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## Study of $W_4$ - Curvature tensor in the space-time of General Relativity

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### Abstract

In this paper, we study the physical properties of  $W_4$ -Curvature tensor and found that it does not satisfy the Bianchi differential identity even with Ricci tensor being of Codazzi type. It is further shown that Einstein field equations with Cosmological constant can be expressed with the help of  $W_{4\beta\gamma}$  and Rainich conditions for the existence of the non-null electrovariance can also be written with the help of  $W_{4ij}$ . Finally, it is shown that  $W_4$ -flat space time signifies empty field.

**Keywords:**  $W_4$  curvature tensor, Einstein field equations, Rainich conditions,  $W_4$ -flat space time. AMS 2021 Subject Classification: 53C50, 53C80

### Introduction

On the line Wyel's of projective curvature tensor Pokhariyal(1973) <sup>[1]</sup> defined  $W_3$  and  $W_4$  curvature tensors, to study geometric and physical properties of these tensors in the 4-dimensional space time of general relativity.

$$W_3(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1} [g(Y, Z)Ric(X, T) - g(Y, T)Ric(X, Z)] \tag{1.1}$$

$$W_4(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{n-1} [g(X, Z)Ric(Y, T) - g(X, Y)Ric(T, Z)] \tag{1.2}$$

Moindi (2007) <sup>[2]</sup> have studied the geometric properties of  $W_3$  tensor in Lorentzian Para-Sasakian and other complex manifolds. The  $W_4$ -curvature tensor, has been studied in Sasakian and LP-Sasakian manifolds(Katende, 2011) <sup>[3]</sup>. Additional physical properties of  $W_4$ -curvature tensor in the space time of general relativity are studied in this paper.

### Properties of $W_4$ -curvature tensor

The  $W_4$ -curvature tensor has been defined by (1.2) where  $R(X, Y, Z, T) = g(R(X, Y, Z), T)$  and  $R(X, Y, Z) = D_X D_Y Z - D_Y D_X Z - D_{[X, Y]} Z$  is the Riemann curvature tensor,  $R(X, Y) = g(R(X), Y)$  is the (0,2)-type Ricci tensor,  $R$  is the scalar curvature and  $D$  denotes the Riemannian connection. It is seen Pokhariyal, 1973, that  $W_4(X, Y, Z, T) \neq -W_4(Y, X, Z, T)$ ,  $W_4(X, Y, Z, T) \neq -W_4(X, Y, T, Z)$ ,  $W_4(X, Y, Z, T) \neq +W_4(Z, T, X, Y)$ .

Therefore, none of symmetric or skew-symmetric properties are satisfied by  $W_4$ -curvature tensor.

From (1.2), it is seen that

$$W_4(X, Y, Z, T) + W_4(X, Z, T, Y) + W_4(X, T, Y, Z) = 0. \tag{2.1a}$$

$$W_4(X, Y, Z, T) + W_4(Y, Z, X, T) + W_4(Z, X, Y, T) = 0 \tag{2.1b}$$

That is, both the cyclic properties are satisfied by  $W_4$ -curvature tensor.

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**Bianchi Differential Identity**

It is shown that the Bianchi differential identity is given by

$$(\nabla_U R)(X, Y, Z, T) + (\nabla_Z R)(X, Y, T, U) + (\nabla_T R)(X, Y, U, Z) = 0 \tag{3.1}$$

For the four-dimensional space time of general relativity, from (1.2), we have

$$W_4(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{3} [g(X, Z) Ric(Y, T) - g(X, Y) Ric(Z, T)] \tag{3.2}$$

In order to check for Bianchi differential identity for  $W_4$ -curvature tensor, we compute

$$\begin{aligned} &(\nabla_U W_4)(X, Y, Z, T) + (\nabla_Z W_4)(X, Y, T, U) + (\nabla_T W_4)(X, Y, U, Z) = \\ &(\nabla_U R)(X, Y, Z, T) + \frac{1}{3} [g(X, Z)(\nabla_U Ric)(Y, T) - g(X, Y)(\nabla_U Ric)(Z, T)] + \\ &(\nabla_Z R)(X, Y, T, U) + \frac{1}{3} [g(X, T)(\nabla_Z Ric)(Y, U) - g(X, Y)(\nabla_Z Ric)(T, U)] + \\ &(\nabla_T R)(X, Y, U, Z) + \frac{1}{3} [g(X, U)(\nabla_T Ric)(Y, Z) - g(X, Y)(\nabla_T Ric)(U, Z)] \end{aligned} \tag{3.3}$$

By virtue of (3.1) and (3.2) we have.

$$\begin{aligned} &(\nabla_U W_4)(X, Y, Z, T) + (\nabla_Z W_4)(X, Y, T, U) + (\nabla_T W_4)(X, Y, U, Z) = \\ &\frac{1}{3} [g(X, Z)(\nabla_U Ric)(Y, T) + g(X, T)(\nabla_Z Ric)(Y, U) + g(X, U)(\nabla_T Ric)(Y, Z) - \\ &g(X, Y) \{ (\nabla_U Ric)(Z, T) + (\nabla_Z Ric)(U, T) + (\nabla_T Ric)(U, Z) \}] \end{aligned} \tag{3.4}$$

The condition for Ricci tensor to be of Codazzi type, Derdzinski and Shen, 1983<sup>[4]</sup> is given by

$$(\nabla_X Ric)(Y, Z) = (\nabla_Y Ric)(X, Z) = (\nabla_Z Ric)(X, Y) \tag{3.5}$$

It is noticed that using (3.5), the equation (3.4) does not vanish. Hence,  $W_4(X, Y, Z, T)$  does not satisfy the Bianchi differential identity, even with Ricci tensor being Codazzi type.

