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Variational iteration method for solving coupled nonlinear system of Klein-Gordon equations

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Abstract
In this paper, variational iteration technique is applied to solve nonlinear coupled Klein-Gordon equations (CNLKGE). The equations arise when dealing with particle physics. The numerical solutions of CNLKGE have been compared to the exact solutions and presented graphically. The results of the proposed method are in good agreement with the exact solutions provided by [1]. The suggested algorithm is efficient and easier to employ compared to other methods like the perturbation method.

Keywords: Variational iteration, LaGrange multiplier, nonlinear Klein-Gordon equations, correction functional approximants

1. Introduction
Over the last few decades, searching for exact and numerical solutions of nonlinear partial differential equations has become an interesting area in the physical, biological sciences and nonlinear sciences [9, 10]. In order to deeper understanding of these nonlinear phenomena, several mathematicians, engineers and scientists make the effort to seek more accurate solutions. The nonlinear coupled Klein-Gordon equation (CNLKGE) arises in the area of particle physics. Spin zero particles can be explained using CNLKGE. That is like π -meson and so on. The Nonlinear coupled Klein-Gordon equation was first studied by [1-3] respectively. Recently, [7] solved closely related equation called Schrodinger-Klein-Gordon equation using Variational Iteration Method (VIM).

In this paper, our aim is to develop VIM for CNLKGE and compare the solutions with the exact solutions given by [1]. In 1999, the Variational Iteration Method (VIM) was first time proposed by J.H. He 1999, 2007, 1998 [4-6]. The method does not require any transformation, discretization, perturbation or any restrictive assumptions that are used to handle nonlinear terms as may be needed by other methods.

The structure of this article are sequenced as; Section II gives basics of VIM. Section III develops the recursive algorithms for CNLKGE. Section IV tabulates and analyze the absolute errors. Section V concludes the study.

The system to be solved is defined by;

$$F_1(u,v,u_{xx},u_{tt}) = u_{xx} - u_{tt} - u + 2u^3 + 2uv = (\Box + 1)u - 2u^3 - 2uv = 0,$$
$$x \in [0,L], L > 0, t \in [0,t_f], t_f > 0 \tag{1}$$
$$F_2(u,u_t,v_x,v_t) = v_x - v_t - 4uu_t = 0, x \in [0,L], L > 0, t \in [0,t_f], t_f > 0 \tag{2}$$

with the derived initial conditions

$$u(x,0) = \psi_1(x) = \sqrt{\frac{(1+c)}{(1-c)}}\sec h\left(\frac{x}{\sqrt{1-c^2}}\right), \quad x \in [0,L], L > 0 \tag{3}$$
$$u_t(x,0) = \psi_2(x) = \left(\frac{c}{1-c}\right)\sec h\left(\frac{x}{\sqrt{1-c^2}}\right)\tanh h\left(\frac{x}{\sqrt{1-c^2}}\right), x \in [0,L], L > 0 \tag{4}$$
$$v(x,0) = \psi_3(x) = \left(\frac{-2c}{1-c}\right)\sec h^2\left(\frac{x}{\sqrt{1-c^2}}\right), \quad x \in [0,L], L > 0 \tag{5}$$

Where $c^2 < 1$

2. Variational Iteration Method (VIM)

2.1 Concepts of VIM

Consider the following generalized system. According to [4, 8], to use variational iteration method (VIM), the system is expressed in the following format.

$$L_t u + P_1(u,v) + Q_1(u,v) = h_1 \quad (6)$$
$$L_t v + P_2(u,v) + Q_2(u,v) = h_2 \quad (7)$$

The correction functional for equations (6) and (7) can be written as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\xi)\left[L_1(u_n(x,\xi)) + P_1(\tilde{u}_n,\tilde{v}_n) + Q_1(\tilde{u}_n,\tilde{v}_n) - h_1(\xi)\right]d\xi, n \geq 0. \quad (8)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\xi)\left[L_2(u_n(x,\xi)) + P_2(\tilde{u}_n,\tilde{v}_n) + Q_2(\tilde{u}_n,\tilde{v}_n) - h_2(\xi)\right]d\xi, n \geq 0. \quad (9)$$

Where $\lambda_1(\xi)$ and $\lambda_2(\xi)$ are Lagrangian multipliers to be determined using variational theory, $\tilde{u}_n, \tilde{v}_n$ are considered as restricted variations i.e. $\delta\tilde{u}_n = 0, \delta\tilde{v}_n = 0$.

