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Ellipse perimeter approximation: New formula with absolute relative error less than one ppm

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Abstract
In this article, the author presents two modifications to his original formula for Ellipse Perimeter Approximation, published two years ago. The objective is to reduce the Absolute Relative Error further. After the first modification, the Absolute Relative Error becomes less than one millimetre per kilometre, (that is to the order of 10^{-7}) for ellipses with eccentricities ranging from zero to 0.92. Though for the remaining ellipses, with eccentricities between 0.92 and 1, this modification increases the Absolute Relative Error up to the order of 10^{-4}, it is reduced and contained within the order of 10^{-6} by an error approximation formula.
The second modification limits the Absolute Relative Error to less than 5 centimetres per kilometre across all eccentricities.

Keywords: Ellipse, major and minor radii, aspect ratio and eccentricity, relative error

Introduction

The ellipse, whose Cartesian equation is: $(x/a)^2 + (y/b)^2 = 1$ is named here as the standard ellipse. 'a' and 'b' ($a \geq b \geq 0$) are the major and minor radii of the ellipse.
Its perimeter P (a, b) is given by the formula

$$P(a,b) = \int_0^{2\pi} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\, d\theta$$

where (a cos θ, b sin θ), $0 \leq \theta < 2\pi$, is a parametric point on the ellipse. Due to the symmetry of the ellipse w. r. t. its axes, it is enough to evaluate the Quarter Perimeter (abbreviated here as QPM), and, denoted by Q (a, b), given by the integral:

$$Q(a, b) = \int_0^{\pi/2} \sqrt{a^2 sin^2\theta + b^2 cos^2\theta}\, d\theta.$$

Obviously, P (a, b) = 4* Q (a, b), Q (a, 0) = a, and, Q (a, a) = πa /2. As the definite integral for Q (a, b) could not be evaluated by direct integration for values of b other than b = 0 and b = a, approximate value for Q (a, b) is obtained by Numerical Methods for given a and b. Simpson's 1/3- Rule Method is popular in this regard. The QPM values used here are obtained by this method, by dividing the interval of integration [0, π /2] into 500 equal sub-intervals, so that the length of each sub-interval is h = π/1000. Then, the Absolute Relative Error vis-vis the exact values is to the order of h^4. Therefore, the Absolute Relative Error in Simpson's 1/3-values used here is less than 10^{-10}.
In an article published two years ago, the author introduced a new method to approximate the perimeter of the ellipse. It was shown there that the Quarter Perimeter Q (a, b) of the standard ellipse follows Lagrange's first order partial differential equation in a and b. Therefore, Q (a, b), as a solution of the partial differential equation, has to be a function of $(a^p + b^p)^{\frac{1}{p}}$, $p \neq 0$ and/or $\sqrt{ab}$, which are two independent particular solutions of the partial differential equation Koshy KI [1]. Further, it was shown that the empirical formula:

$$Q(a, b; Ko) := (a^p + b^p)^{\frac{1}{p}} + k\,(ab)^2/(a+b)^3 \quad (1)$$

approximates the perimeter, where k ≈ 0.4825159, and, p = ln (2) / ln (π/2 – k/8) ≈ 1.680647, with maximum Absolute Relative Error less than $6.0*10^{-5}$, that is, 'less than six centimetres per kilometre', for all eccentricities.
The outcome of further research on the topic is presented in this article.

Notations
Conventional notations and terminologies related to the standard ellipse are used in this article. Aspect Ratio of the ellipse is the ratio (b/a). Obviously, the eccentricity 'e' is

$$\sqrt{(1-\left(\frac{b}{a}\right)*\left(\frac{b}{a}\right)}$$

QPM and ARE are abbreviations used to denote Quarter Perimeter and Absolute Relative Error respectively. Approximations to the QPM obtained by different authors are identified by adding name-indicative characters after the parameter 'b'. For example, Q (a, b; Sm) indicates the QPM approximated by Simpson's (1/3) Rule; Q (a, b; Sm) values presented here are derived with step-width h = π/1000.
Q (a, b; Ko) denotes the QPM by the author's Formula given in equation (1) above.
Q (a, b; K1) is the QPM given by the first modification of equation (1);
Q (a, b; Kc) is the formula obtained by subtracting a Correction Term (CT) from Q (a, b, K1) to reduce the relative error further. (See equation (3) below.
Q (a, b; K2) is the QPM given by the second modification of equation (1).

