behaviors of various classes of operators. Properties of n-normal operators was intensively explored [2], K^*quasi-normal operators of order n was covered in [5] and quasi-normal operators [1]. Wanjala Victor and A.M. Nyongesa [8] explored properties of (n, m)-Metric equivalence of operators in relationship between (n, m)-metrically equivalent operators and other classes like (n, m)-class (Q) and n-metrically equivalent operators were covered by Wanjala Victor and A.M. Nyongesa [8]. N-metrically equivalent as a new equivalence relation was introduced and sheltered by Wanjala Victor *et al*. 2020 [7]. Wanjala Victor *et al*. 2020 [7] striked results relating the class of n-metrically equivalent operators to supplementary classes such as quasinormal and k-quasinormal operators.

2. Preliminaries
Definition 2.1.1: [6] An operator $V \in L(H)$ is said to be normal operator of order n if $V^{*n}V^n = V^nV^{*n}$.
Definition 2.1.2. [7] Operators $V \in L(H)$ and $Q \in L(H)$ are presumablyn-metrically equivalent, denoted by $V \sim_{n-m} Q$, provided $V^*V^n = Q^*Q^n$ for any positive integer $n \in \mathbb{N}$.
Definition 2.1.3. [8] Operators $V \in L(H)$ and $Q \in L(H)$ are presumably (n, m)-metrically equivalent, provided $V^{*m}V^n = Q^{*m}Q^n$ for any positive integer $n, m \in \mathbb{N}$.
Definition 2.1.4. [4] Two operators $V \in L(H)$ and $Q \in L(H)$ are presumablymetrically equivalent, denoted by $V \sim_m Q$, provided $V^*V = Q^*Q$.
Definition 2.1.5. Two operators $V \in L(H)$ and $Q \in L(H)$ are presumably metrically equivalent of order n, denoted by $V \sim_m^n Q$, provided $V^{*n}V^n = Q^{*n}Q^n$ for any positive integer n.

3. Main results
Theorem 3.1.1. Let $V \in L(H)$bea normal operator of order n and $Q \in L(H)$beunitarily equivalent to V, it follows Qis a normal of order n. Proof:
Since $Q = U^*VU$ and U being a unitary and V normal operator of order n, we have:
$Q^{*n}Q^n = U^*V^{*n}UU^*V^nU$
$= U^*V^{*n}V^nU$
$= U^*V^nV^{*n}U$
$= Q^nU^*Q^{*n}U$
$= Q^nU^*UQ^{*n}$
$= Q^nQ^{*n}$
Hence the proof.

Corollary 3.1.2. Operator Q∈L (H) is a normal of order n if and only if Q and Q^* are metrically equivalent of order n.
Proof.
The proof trickles down from Theorem 2.1.1 above.

Proposition 3.1.3. Suppose V ∈L (H) and Q∈L (H) are metrically equivalent of order n, it follows V^* and Q^* are co-metrically equivalent of order n.
Proof.
Since V and Q are metric equivalence relation of order n, we have;

$V^{*n}V^n = Q^{*n}Q^n$, taking adjoint on both sides we have;
$= (V^{*n}V^n)^* = (Q^{*n}Q^n)^*$
$= (V^{*n})^*(V^n)^* = (Q^{*n})^*(Q^n)^*$
$= V^n (V^n)^* = Q^n (Q^n)^*$.

Thus V^* and Q^* are co-metric equivalence of order n.

Proposition 3.1.4. Let V ∈L (H) and Q ∈L (H) be metric equivalence relation of order n+q, it follows V and S are metric equivalence relation of order n+q+1and thus V and Q are metric equivalence of order n+k for every k≥q for all n, k, q which are reals.
Proof.
V and Q are metric equivalence of order n+q, we have;

$= V^{*(n+q)} V^{n+q} = Q^{*(n+q)} Q^{n+q}$ ……………………… (F').

Pre-multiplying by V^* and Q^* on the left-hand side and post-multiplying (F') by V and by Q on the right-hand side;

$= V^* V^{*(n+q)} V^{n+q}V = Q^* Q^{*(n+q)} Q^{n+q} Q$
$= V^* V^{*(n+q)} V^{n+q+1} = Q^* Q^{*(n+q)} Q^{n+q+1}$
$= V^{*(n+q+1)} V^{n+q+1} = Q^{*(n+q+1)} Q^{n+q+1}$

Thus V and Q are metrically equivalent operators of order n+q+1.