Equations (8) and (9) are called correction functional for equations (6) and (7) respectively. The solutions can now be found using limit definitions;

$u(x,t) = \lim_{n\to\infty} u_n(x,t)$ and $v(x,t) = \lim_{n\to\infty} v_n(x,t)$.

3. Numerical Solution of CNLKGE using VIM

Particularly, the correction functionals for our equations (1) and (2) are given by

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda_1(\xi)\left[\frac{\partial^2 \tilde{u}_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + \tilde{u}_n(x,\xi) - 2\tilde{u}_n{}^3(x,\xi) - 2\tilde{u}_n(x,\xi)\tilde{v}_n(x,\xi)\right]d\xi$$
$$n \geq 0 \quad (10)$$

$$v_{n+1}(x,t) = v_n(x,t) + \int_0^t \lambda_2(\xi)\left[\frac{\partial v_n(x,\xi)}{\partial \xi} - \frac{\partial \tilde{v}_n(x,\xi)}{\partial x} + 4\tilde{u}_n(x,\xi)\frac{\partial \tilde{u}_n(x,\xi)}{\partial \xi}\right]d\xi$$
$$n \geq 0 \quad (11)$$

By making the correction functionals stationary, we get

$\lambda_1(\xi) = \xi - t$ and $\lambda_2(\xi) = -1$

We obtain the iterative formulae

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\xi - t)\left[\frac{\partial^2 u_n(x,\xi)}{\partial \xi^2} - \frac{\partial^2 u_n(x,\xi)}{\partial x^2} + u_n(x,\xi) - 2u_n{}^3(x,\xi) - 2u_n(x,\xi)v_n(x,\xi)\right]d\xi$$
$$n \geq 0 \quad (12)$$

$$v_{n+1}(x,t) = v_n(x,t) - \int_0^t \left[\frac{\partial v_n(x,\xi)}{\partial \xi} - \frac{\partial v_n(x,\xi)}{\partial x} + 4u_n(x,\xi)\frac{\partial u_n(x,\xi)}{\partial \xi}\right]d\xi$$
$$n \geq 0 \quad (13)$$

By taking $c = 0.5$

$$u(x,0)=\sqrt{3}\sec h(\frac{2x}{\sqrt{3}}), u_t(x,0)=\sec h(\frac{2x}{\sqrt{3}})\tan h(\frac{2x}{\sqrt{3}}) \quad (14)$$

And

$$v(x,0)=-2\sec h^2(\frac{2x}{\sqrt{3}}) \quad (15)$$

Are the initial conditions of our system given by equations (1) and (2) Using the variational iteration formulae equations (12) and (13) along the $t-$direction we generated the approximants.

$$u_0(x,0)=u(x,0)+tu_t(x,0)=\sqrt{3}\sec h(\frac{2x}{\sqrt{3}})+t\sec h(\frac{2x}{\sqrt{3}})\tan h(\frac{2x}{\sqrt{3}}), \quad (16)$$

$$v_0(x,t)=u(x,0)=-2\sec h^2(\frac{2x}{\sqrt{3}}), \quad (17)$$

$$u_1(x,t)=u_0(x,t)+\int_0^t(\xi-t)\left[\frac{\partial^2u_0(x,\xi)}{\partial\xi^2}-\frac{\partial^2u_0(x,\xi)}{\partial x^2}+u_0(x,\xi)-2u_0{}^3(x,\xi)-2u_0(x,\xi)v_0(x,\xi)\right]d\xi \quad (18)$$

$$v_1(x,t)=v_0(x,t)-\int_0^t\left[\frac{\partial v_0(x,\xi)}{\partial\xi}-\frac{\partial v_0(x,\xi)}{\partial x}+4u_0(x,\xi)\frac{\partial u_0(x,\xi)}{\partial\xi}\right]d\xi \quad (19)$$

$$u_2(x,t)=u_1(x,t)+\int_0^t(\xi-t)\left[\frac{\partial^2u_1(x,\xi)}{\partial\xi^2}-\frac{\partial^2u_1(x,\xi)}{\partial x^2}+u_1(x,\xi)-2u_1{}^3(x,\xi)-2u_1(x,\xi)v_1(x,\xi)\right]d\xi \quad (20)$$

$$v_2(x,t)=v_1(x,t)-\int_0^t\left[\frac{\partial v_1(x,\xi)}{\partial\xi}-\frac{\partial v_1(x,\xi)}{\partial x}+4u_1(x,\xi)\frac{\partial u_1(x,\xi)}{\partial\xi}\right]d\xi \quad (21)$$

We iterated two times to get; u_1, v_1, u_2 and v_2. From equations (16), (17), (18), (19), (20) and (21) solutions are can now be found by the limits;

$$u(x,t)=\lim_{n\to\infty}u_n(x,t) \quad (22)$$

$$v(x,t)=\lim_{n\to\infty}v_n(x,t) \quad (23)$$

These solutions obtained from equations (22) and (23) are used for numerical study of convergence of VIM [11].

