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Finite single integral representation for the polynomial set $M_n(x_1, x_2, x_3 \text{ and } x_4)$

Mukesh Kumar and Brijendra Kumar Singh

Abstract
In the present paper an attempt has been made to express a Finite Single Integral Representation for the quadruple hypergeometric polynomial set $Mn(x_1, x_2, x_3, x_4)$. Many interesting new results may be obtained as particular cases on separating the parameter.

Keywords: hypergeometric polynomial, generating functions, Lauricella functions, integral representation

1. Introduction

We defined the quadruple hypergeometric polynomial set $Mn(x_1, x_2, x_3, x_4)$ by means of the generating functions,

$$(\xi-\lambda t)^{\sigma} F\left[\begin{matrix}\lambda_1,(\alpha_{u_1}); & \\ & \mu_1 x_1^{-e_1} t^{e_1} \\ (\beta_{v_1}); & \end{matrix}\right] \times F\left[\begin{matrix}(A_p);(C_r);(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4}) & \\ & \mu x_1^{e} t, \mu_2 x_2^{-e_2} t^{e_2}, \mu_3 x_3^{e_3} t^{e_3}, \mu_4 x_4^{e_4} t^{4} \\ (B_q);(D_s);(\beta_{v_2});(\beta_{v_3});(\beta_{v_4}) & \end{matrix}\right]$$

$$= M_{n,e;e_1;e_2;e_3;e_4;(B_q);(D_s);(\beta_{v_1});(\beta_{v_2});(\beta_{v_3});(\beta_{v_4})}^{\lambda;\lambda_1;\xi_1;\sigma;\mu;\mu_1;\mu_2;\mu_3;\mu_4;(A_p);(C_r);(\alpha_{u_1});(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4})}(x_1, x_2, x_3, x_4)\, t^n \tag{1.1}$$

Where

$\lambda, \lambda_1, \xi, \sigma, \mu, \mu_1, \mu_2, \mu_3, \mu_4$ are real and e, $e1$, $e2$, $e3$, $e4$ are non-negative integer.

The left hand side of (1.1) contains the product of generalized hypergeometric function and Lauricella function in the notation of Burchnall and Chaundy [1]. The polynomial set contains number of parameters. For simplicity we shall denote

$$M_{n,e;e_1;e_2;e_3;e_4;(B_q);(D_s);(\beta_{v_1});(\beta_{v_2});(\beta_{v_3});(\beta_{v_4})}^{\lambda;\lambda_1;\xi_1;\sigma;\mu;\mu_1;\mu_2;\mu_3;\mu_4;(A_p);(C_r);(\alpha_{u_1});(\alpha_{u_2});(\alpha_{u_3});(\alpha_{u_4})}(x_1, x_2, x_3, x_4)$$

by $Mn\ (x_1, x_2, x_3, x_4)$.

Where n is the order of the polynomial set. After little simplification (1.1) gives

$$M_n(x_1, x_2, x_3, x_4) = \sum_{h=0}^{n} \sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]} \sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]} \sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]} \sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}$$

$$\times\frac{\left[\left(A_p\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{\left[\left(B_q\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}\left[\left(\alpha_{u_4}\right)\right]_{h_4}}{\left[\left(D_s\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}\left[\left(\beta_{v_4}\right)\right]_{h_4}}$$

$$\times\frac{(\sigma)_h\,\lambda^h\,(\lambda_1)_{h_1}\;\mu_1^{h_1}\;\mu_2^{h_2}\;\left(\mu_3x_3^{e_3}\right)^{h_3}}{\xi^{\sigma+h}\;\;h!\;\;h_1!\;\;x_2^{e_1h_1+e_2h_2}!\;\;h_3!}\times\frac{\left(\mu_4x_4^{e_4}\right)^{h_4}\qquad\left(\mu x_1^{e}\right)^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}}{h_4!\;\left(n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4\right)!}\quad\ldots\qquad(1.2)$$