Materials and Methods
In all forms, the Relative Error is calculated based on Q (a, b; Sm). It is well-known from Numerical Mathematics [2], that the Absolute Relative Error of Q (a, b; Sm), vis-a-vis the actual quarter perimeter value Q (a, b), is of the order of h^4 or $10^{(-10)}$ Erwin Kreiszig 2010 [2].

Modified Koshy's Formula # I

$$Q\ (a, b; K1): = (a^p + b^p)^{\frac{1}{p}} + k\ (ab)^{2.1}/(a+b)^{3.2} \qquad (2)$$

where k = 0.440143 and p = ln (2)/ ln (π/2 -0.440143/$2^{3.2}$).

This formula approximates the Quarter Perimeter of the ellipse with Relative Error limits as follows: between -$9.6647809*10^{-7}$ and $9.6781748*10^{-7}$, for b/a ∈ [0.37, 1]; between $1.514497*10^{-6}$ and $8.3793697*10^{-6}$, for b/a ∈ [0.31, 0.37)
Between 0.31 and 0, the Relative Error increases and reaches to the order of 10^{-4}. (Table 1)
In order to reduce the Relative Error for b/a < 0.37, we introduce the Correction Term (CT): f (b/a): = 2.1*b/a*(0.3848- b/a)*exp(-12.95*b/a), for b/a < 0.37.

Then the corrected Koshy Formula is Q (a, b; Kc): =

$$\begin{cases} Q\ (a, b; K1), if\ \frac{b}{a} \geq 0.37 \\ Q\ (a, b; K1) - f\left(\frac{b}{a}\right) if\ \frac{b}{a} < 0.37 \end{cases} \qquad (3)$$

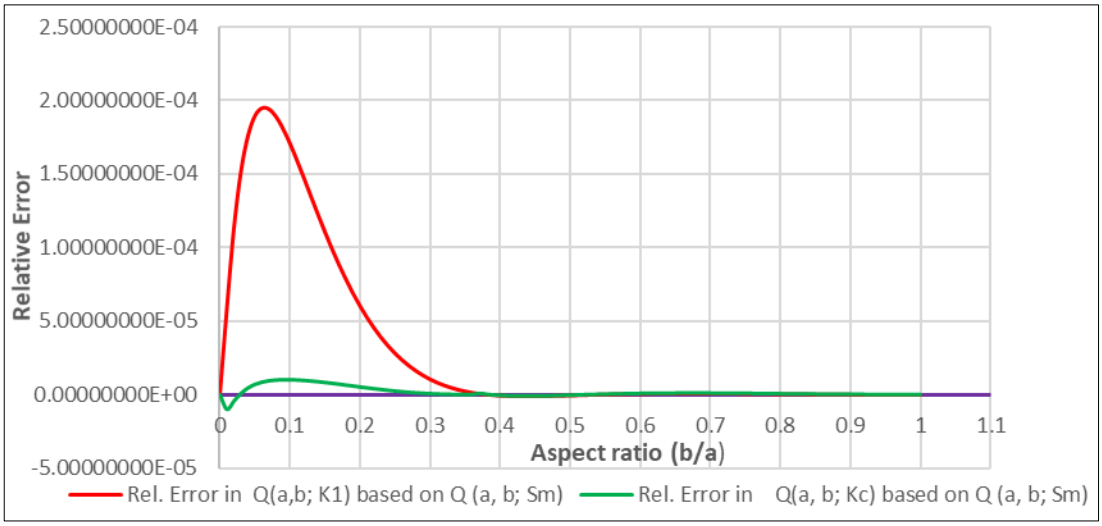
The maximum Absolute Relative Error in Q (a, b; Kc) is of the order of 10^{-6} for all b/a from 0 to 1, and of the order of 10^{-7} for all b/a ≥ 0.29. (Table 2)