4. Numerical Results and Discussions

In this proposed method, $c=0.5,$ for analysis purposes. The actual expressions for equations (22) and (23) are extremely long and cannot be shown here.

4.1 First iteration of VIM

Table 1: Values of $u(x,t)$ using first iteration of Variational Iteration Method (VIM) $c=0.5, t=0.1$.

x	u (First iteration)	u (exact)	Absolute error
0	1.729164056	1.72916806	4.00395E-06
0.1	1.729315686	1.72916806	0.000147626
0.2	1.706671583	1.706390891	0.000280692
0.3	1.662684418	1.662305739	0.000378679
0.4	1.600060559	1.599626806	0.000433754
0.5	1.522373316	1.521926123	0.000447193
0.6	1.433607129	1.433180153	0.000426976
0.7	1.337730409	1.33734623	0.000384179
0.8	1.23836548	1.238035736	0.000329744
0.9	1.138583593	1.138311112	0.000272481
1	1.040818342	1.040599983	0.000218358

Table 2: Values of $v(x,t)$ using first iteration of Variational Iteration Method (VIM); $c = 0.5, t = 0.1$.

x	V (First iteration)	V (exact)	Absolute error
0	-2	-1.99334812	0.00665188
0.1	-2.000027329	-1.99334812	0.006679209
0.2	-1.94769774	-1.941179914	0.006517826
0.3	-1.848349568	-1.84217358	0.006175988
0.4	-1.711556276	-1.705870612	0.005685664
0.5	-1.549267667	-1.54417275	0.005094917
0.6	-1.373793506	-1.369336901	0.004456605
0.7	-1.196148064	-1.192329959	0.003818105
0.8	-1.025036757	-1.021821655	0.003215101
0.9	-0.866504698	-0.863834792	0.002669906
1	-0.7240919	-0.721898883	0.002193017

4.2 Second Iteration of VIM

Table 3: Values of $u(x,t)$ using second iteration of Variational Iteration Method (VIM); $c = 0.5, t = 0.1$.

x	u (Second iteration)	u (exact)	Absolute error
0	1.729148926	1.72916806	1.91337E-05
0.1	1.729148918	1.72916806	1.91426E-05
0.2	1.706372583	1.706390891	1.83071E-05
0.3	1.662289044	1.662305739	1.6695E-05
0.4	1.599612284	1.599626806	1.45219E-05
0.5	1.521914041	1.521926123	1.20822E-05
0.6	1.433170493	1.433180153	9.6599E-06
0.7	1.337338766	1.33734623	7.46444E-06
0.8	1.238030127	1.238035736	5.6091E-06
0.9	1.138306988	1.138311112	4.12352E-06
1	1.040597001	1.040599983	2.98189E-06

Table 4: Values of $v(x,t)$ using second iteration of Variational Iteration Method (VIM); $c = 0.5, t = 0.1$.

x	V (Second iteration)	V (exact)	Absolute error
0	-1.99335	-1.99334812	1.87979E-06
0.1	-1.994337986	-1.99334812	0.000989865
0.2	-1.943004316	-1.941179914	0.001824402
0.3	-1.84454773	-1.84217358	0.00237415
0.4	-1.708463736	-1.705870612	0.002593124
0.5	-1.546690895	-1.54417275	0.002518145
0.6	-1.371574314	-1.369336901	0.002237413
0.7	-1.194181569	-1.192329959	0.00185161
0.8	-1.023266761	-1.021821655	0.001445106
0.9	-0.864908258	-0.863834792	0.001073466
1	-0.72266299	-0.721898883	0.000764106