2. Notations

(A) (i) $(n) = 1, 2, \ldots\ldots n-1, n.$

(ii) $(ap) = a1, a2, a3, \ldots\ldots ap.$

(iii) $(ap; i) = a1, a2, a3, \ldots\ldots ai-1, ai+1 \ldots\ldots ap.$

(B) (i) $[(ap)] = a1, a2, a3 \ldots\ldots ap.$

(ii) $$\left[\left(a_p\right)\right]_n=\prod_{i=1}^{p}\left(a_i\right)_n=\left(a_1\right)_n\left(a_2\right)_n\left(a_3\right)_n\ldots\left(a_p\right)_n$$

(iii) $$\left[\left(a_p\right)+m\left(p\right)\right]_n=\prod_{i=1}^{p}\left(a_i+m_i\right).$$

(C) (i) $$\Delta\left(a;b\right)=\frac{b}{a},\frac{b+1}{a},\frac{b+2}{a}+\ldots\ldots\frac{b+a-1}{a}.$$

(ii) $$\Delta\left(a\left(1\right);b\right)=\frac{b}{a},\frac{b+1}{a},\frac{b+2}{a},\ldots\ldots\frac{b+a-2}{a}$$

(iii) $$\Delta\left(m;\left(a_p\right)\right)=\left(\frac{a_1+r-1}{m}\right),\begin{matrix}r=1;\ldots\ldots m\\ i=1;\ldots\ldots p\end{matrix}$$

(iv) $\Delta(a; b \pm c \pm d) = \Delta(a; b+c+d), \Delta(a; b+c-d), \Delta(a; b-c+d), \Delta(a; b-c-d),$

(D) (i) $$\Delta_k\left[a;b\right]=\prod_{r=1}^{a}\left(\frac{b+r-1}{a}\right)_k=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\ldots\left(\frac{b+a-1}{a}\right)_k.$$

(ii) $$\Delta_k\left[a\left(1\right);b\right]=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\ldots\left(\frac{b+a-2}{a}\right)_k.$$

(iii) $$\Delta_k\left[m\left(a_p\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{a}\left(\frac{a_i+r-1}{m}\right)_k.$$

$$\Gamma\left[\left(a_p\right)\right]=\prod_{i=1}^{p}\left(a_i\right).$$

(E) (i)

$$\Gamma\left[a+\frac{(m)}{m}\right]=\prod_{r=1}^{m}\left(a+\frac{r}{m}\right).$$

(ii)

$$\Gamma\left[(a,b)\right]=\prod_{r=1}^{a}\Gamma\left(\frac{b+r-1}{a}\right)$$

(iii)

$$\Gamma\left[\Delta(m);\left(a_p\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{m}\Gamma\left(\frac{a_i+r-1}{m}\right)$$

(iv)

(F) (I) $\Gamma_{*}\left(a\pm b\right)=\Gamma\left(a+b\right)\Gamma\left(a-b\right).$

(ii) $\Gamma_{**}\left(a+b\right)=\Gamma\left(a+b\right)\Gamma\left(a+b\right)$

$$K=\frac{\left[\left(A_p\right)\right]_n\left[\left(C_r\right)\right]_n\left(\mu x_1^e\right)^n}{\left[\left(B_q\right)\right]_n\left[\left(D_k\right)\right]_n\xi^{\sigma}n!}$$