Modified Koshy's Formula # II

$$Q\ (a, b; K2): = (a^p + b^p)^{\frac{1}{p}} + k\ (ab)^{1.97}/(a+b)^{2.94} \qquad (4)$$

where k = 0.49496 and p = ln (2) /ln (π/2 -0.49496/$2^{2.94}$).
Equation (4) approximates the Quarter Perimeter of the ellipse with Relative Error lying in the interval (-$4.9038243*10^{-5}$, $4.9045481*10^{-5}$). Thus, the Absolute Relative Error is less than $5*10^{-5}$. Therefore, Q (a, b; K2) is a better formula than Q (a, b; Ko) for EPM Approximation. (Table 3.)

Discussion
Now that there are four formulae before us, academics may make their choice depending upon the precision preferred. Obviously, Q (a, b; K1) is the choice formula for QPM of the ellipse, if b/a ≥ 0.37, as then the Absolute Relative Error is less than one millimetre per kilometre.
If b/a < 0.37, choose Q (a, b; K2) or Q (a, b; Ko). The Absolute Relative Error of the former is less than 5 cm per km and of the latter is 6 cm per km; otherwise, if higher accuracy is required, choose Q (a, b; Kc).
Figure below compares the two Absolute Relative Errors in Q (a, b; K1) and Q (a, b; Kc).



Graph 1: Graphs of Relative Errors in Q (a, b; K1) and Q (a, b; Kc)

It is pointed out here that formulae (1) to (4) are all functions of $(a^p + b^p)^{\frac{1}{p}}$ and $\sqrt{ab}$ and can be re-written in terms the Geometric Mean (GM:$\sqrt{ab}$) and the Arithmetic Mean (AM: (a+b)/2) of a and b. For, In (1), $(ab)^2/(a+b)^3 = (ab)^{1.5}/(a+b)^3 * \sqrt{ab} = (2GM/AM)^3 * \sqrt{ab}$; in (2), $(ab)^{2.1}/(a+b)^{3.2} = (ab)^{1.6}/(a+b)^{3.2} * \sqrt{ab} = (2GM/AM)^{3.2} * \sqrt{ab}$, and, in (4), $(ab)^{1.97}/(a+b)^{2.94} = (ab)^{1.47}/(a+b)^{2.94} * \sqrt{ab} = (2GM/AM)^{2.94} * \sqrt{ab}$.

Table 1: Ellipse QPM Values of Modified Koshy Formula # I; Col. 3: Simpson values; Col. 5. Modified values.