4. Relationship between metric equivalence of order n+q and other classes of operators
Definition 4.1.1: V ∈L (H) is presumably normal operator of order n+q for n, q > 0 if $V^{*n+q}V^{n+q} = V^{n+q}V^{*n+q}$

Definition 4.1.2: S, T ∈L (H) are presumably metrically equivalent of order n+q if $V^{*n+q}V^{n+q} = Q^{n+q}Q^{*n+q}$ for n, q > 0.

Remark 4.1.3: We note that normal operator of order n+q is $V^{*n} V^n = V^n V^{*n}$ whenever q =1 and it's a normal operator whenever n=q=1.

Proposition 4.1.4: LetV ∈L (H)and Q ∈L (H)beboth unitarilymetrically equivalent operators of order n+q+1 and metrically equivalent, tit follows V and Q are metrically equivalent of order n+q for n, q > 0, and if Vis normal operator of order n+q, it follows Qisnormal operator of order n+q for all n, q > 0.
Proof.
V and Q being unitarily metrically equivalent of order n+q+1 implies; $V^{*(n+q+1)}V^{n+q+1} = UQ^{*(n+q+1)}Q^{n+p+1} U^*$; and from Proposition 2.2.4, V and Q are metrically equivalence of order n+k since they are metrically equivalent of order n+q+1.Hence $V^{*(n+q+1)} V^{n+q+1} = UQ^{*(n+q+1)} Q^{n+q+1} U^*$ gives us;

$V^{*(n+q+1)} V^{n+q+1} = V^{*(n+q)}V^*V^{n+q}V = UQ^{*(n+q)}Q^*Q^{n+q} QU^*$…………………………. (1)

(Since V and Q are unitarily metric equivalence of order n+q+1)

$= V^{*(n+q)} V^{n+q} V^*V = UQ^{*(n+q)} Q^{n+q} Q^*QU^*$……………………… (2) (Since V and Q are unitarily metric equivalence)

$= V^{*(n+q)}V^{n+q} = UQ^{*(n+q)}Q^{n+q}U^*$…………………………………. (3)

$= V^{n+q} V^{*(n+q)} = U Q^{n+q} Q^{*(n+q)} U^*$……………………………….. (4)

From (3) and (4) we have;

$= UQ^{*(n+q)}Q^{n+q}U^* = U Q^{n+q} Q^{*(n+q)} U^*$
$= Q^{*(n+q)}Q^{n+q} =Q^{n+q} Q^{*(n+q)}$

Thus Q is a normal operator of order n+q.

Theorem 4.1.5. Let$V \in L(H)$ and $Q \in L(H)$ be both subprojections and idempotent. Suppose V and Q are unitarily metric equivalence of order 2 +p and V is K^* quasinormal of order 1+p, then Q is K^* quasinormal of order 1+p.
Proof.

$V^{*(2+p)}V^{2+p} = UQ^{*(2+p)}Q^{2+p}U^*$
$= UQ^{*(2+p)}Q^{2+p}U^* = Q^{*(2+p)}Q^{2+p}$
$= UQ^{*(2+p)}Q^p Q^2U^* = Q^{*(2+p)}Q^pQ^2$
$= Q^* QQ^{*(1+p)}Q^{1+p}= Q^{*(1+p)}Q^{1+p} Q^*Q$
$= Q^* Q^* Q^{*(1+p)}Q^{1+p} = Q^{*(1+p)}Q^{1+p} Q^* Q^*$
$^= (Q^*)^2 (Q^{*(1+p)}Q^{1+p}) = (Q^{*(1+p)}Q^{1+p}) (Q^*)^2$

Hence Q is K^* Quasi normal of order n for K=2.

Theorem 4.1.6. If $V \in L(H)$ and $Q \in L(H)$ are unitarily metrically equivalent of order 2+n then they are metric equivalence of order n if they are Isometrics.
Proof.
V and Q being metrically equivalent operators of order 2+n, we have;

$V^{*(2+n)} V^{2+n} = Q^{*(2+n)} Q^{2+n}$
$^= V^{*2} V^2 V^{*n} V^n = Q^{*2} Q^2 Q^{*n} Q^n$
$= V^* V V^{*n} V^{nV^*}V = Q^* Q Q^{*n} Q^{nQ^*}Q$
$= I V^{*n} V^n I = I Q^{*n} Q^n I$
$= V^{*n} V^n= Q^{*n} Q^n.$