3. Theorem: For $e_2 > 1$, $e_3 > 1$ and $e_4 > 1$, then

$$M_n\left(x_1,x_2,x_3,x_4\right)=\frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}{\Gamma(d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}$$

$$\times K\int_0^1 x_1^{d-1}\left(1-x_1\right)^{(a-2d)}\ {}_4F_3\left[\begin{matrix}a,1+\frac{a}{2},b,c; \\ & x_1 \\ \frac{a}{2},1+a-b,1+a-c;\end{matrix}\right]F^{1+q+s\ :\ u_1;u_2,u_3,u_4}_{p+r\ :\ v_1;v_2;v_3;v_4}$$

$$\left[\begin{matrix}\left[-n;e,e_1,e_2,e_3,e_4\right],\left[\left(1-\left(B_q\right)-n\right):e,e_1,e_2-1,e_3-1,e_4-1\right], \\ \text{——},\qquad \left[\left(1-\left(A_p\right)-n\right):e,e_1,e_2-1,e_3-1,e_4-1\right],\end{matrix}\right.$$

$$\begin{matrix}\left[\left(1-\left(D_s\right)-n\right):e,e_1,e_2,e_3,e_4\right],\left[\left(\alpha_{u_1}\right):1\right],\left[\left(\alpha_{u_2}\right):1\right],\left[\left(\alpha_{u_3}\right):1\right],\left[\left(\alpha_{u_4}\right):1\right] \\ \left[\left(1-\left(C_r\right)-n\right):e,e_1,e_2,e_3,e_4\right],\left[\left(\beta_{v_1}\right):1\right],\left[\left(\beta_{v_2}\right):1\right],\left[\left(\beta_{v_3}\right):1\right],\left[\left(\beta_{v_4}\right):1\right]\end{matrix}$$

$$\begin{matrix}[\sigma:1],[\lambda:1]\left[(1+a-2d):2\right],\left[(1+a-b-c-d):1\right] \\ [d:1],[1+a-b-d:1],\left[(1+a-c-d):1\right]\end{matrix};\frac{\lambda(-1)^{(p+q+r+s+1)}}{\mu\xi x_1^e},$$

$$\frac{\mu_1(-1)^{e_1(p+q+r+s+1)}}{\left(\mu x_1^e x_2\right)^{e_1}},\frac{\mu_2(-1)^{e_2(p+q+r+s+1)+p+q}}{\left(\mu x_1^e x_2\right)^{e_2}},\frac{\mu_3 x_3^{e_3}(-1)^{e_3(p+q+r+s+1)+p+q}}{\left(\mu x_1^e\right)^{e_3}},$$

$$\left.\frac{\mu_4 x_4^{e_4}\left(-1\right)^{e_4(p+q+r+s+1)+p+q}}{\left(\mu x_1^{e}\right)^{e_4}}\right]dx \tag{3.1}$$

Proof: we have

$$I_1=\int_0^1 x_1^{d-1}\left(1-x_1\right)^{a-2d}\ {}_4F_3\left[\begin{matrix} a,1+\frac{a}{2},b,c; \\ & x_1 \\ \frac{a}{2},1+a-b,1+a-c; \end{matrix}\right]$$

$$\times\sum_{h=0}^{n}\sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]}\sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]}\sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]}\sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}\frac{\left[\left(A_p\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{\left[\left(B_q\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}}{\left[\left(D_s\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}}$$

$$\times\frac{\left[\left(\alpha_{u_4}\right)\right]_{h_4}\left(\sigma\right)_h\lambda^h\left(\lambda_1\right)_{h_1}\ \mu_1^{h_1}\ \mu_2^{h_2}\ \left(\mu_3 x_3^{e_3g_3}\right)^{h_3}\left(\mu_4 x_4^{e_4}\right)^{h_4}}{\left[\left(\beta_{v_4}\right)\right]_{h_4}\xi^{\sigma+h}\ \ h!\ \ h_1!\ \ h_2!\ \ x_2^{e_1h_1+e_2h_2}\ \ h_3!\ \ h_4!}$$

$$\times\frac{\left(\mu x_1^{e}\right)^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left(1+a-2d\right)_{2h_1}\left(1+a-b-c-d\right)_{h_1}\ dx_1}{\left(n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4\right)!\left(d\right)_{h_1}\left(1+a-b-d\right)_{h_1}\left(1+a-c-d\right)_{h_1}}$$