**Einstein Field Equations**

The Einstein field equations with cosmological term is given by

$$R_{\beta\gamma} - \frac{1}{2} R g_{\beta\gamma} + \Lambda g_{\beta\gamma} = k T_{\beta\gamma} \tag{3.6}$$

where  $\Lambda$  is the cosmological constant,  $K$  is the non-zero gravitational constant and  $T_{\beta\gamma}$  is the energy momentum tensor. Writing  $W_4$  tensor in the local coordinates, we have from (3.2)

$$W_{4\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{3} [g_{\alpha\gamma} R_{\beta\delta} - g_{\alpha\beta} R_{\gamma\delta}] \tag{3.7}$$

$$W_{4\beta\gamma\delta}^\alpha = R_{4\beta\gamma\delta}^\alpha + \frac{1}{3} [g_\gamma^\alpha R_{\beta\delta} - g_\beta^\alpha R_{\gamma\delta}] \tag{3.8}$$

Contracting for  $\alpha$  and  $\delta$ , we get

$$W_{4\beta\gamma} = R_{\beta\gamma} + \frac{1}{3} [R_{\beta\gamma} - R_{\beta\gamma}] = R_{\beta\gamma} \tag{3.9}$$

Further contraction gives

$$W_4 = R \tag{3.10}$$

Hence, the Einstein tensor and the Einstein field equations with cosmological constant can alternatively be expressed using (3.9) and (3.10) with the help of  $W_{4\beta\gamma}$  and  $W_4$  as follows:

$$E_{\beta\gamma} = W_{4\beta\gamma} - \frac{1}{2} W_4 g_{\beta\gamma} \tag{3.11}$$

and

$$W_{4\beta\gamma} - \frac{1}{2} W_4 g_{\beta\gamma} + \Lambda g_{\beta\gamma} = k T_{\beta\gamma} \tag{3.12}$$

**$W_4$  – Flat Space Time**

We now study  $W_4$  flat Space Time and some other physical properties.

**Definition (3.3);** A space time is said to be  $W_4$  – flat if tensor  $W_4$  vanishes in it. Thus, for a  $W_4$  flat Space Time we have from (3.8)

$$R_{\beta\gamma\delta}^{\alpha} = \frac{1}{3} [g_{\beta}^{\alpha} R_{\gamma\delta} - g_{\gamma}^{\alpha} R_{\beta\delta}] \quad (3.13)$$

On contracting  $\alpha$  and  $\delta$ , we get

$$R_{\beta\gamma} = \frac{1}{3} [R_{\beta\gamma} - R_{\beta\gamma}] = 0 \quad (3.14)$$

Thus, a  $W_4$  flat space time signifies empty field, which may contain isolated masses as singularities. It can also be said that empty space can still be curved and gravity can still be felt.

The vector

$$\theta_i = \frac{g_{ij} \epsilon^{jklm} R_k^p R_{plm}}{\sqrt{-g} R_{ab} R^{ab}}, \quad (3.15)$$

known as complexion vector of a non-null electromagnetic field with no matter (Misner and Wheeler, 1954 <sup>[5]</sup> and its vanishing implies that the field is purely electrical. It was shown <sup>[1]</sup> that the vanishing of the divergence of  $W_{4ij}^h$  in an electromagnetic field implies purely electric field.

Rainich (1952) <sup>[6]</sup> has shown that the necessary and sufficient condition for the existence of the non-null electrovariance are

$$R = 0 \quad (3.16)$$

$$R_j^i R_k^j = \frac{1}{4} \delta_k^i R_{ab} R^{ab} \quad (3.17)$$

$$\theta_{i;j} = \theta_{j;i} \quad (3.18)$$

From (3.9) and (3.10), it is noticed that  $R_{ij}$  can very well be replaced by  $W_{4ij}$  and  $R$  be replaced by  $W_4$  in expressing the Rainich conditions.

### Discussion

One of the advantages of  $W_4$ -Curvature tensor is that its contracted part is equal to Ricci tensor and the scalar  $W_4$  is equal to  $R$ , the scalar curvature. Therefore, these results can be alternatively used for getting various geometrical and physical results. Further, the  $W_4$  – flat space time signifies empty field.  $W_4$ -Curvature tensor has additional terms as compared to Riemann curvature tensor and is likely to provide additional interpretations of geometrical and physical phenomena.

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### References

1. Pokhariyal GP. Curvature Tensors and their Relativistic Significance III: Yokohama math Journal. 1973;2(2):115-119.
2. Moindi S. PhD thesis, University of Nairobi, Kenya, 2007.
3. Katende J. PhD thesis, University of Nairobi, Kenya, 2011.
4. Derdzinski A, Shen CL. Codazzi tensor fields, Curvature and Pontryagin form: Pre London Math. Society. 1983;47(3):15-26.
5. Misner CW, Wheeler JA. Classical physics as Geometry: Gravitation, electromagnetism, unquantized charge and mass as properties of curved empty space; Ann. Physics. 1957;2:525-603.
6. Rainich GY. Electrodynamics in the general relativity theory; Trans. Amer. Mat. 1952;27:106-136.