a	b	Q (a, b; Sm): n = 500	LN(2)/LN(π/2-0.440143/2^3.2)	Q (a, b; K1)	Relative Error
100	100	157.079632679490000	1.6479306001	157.0796326795	0.00000000E+00
100	99	156.295221198759000	1.6479306001	156.2952214162	1.39097480E-09
100	98	155.512803035381000	1.6479306001	155.5128039083	5.61298777E-09
100	97	154.732408602913000	1.6479306001	154.7324105726	1.27299024E-08
100	96	153.954068977126000	1.6479306001	153.9540724860	2.27914371E-08
100	95	153.177815915124000	1.6479306001	153.1778214037	3.58317218E-08
100	94	152.403681875157000	1.6479306001	152.4036897800	5.18678247E-08
			etc.		
100	70	134.559224536798000	1.6479306001	134.5593502851	9.34520077E-07
100	69	133.851177790929000	1.6479306001	133.8513052672	9.52372931E-07
100	68	133.146427448379000	1.6479306001	133.1465557626	9.63707517E-07
100	67	132.445036148041000	1.6479306001	132.4451643307	9.67817483E-07
100	66	131.747068267573000	1.6479306001	131.7471952731	9.64010584E-07
100	65	131.052589989168000	1.6479306001	131.0527147015	9.51620640E-07
100	64	130.361669368570000	1.6479306001	130.3617906077	9.30021381E-07
100	63	129.674376407550000	1.6479306001	129.6744929384	8.98642404E-07
100	62	128.990783130063000	1.6479306001	128.9908936735	8.56987430E-07
100	61	128.310963662313000	1.6479306001	128.3110669084	8.04655127E-07
			etc.		
100	45	118.017231101804000	1.6479306001	118.0171205639	-9.36625070E-07
100	44	117.415435914284000	1.6479306001	117.4153224348	-9.66478092E-07
100	43	116.819231774770000	1.6479306001	116.8191206076	-9.51616652E-07
100	42	116.228755811240000	1.6479306001	116.2286533874	-8.81226361E-07
100	41	115.644150306677000	1.6479306001	115.6440643806	-7.43021648E-07
100	40	115.065562978324000	1.6479306001	115.0655027855	-5.23117527E-07
100	39	114.493147277843000	1.6479306001	114.4931237038	-2.05898768E-07
100	38	113.927062714469000	1.6479306001	113.9270884747	2.26111649E-07
100	37	113.367475203505000	1.6479306001	113.3675650339	7.92382216E-07
100	36	112.814557442816000	1.6479306001	112.8147283002	1.51449747E-06
100	35	112.268489320333000	1.6479306001	112.2687605917	2.41627328E-06
			etc.		
100	32	110.673299066356000	1.6479306001	110.6740154712	6.47314945E-06
100	31	110.156588483269000	1.6479306001	110.1575115261	8.37936972E-06
100	30	109.647751739223000	1.6479306001	109.6489162412	1.06203907E-05
100	29	109.147022658760000	1.6479306001	109.1484671596	1.32344506E-05
			etc.		
100	17	103.874232134835000	1.6479306001	103.8834321482	8.85687738E-05
100	16	103.506806897050000	1.6479306001	103.5170928609	9.93747578E-05
100	15	103.152507352688000	1.6479306001	103.1639375670	1.10808885E-04
100	14	102.811895824480000	1.6479306001	102.8245164924	1.22754938E-04
			etc.		
100	3	100.197736240267000	1.6479306001	100.2129640000	1.51977083E-04
100	2	100.095978995359000	1.6479306001	100.1073671891	1.13772740E-04
100	1	100.027458255494000	1.6479306001	100.0333913759	5.93149176E-05
100	0	100.000000000000000	1.6479306001	100.0000000000	4.26325641E-16

Table 2: Ellipse QPM Values of Modified Koshy Formula # I and its corrected form; Col. 7: Corrected values. (Corrections made for b/a less than 0.37; $ CT: f(b/a) =2.1*b/a*(0.3848-b/a)*exp(-12.95*b/a)

a	b	b/a	Q (a, b; Sm): Simpson's (1/3) Rule	Q (a, b; K1): Modified Koshy Formula # I	$ CT: f (b/a) = values to be reduced from Q (a, b: K1)	Q (a, b; Kc): = Q (a, b; K1) - f (b/a)	Rel. Error in Q (a, b; Kc) based on Q (a, b; Sm)
100	40	0.4	115.065562978324000	115.0655027855	0	115.0655027855	-5.23117527E-07
100	39	0.39	114.493147277843000	114.4931237038	0	114.4931237038	-2.05898768E-07
100	38	0.38	113.927062714469000	113.9270884747	0	113.9270884747	2.26111649E-07
100	37	0.37	113.367475203505000	113.3675650339	0	113.3675650339	7.92382216E-07
100	36	0.36	112.814557442816000	112.8147283002	1.77130195E-04	112.8145511700	-5.56030469E-08
100	35	0.35	112.268489320333000	112.2687605917	2.75059623E-04	112.2684855321	-3.37429720E-08
100	34	0.34	111.729458356001000	111.7298520747	3.91541339E-04	111.7294605333	1.94876788E-08
100	33	0.33	111.197660182088000	111.1982012487	5.29122616E-04	111.1976721261	1.07412309E-07
100	32	0.32	110.673299066356000	110.6740154712	6.90602182E-04	110.6733248690	2.33142264E-07
100	31	0.31	110.156588483269000	110.1575115261	8.79037472E-04	110.1566324886	3.99479598E-07