$$=\int_0^1 x_1^{d+h_1-1}\left(1-x_1\right)^{a-2d-2h_1}\ {}_4F_3\left[\begin{matrix} a,1-\frac{a}{2},b,c; \\ & x_1 \\ \frac{a}{2},1+a-b,1+a-c; \end{matrix}\right]$$

$$\times\sum_{h=0}^{n}\sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]}\sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]}\sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]}\sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}\frac{\left[\left(A_p\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{\left[\left(B_q\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times\frac{\left[\left(C_r\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}}{\left[\left(D_s\right)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}}$$

$$\times\frac{\left[\left(\alpha_{u_4}\right)\right]_{h_4}(\sigma)_h\lambda^h(\lambda_1)_{h_1}\ \mu_1^{h_1}\ \mu_2^{h_2}\ \left(\mu_3x_3^{e_3g_3}\right)^{h_3}\left(\mu_4x_4^{e_4}\right)^{h_4}}{\left[\left(\beta_{v_4}\right)\right]_{h_4}\xi^{\sigma+h}\ h!\ h_1!\ h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!\ h_4!}$$

$$\times\frac{\left(\mu x_1^e\right)^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\ (1+a-2d)_{2h_1}(1+a-b-c-d)_{h_1}\ dx_1}{(n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4)!\ (d)_{h_1}(1+a-b-d)_{h_1}(1+a-c-d)_{h_1}}$$

$$\times\sum_{h=0}^{n}\sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]}\sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]}\sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]}\sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}\frac{\left[\left(A_p\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}{\left[\left(B_q\right)\right]_{n-h-e_1h_1-(e_2-1)h_2-(e_3-1)h_3-(e_4-1)h_4}}$$

$$\times\frac{\left[(C_r)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}\left[\left(\alpha_{u_3}\right)\right]_{h_3}}{\left[(D_s)\right]_{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}\left[\left(\beta_{v_3}\right)\right]_{h_3}}$$

$$\times\frac{\left[\left(\alpha_{u_4}\right)\right]_{h_4}(\sigma)_h\ \lambda^h\ (\lambda_1)_{h_1}\ \mu_1^{h_1}\ \mu_2^{h_2}\ \left(\mu_3x_3^{e_3g_3}\right)^{h_3}}{\left[\left(\beta_{v_4}\right)\right]_{h_4}\xi^{\sigma+h}\ h!\ h_1!\ h_2!\ x_2^{e_1h_1+e_2h_2}\ h_3!}$$

$$\times\frac{\left(\mu_4x_4^{e_4}\right)^{h_4}\ \left(\mu x_1^e\right)^{n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4}}{h_4!\ (n-h-e_1h_1-e_2h_2-e_3h_3-e_4h_4)!}\times\frac{(1+a-2d)_{2h_1}(1+a-b-c-d)_{h_1}\Gamma(d+h_1)\Gamma(1+a+2d-2h_1)}{(d)_{h_1}\ (1+a-b-d)_{h_1}\ (1+a-c-d)h_1\Gamma(1+a)}$$

$$\times\frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-b-c-d-h_1)}{\Gamma(1+a-c-d)\Gamma(1+a-b-d-h_1)\Gamma(1+a-c-d-h_1)}$$

$$\times\frac{\Gamma(d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

$$\times\sum_{h=0}^{n}\sum_{h_1=0}^{\left[\frac{n-h}{e_1}\right]}\sum_{h_2=0}^{\left[\frac{n-h-e_1h_1}{e_2}\right]}\sum_{h_3=0}^{\left[\frac{n-h-e_1h_1-e_2h_2}{e_3}\right]}\sum_{h_4=0}^{\left[\frac{n-h-e_1h_1-e_2h_2-e_3h_3}{e_4}\right]}\frac{\left[1-\left(B_q\right)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}}{\left[1-\left(A_p\right)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}}$$