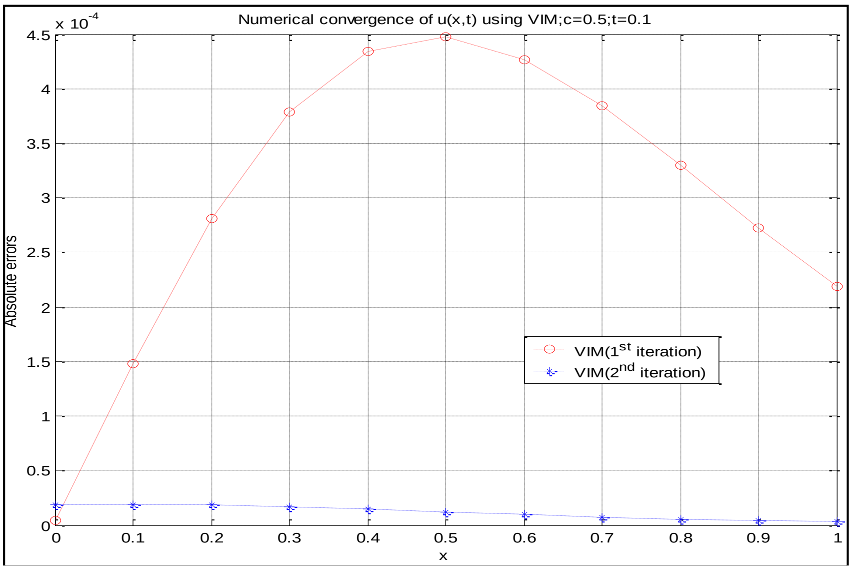

Fig 1: Numerical convergence of $u(x,t)$ using VIM

From Fig. 1, the convergence of VIM is rapid. The errors are in the order of 1E-4. It is computationally efficient. At the second iteration, the absolute errors dropped drastically to almost zero.

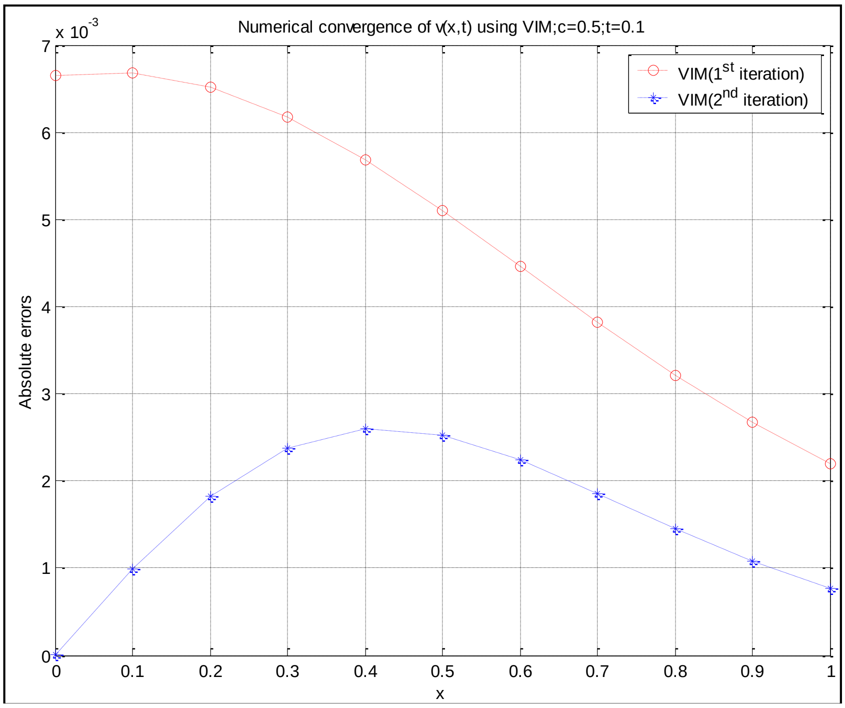

Fig 2: Numerical convergence of $v(x,t)$ using VIM

From Fig. 2, the convergence is good. When Fig. 1 and Fig. 2 are compared, it was noted that $u(x,t)$ values converged more rapidly than the $v(x,t)$ values. Absolute errors become smaller as you increase the number of iterations.

5. Summary of Findings

In this study, VIM has been used to compute the semi-analytic solution of the simultaneous nonlinear Klein-Gordon equations. Results obtained by VIM were in an good agreement with the exact results obtained by [1]. This therefore showed that the method is efficient and reliable. VIM do not use small parameter like in homotopy method. VIM reduces the errors as we progress in the iterations. VIM can be used to solve nonlinear problems efficiently compared to Finite Difference method. Based on its accuracy, VIM can be utilized to solve linear and non-linear problems. This explains why a few approximations can be used to achieve a high degree of accuracy. From the tables 1-4 and Fig.1 and Fig. 2 we concluded that accuracy is improved as more iterations are used. The computations in this paper were performed using MATLAB 6.1 symbolic toolbox.

6. Conflict of interests

The author (s) declare that there is no conflict of interests.

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