100	30	0.3	109.647751739223000	109.6489162412	1.09774720E-03	109.6478184940	6.08811039E-07
100	29	0.29	109.147022658760000	109.1484671596	1.35030757E-03	109.1471168521	8.62994740E-07
100	28	0.28	108.654646339879000	108.6564132715	1.64053998E-03	108.6547727315	1.16324213E-06
100	27	0.27	108.170879987965000	108.1730158134	1.97248764E-03	108.1710433258	1.50999795E-06
100	26	0.26	107.695993839586000	107.6985491439	2.35037797E-03	107.6961987659	1.90282251E-06
100	25	0.25	107.230272189460000	107.2333017050	2.77856660E-03	107.2305231384	2.34028116E-06
100	24	0.24	106.774014536549000	106.7775770813	3.26145844E-03	106.7743156229	2.81984673E-06
100	23	0.23	106.327536868368000	106.3316951705	3.80339974E-03	106.3278917707	3.33782183E-06
100	22	0.22	105.891173106713000	105.8959934821	4.40853410E-03	105.8915849480	3.88928787E-06
100	21	0.21	105.465276743060000	105.4708285849	5.08061369E-03	105.4657479712	4.46808805E-06
100	20	0.2	105.050222698445000	105.0565777276	5.82275543E-03	105.0507549722	5.06685010E-06
100	19	0.19	104.646409451062000	104.6536406634	6.63712926E-03	104.6470035341	5.67705180E-06
100	18	0.18	104.254261485833000	104.2624417177	7.52456356E-03	104.2549171542	6.28912740E-06
100	17	0.17	103.874232134835000	103.8834321482	8.48404945E-03	103.8749480988	6.89260368E-06
100	16	0.16	103.506806897050000	103.5170928609	9.51212213E-03	103.5075807388	7.47624001E-06
100	15	0.15	103.152507352688000	103.1639375670	1.06020931E-02	103.1533354738	8.02812424E-06
100	14	0.14	102.811895824480000	102.8245164924	1.17431024E-02	102.8127733900	8.53564150E-06
100	13	0.13	102.485580990886000	102.4994207953	1.29189528E-02	102.4865018425	8.98518204E-06
100	12	0.12	102.174224732294000	102.1892879067	1.41066827E-02	102.1751812240	9.36137994E-06
100	11	0.11	101.878550604060000	101.8948081047	1.52748241E-02	101.8795332806	9.64556814E-06
100	10	0.1	101.599354502522000	101.6167327793	1.63812835E-02	101.6003514959	9.81298895E-06
100	9	0.09	101.337518361821000	101.3558850876	1.73707698E-02	101.3385143178	9.82810680E-06
100	8	0.08	101.094028165077000	101.1131741064	1.81716834E-02	101.0950024230	9.63714589E-06
100	7	0.07	100.869998319400000	100.8896143245	1.86923589E-02	100.8709219656	9.15679815E-06
100	6	0.06	100.666705836685000	100.6863537026	1.88165394E-02	100.6675371633	8.25820775E-06
100	5	0.05	100.485640478649000	100.5047163734	1.83979346E-02	100.4863184388	6.74683619E-06
100	4	0.04	100.328582826684000	100.3462724635	1.72536904E-02	100.3290187731	4.34518631E-06
100	3	0.03	100.197736240267000	100.2129640000	1.51565639E-02	100.1978074361	7.10553014E-07
100	2	0.02	100.095978995359000	100.1073671891	1.18255636E-02	100.0955416255	-4.36950506E-06
100	1	0.01	100.027458255494000	100.0333913759	6.91477004E-03	100.0264766059	-9.81380128E-06
100	0	0	100.000000000000000	100.0000000000	0.00000000E+00	100.0000000000	4.26325641E-16