$$\times\frac{\left[1-(D_s)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[\left(\alpha_{u_1}\right)\right]_{h_1}\left[\left(\alpha_{u_2}\right)\right]_{h_2}}{\left[1-(C_r)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[\left(\beta_{v_1}\right)\right]_{h_1}\left[\left(\beta_{v_2}\right)\right]_{h_2}}\times\frac{\left[\left(\alpha_{u_3}\right)\right]_{h_3}\left[\left(\alpha_{u_4}\right)\right]_{h_4}(\sigma)_h(\lambda_1)_{h_1}(n)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}}{\left[\left(\beta_{v_3}\right)\right]_{h_3}\left[\left(\beta_{v_4}\right)\right]_{h_4}\ h!\ \xi^h\ h_1!\ h_2!}$$

$$\times\frac{\lambda^h(-1)^{(p+q+r+h+1)h}}{\left(\mu x_1^e\right)^h},\frac{\mu_1(-1)^{e_1(p+q+r+s+1)h_1}}{\left(\mu x_1^ex_2\right)^{e_1h_1}},\frac{\mu_2(-1)^{e_2(p+q+r+s+1)+p+q}}{\left(\mu x_1^ex_2\right)^{e_2h_2}}$$

$$\frac{\left(\mu_3 x_3^{e_3}\right)^{h_3}(-1)^{\{e_3(p+q+r+s+1)+p+q\}h_3}}{\left(\mu x_1^e x_2\right)^{h_3 e_3}}, \frac{\left(\mu_4 x_4^{e_4}\right)^{h_4}(-1)^{\{e_4(p+q+r+s+1)+p+q\}h_4}}{\left(\mu x_1^e x_2\right)^{e_4 h_4}} \tag{3.2}$$

The single terminating factor makes all summation in (3.2) runs up to □.

$$= K\frac{\Gamma(d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1-a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

$$\times \sum_{h,h_1,h_2,h_3,h_4=0}^{\infty} \frac{\left[1-(B_q)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}\left[1-(D_s)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[(\alpha_{u_1})\right]_{h_1}}{\left[1-(A_p)-n\right]_{h+e_1h_1+(e_2-1)h_2+(e_3-1)h_3+(e_4-1)h_4}\left[1-(C_r)-n\right]_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\left[(\beta_{v_1})\right]_{h_1}}$$

$$\times \frac{\left[(\alpha_{u_2})\right]_{h_2}\left[(\alpha_{u_3})\right]_{h_3}\left[(\alpha_{u_4})\right]_{h_4}(\sigma)_h(\lambda_1)_{h_1}}{\left[(\beta_{v_2})\right]_{h_2}\left[(\beta_{v_3})\right]_{h_3}\left[(\beta_{v_4})\right]_{h_4} h!\ h_1!\ h_2!} \times \frac{(-n)_{h+e_1h_1+e_2h_2+e_3h_3+e_4h_4}\ \lambda^h(-1)^{(p+q+r+s+1)h}\ \mu_1(-1)^{e_1(p+q+r+s+1)h_1}}{h_3!h_4!\ \left(\mu x_1^e\right)^h\ \left(\mu_1 x_1^e x_2\right)^{e_2h_2}}$$

$$\times \frac{\mu_2(-1)^{\{e_2(p+q+r+s+1)+p+q\}h_2}}{\left(\mu x_1^e x_2\right)^{e_2h_2}}, \frac{\left(\mu_3 x_3^{e_3}\right)^{h_3}(-1)^{\{e_3(p+q+r+s+1)+p+q\}h_3}}{\left(\mu x_1^e\right)^{e_3h_3}} \times \frac{\left(\mu_4 x_4^{e_4}\right)^{h_4}(-1)^{\{e_4(p+q+r+s+1)+p+q\}h_4}}{\left(\mu x_1^e\right)^{e_4h_4}}$$

$$= \frac{\Gamma(a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-d)}{\Gamma(1+a)\Gamma(1+a-b-d)\Gamma(1+a-b-d)\Gamma(1+a-c-d)} M_n(x_1,x_2,x_3,x_4)$$