Theorem 4.1.7. Let $V \in L(H)$ and $Q \in L(H)$ be metrically equivalent operators of order 2, it follows they are metric equivalence if they are Quasi-isometrics.
Proof.
The proof is trifling and trickles directly from definition. Since V and Q are metrically equivalent of order 2, we obtain;

$V^{*2}V^2 = Q^{*2}Q^2$
$= V^* V ^= Q^* Q.$

Theorem 4.1.8. Let $V \in L(H)$ and $Q \in L(H)$ be metrically equivalent operators of order 3, then they are (3, 4)-power-metrically equivalent operators if they are idempotent.
Proof.
V and Q being metrically equivalent operators of order 3 implies;

$V^{*3} V^{3=} Q^{*3} Q^3$
$^=V^{*3} V^2 V = Q^{*3} Q^2 Q$; since they are idempotent $V^2 = V$ and $Q^2 = Q$, hence;
$^=V^{*3} V^2 V^2 = Q^{*3} Q^2 Q^2$
$^=V^{*3} V^4= Q^{*3} Q^4$as desired.

Theorem 4.1.9. Let $V \in L(H)$ and $Q \in L(H)$ be metrically equivalent operators of order 4, then they are metric equivalence of order 3 if they are quasi-isometrics.
Proof.
V and Q being metrically equivalent of order 4 imply;

$V^{*4} V^4= Q^{*4} Q^4$
$= V^{*2} V^2 V^{*2}V^{2\,=} Q^{*2} Q^2 Q^{*2}Q^2$
$= V^{*2} V^2 V^*V ^= Q^{*2} Q^2 Q^*Q$
$=V^{*3} V^3= Q^{*3} Q^3$

Theorem 4.1.10. Let $V \in L(H)$ and $Q \in L(H)$ be both metrically equivalent of order 2 and order 3, then they are metrically equivalent of order n.
Proof.
Proof is given by induction, first by showing that the result holds whenever n=4. Since V and Q are metric equivalence of order 2 we have;

$V^{*2} V^2= Q^{*2} Q^2$……………………………………. (1)

Pre-multiplying and post-multiplying left and right hand sides of (1) by V* and V and by Q* and Q respectively we obtain;

$V^{*4} V^4= Q^{*4} Q^4$……………………………………. (2).

Let the result hold whenever n > 4, that is

$V^{*n} V^n = Q^{*n} Q^n$ it follows;

$V^{*n+1} V^{n+1} = Q^{*n+1} Q^{n+1}$
$= V^* V^{*n} V^n V = Q^* Q^{*n} Q^n Q$
$= V^* V^{*n-2} V^{*2} V^{n-2} V^2 V = Q^* Q^{*n-2} Q^{*2} Q^{n-2} Q^2 Q$
$= V^{*3} V^{*n-2} V^{*n-2} V^3 = Q^{*3} Q^{*n-2} Q^{*n-2} Q^3$
$= V^{*n+1} V^{n+1} = Q^{*n+1} Q^{n+1}$

Hence, V and Q are metrically equivalent of order n+1 and the result follows by induction.

5. References

1. Bala A. A note on quasi normal operator, University of Delhi; c1976.
2. Alzuraiqi SA. On n-normal operators, General math. Notes. 2010;1(2):61-73.
3. Senthil Kumar D, Masheswari Naik P, Santhi R. K-quasi-normal operators. International Journal of Mathematics and Computation. 2012;15(2):0974-5718.
4. Nzimbi BM, Pokhariyal GP, Moindi SK. A note on Metric Equivalence of some operators, Far East Journal of Mathematical Sciences. 2013;75(2):301-318.
5. Abdul Ali N, Prasanna A, Mathura G, Keerthiga A, Gayathri T. On k* Quasi Normal Operator of Order n, International Journal of Mathematics Trends and Technology. 2019;65(8):2231-5373.
6. Shaakir Laith K, Marai Saad. Quasi-normal operator of order n. Tikrit Journal of Pure Sciences. 2010. ISSN 1662-1813.
7. Wanjala Victor, Obogi RK, Okoya M. N-metric equivalence of operators, International Journal of mathematics and its applications. 2020;8(1):107-109.
8. Wanjala Victor, Nyongesa AM. On (n, m)-metrically equivalent operators, International Journal of mathematics and its applications. 2021;9(2):101-104.