Table 3: Ellipse QPM Values of Modified Koshy Formula # II; Col. 3: Simpson values; Col. 5: Modified values.

a	b	Q (a, b; Sm) by Simpson's (1/3)-Rule	p	Q (a, b; K2): Modified Koshy Formula # II	Relative Error	Remarks
100	100	157.0796326795	1.69202406	157.0796326795	3.61876444E-16	
100	99	156.2952211988	1.69202406	156.2952248131	2.31248444E-08	
100	98	155.5128030354	1.69202406	155.5128175595	9.33952715E-08	
100	97	154.7324086029	1.69202406	154.7324414250	2.12121350E-07	
100	96	153.9540689771	1.69202406	153.9541275663	3.80562364E-07	
100	95	153.1778159151	1.69202406	153.1779078094	5.99919225E-07	
100	94	152.4036818752	1.69202406	152.4038146685	8.71326403E-07	
100	93	151.6317000372	1.69202406	151.6318813649	1.19584334E-06	
100	92	150.8619043241	1.69202406	150.8621418479	1.57444535E-06	
100	91	150.0943294240	1.69202406	150.0946308156	2.00801400E-06	
			Etc.			
100	82	143.2922252901	1.69202406	143.2934387731	8.46858894E-06	
100	81	142.5489523035	1.69202406	142.5503024791	9.47166271E-06	
100	80	141.8083394449	1.69202406	141.8098326708	1.05298878E-05	
100	79	141.0704326576	1.69202406	141.0720750083	1.16420613E-05	
			etc.			
100	50	121.1056027568	1.69202406	121.1115065683	4.87492846E-05	
100	49	120.4777461026	1.69202406	120.4836484487	4.89911731E-05	
100	48	119.8548636541	1.69202406	119.8607419935	4.90454808E-05	Maximum
100	47	119.2370703402	1.69202406	119.2429008384	4.88983682E-05	
			etc.			
100	31	110.1565884833	1.69202406	110.1577819570	1.08343381E-05	
100	30	109.6477517392	1.69202406	109.6484263223	6.15227451E-06	
100	29	109.1470226588	1.69202406	109.1471630651	1.28639673E-06	
100	28	108.6546463399	1.69202406	108.6542417430	-3.72369571E-06	
100	27	108.1708799880	1.69202406	108.1699247474	-8.83084793E-06	
100	26	107.6959938396	1.69202406	107.6944882753	-1.39797609E-05	
			etc			
100	17	103.8742321348	1.69202406	103.8691913849	-4.85274336E-05	
100	16	103.5068068971	1.69202406	103.5017311051	-4.90382427E-05	Minimum
100	15	103.1525073527	1.69202406	103.1475140228	-4.84072566E-05	
100	9	101.3375183618	1.69202406	101.3359864879	-1.51165530E-05	
100	8	101.0940281651	1.69202406	101.0935557317	-4.67320732E-06	
100	7	100.8699983194	1.69202406	100.8706634677	6.59411384E-06	
100	6	100.6667058367	1.69202406	100.6685240636	1.80618500E-05	

100	5	100.4856404786	1.69202406	100.4885318152	2.87736295E-05	
100	4	100.3285828267	1.69202406	100.3323258311	3.73074585E-05	
100	3	100.1977362407	1.69202406	100.2019036112	4.15914634E-05	
100	2	100.0959790450	1.69202406	100.0998501085	3.86735163E-05	
100	1	100.0274635978	1.69202406	100.0299257399	2.46146614E-05	
100	0	100.0000000000	1.69202406	100.0000000000	4.26325641E-16	

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