The above result is obtained on using the result (q)

$$\int_0^1 x^{d-1}(1-x)^{a-2d}\ {}_4F_3\left[\begin{matrix} a,1+\frac{a}{2},b,c; \\ & x \\ \frac{a}{2},1+a-b,1+a-c; \end{matrix}\right]dx$$

$$= \frac{\Gamma(d)\Gamma(1+a-2d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-b-c-d)}{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}$$

Provided *Re* (*d*), *Re* (*a*–2*d*) > –1 and *Re* (*b* + *c* + *d* –*a*) > –1.

4. Particular Cases of (3.1)

Separating the term corresponding to $h_2 = 0 = h_3 = h_4 \Rightarrow x_3 = 0 = x_4$ in (3.1) we obtain a number of results on specializing the remaining parameters:

(i) If we put $p = 0 = q = r = s = u_1 = \lambda_1 = h$; $\mu_1 = -1 = \sigma = \lambda = 1 = v_1 = e = \mu = h_1 = \xi$; $B_1 = 1 + \alpha$; $x_1 = \frac{1}{y}$ we get

$$L_n^{\alpha}(y) = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)(1+\alpha)}{\Gamma(d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)n!y^n}$$

$$\times\int_0^1 x_1^{d-1}(1-x_1)^{a-2d}\ {}_4F_3\left[\begin{matrix} a,1+\frac{a}{2},b,c; \\ \frac{a}{2},1+a-b,1+a-c; \end{matrix}\ x_1\right]$$

$$\times F\left[\begin{matrix} -n,(1+a-2d;2),(1+a-b-c-d:1); \\ 1+\alpha,(d:1),(1+a-b-d:1),(1+a-c-d:1); \end{matrix}\ y\right]dx_1$$

(ii) For $p = 0 = q = r = u_1 = h = \lambda_1$; $\lambda = 1 = \sigma = s = v_1 = e = \xi = x_2$;

$\mu = \frac{1}{2} = \mu_1$; $B_1 = \frac{1}{2} = D_1$ and $x_1 = \frac{x-1}{x+1}$

$$T_n(x) = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)(1+a-c-d)\left(\frac{x-1}{2}\right)^n}{\Gamma(d)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)n!}$$

$$\times\frac{1}{\left(\frac{1}{2}\right)_n(x+1)^n}\int_0^1 x_1^{d-1}(1-x_1)^{a-2d}\ {}_4F_3\left[\begin{matrix} a,1+\frac{a}{2},b,c; \\ \frac{a}{2},1+a-b;1+a-c; \end{matrix}\ x_1\right]$$

$$\times F\left[\begin{matrix} -n,\frac{1}{2}-n,(1+a-2d;2),(1+a-b-c-d;1) \\ \frac{1}{2},(d;1),(1+a-b-d;1)[(1+a-c-d);1]; \end{matrix}\ \frac{x-1}{x+1}\right]dx_1$$

(iii) On making the substitution $p = 0 = q = u_1 = h = \lambda_1 = r$; $\lambda = 1 = \sigma = \xi = s = v_1 = e = x_2 = \mu$;

$\mu = \frac{1}{2} = \mu_1$; $B_1 = \frac{3}{2} = D_1$ and $\frac{x-1}{x+1}$

Instead of x, we arrive at

$$\cup_n(x) = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)}{\Gamma d\ \Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}$$

$$\times\frac{\left(\frac{x-1}{2}\right)^n}{(x+1)^n\left(\frac{3}{2}\right)_n n!}\int_0^0 x_1^{d-1}(1-x)^{a-2d}\ {}_4F_3\left[\begin{matrix} a,1+\frac{a}{2},b,c; \\ \frac{a}{2},1+a-b,1+a-c; \end{matrix}\ x\right]$$

$$\times F\left[\begin{matrix} -n, \frac{1}{2}-n, (1+a-2d;2), (1+a-b-c-d;1); \\ \qquad\qquad\qquad\qquad \frac{x+1}{x-1} \\ \frac{3}{2}, (d;1), (1+a-b-d;1), (1+a-c-d;1); \end{matrix}\right] dx_1$$

(iv) On taking $p = 0 = q = u_1 = r = h = \lambda_1$; $v_1 = 1 = s = e = e_1 = \lambda = \sigma = \xi$;

$\mu = \frac{1}{2} = \mu_1$; $D_1 = 1 + \beta$; $\beta_1 = 1 + \alpha$ and $\frac{x+1}{x-1}$, we have

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)(1+\alpha)^n}{\Gamma(d)\ \Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)n!}$$

$$\times\left(\frac{x+1}{2}\right)^n \int_0^1 x_1^{d-1}(1-x_1)^{a-2d}\ {}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, c; \\ \qquad\qquad x_1 \\ \frac{a}{2}, 1+a-b, 1+a-c; \end{matrix}\right]$$

$$\times F\left[\begin{matrix} -n, -\beta-n, (1+a-2d;2), (1+a-b-c-d;1), \\ \qquad\qquad\qquad\qquad \frac{x-1}{x+1} \\ 1+\alpha, (d:1), (1+a-b-d;1)(1+a-c-d;1); \end{matrix}\right] dx_1$$

(v) On Putting $p = 0 = q = r = u_1 = \lambda_1 = h$; $v_1 = 1 = s = e = e_1 = \lambda = \sigma$; $D_1 = 1+\alpha$, $\beta_1 = 1+\beta$,

$\mu = \frac{1}{2} = \mu_1$ and $x_1 = \frac{x-1}{x+1}$,

We achieve

$$P_n^{(\alpha,\beta)}(x) = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)(1+\beta)_n}{\Gamma(d)\ \Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}$$

$$\times\left(\frac{x-1}{2}\right)^n \int_0^1 x_1^{d-1}(1-x)^{a-2d}\ {}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, c; \\ \qquad\qquad x_1 \\ \frac{a}{2}, 1+a-c, 1+a-c; \end{matrix}\right]$$

$$\times F\left[\begin{matrix} -n, -\alpha-n, (1-a-2d;2), (1+a-b-c-d;1), \\ \qquad\qquad\qquad\qquad \frac{x+1}{x-1} \\ 1+\beta, (d:1), (1+a-b-d;1)(1+a-c-d;1); \end{matrix}\right] dx_1$$

(vi) On making the substitution $p = 0 = q = r = u_1 = \lambda_1 = h$; $\xi = v_1 = 1 = s = e = e_1 = \sigma = \lambda$; $D_1 = 1 + \alpha = \beta_1$; $\mu = \frac{1}{2} = \mu_1$; and writing $\frac{x+1}{x-1}$ for x_1, we get

For x_1, we get

$$P_n^{(\alpha,\alpha)} = \frac{\Gamma(1+a)\Gamma(1+a-b-c)\Gamma(1+a-b-d)\Gamma(1+a-c-d)(1+\alpha)_n}{\Gamma(d)\ \Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+a-2d)\Gamma(1+a-b-c-d)}$$

$$\times\left(\frac{x+1}{2}\right)^n \frac{1}{n!}\int_0^1 x_1^{d-1}(1-x)^{a-2d}\ {}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, c; \\ & x_1 \\ \frac{a}{2}, 1+a-b, 1+a-c; \end{matrix}\right]$$

$$\times F\left[\begin{matrix} -n, -\alpha-n, (1+a-2d;2), (1+a-b-c-d;1), \\ & \frac{x-1}{x+1} \\ 1+\alpha, (d;1), (1+a-b-d;1)(1+a-c-d;1); \end{matrix}\right] dx_1$$

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