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Study of W_4- curvature tensor in the space-time of general relativity

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Abstract
In this paper, we study the physical properties of W_4- Curvature tensor and found that it does not satisfy the Bianchi differential identity even with Ricci tensor being of Codazzi type. It is further shown that Einstein field equations with Cosmological constant can be expressed with the help of $W_{4\beta\gamma}$ and Rainich conditions for the existence of the non-null electrovariance can also be written with the help of W_{4ij}. Finally, it is shown that W_4-flat space time signifies empty field.

Keywords: W_4 curvature tensor, Einstein field equations, Rainich conditions, W_4-flat space time. AMS 2021 subject classification: 53C50, 53C80

Introduction
On the line Wyel's of projective curvature tensor Pokhariyal (1973) [1] defined W_3 $and W_4$curvature tensors, to study geometric and physical properties of these tensors in the 4-dimensional space time of general relativity.

$$W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)] \quad (1.1)$$

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(T,Z)] \quad (1.2)$$

Moindi (2007) [2] have studied the geometric properties of W_3 tensor in Lorentzian Para-Sasakian and other complex manifolds. The W_4-curvature tensor, has been studied in Sasakian and LP-Sasakian manifolds (Katende, 2011) [3]. Additional physical properties of W_4-curvature tensor in the space time of general relativity are studied in this paper.

Properties of W_4-curvature tensor
The W_4-curvature tensor has been defined by (1.2) where $R(X,Y,Z,T)=g(R(X,Y,Z),T)$ and $R(X,Y,Z)=D_XD_YZ- D_YD_XZ-D_{[X,Y]}Z$ is the Riemann curvature tensor, $R(X,Y)=g(R(X),Y)$ is the (0,2)-type Ricci tensor, R is the scalar curvature and D denotes the Riemannian connection. It is seen Pokhariyal, 1973 [1], that $W_4(X,Y,Z,T) \neq -W_4(Y,X,Z,T)$, $W_4(X,Y,Z,T) \neq -W_4(X,Y,T,Z)$, $W_4(X,Y,Z,T) \neq +W_4(Z,T,X,Y)$.

Therefore, none of symmetric or skew-symmetric properties are satisfied by W_4-curvature tensor.
From (1.2), it is seen that

$$W_4(X,Y,Z,T) + W_4(X,Z,T,Y) + W_4(X,T,Y,Z) = 0. \quad (2.1)a$$

$$W_4(X,Y,Z,T) + W_4(Y,Z,X,T) + W_4(Z,X,Y,T)0 \quad (2.1)b$$

That is, both the cyclic properties are satisfied by W_4-curvature tensor.

Bianchi Differential Identity
It is shown that the Bianchi differential identity is given by

$$(\nabla_U R)(X,Y,Z,T)+(\nabla_Z R)(X,Y,T,U)+(\nabla_T R)(X,Y,U,Z)=0 \tag{3.1}$$

For the four-dimensional space time of general relativity, from (1.2), we have

$$W_4(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{3}[g(X,Z)Ric(Y,T) - g(X,Y)Ric(Z,T)] \tag{3.2}$$

In order to check for Bianchi differential identity for W_4-curvature tensor, we compute

$$\begin{aligned}&(\nabla_U W_4)(X,Y,Z,T)+(\nabla_Z W_4)(X,Y,T,U)+(\nabla_T W_4)(X,Y,U,Z)=\\ &(\nabla_U R)(X,Y,Z,T)+\frac{1}{3}[g(X,Z)(\nabla_U Ric)(Y,T) - g(X,Y)(\nabla_U Ric)(Z,T)]+\\ &(\nabla_Z R)(X,Y,T,U)+\frac{1}{3}[g(X,T)(\nabla z\ Ric)(Y,U) - g(X,Y)(\nabla z\ Ric)(T,U)]+\\ &(\nabla_T R)(X,Y,U,Z)+\frac{1}{3}[g(X,U)(\nabla_T Ric)(Y,Z) - g(X,Y)(\nabla_T Ric)(U,Z)]\end{aligned} \tag{3.3}$$

By virtue of (3.1) and (3.2) we have.

$$\begin{aligned}&(\nabla_U W_4)(X,Y,Z,T)+(\nabla_Z W_4)(X,Y,T,U)+(\nabla_T W_4)(X,Y,U,Z)=\\ &\frac{1}{3}[g(X,Z)(\nabla_U Ric)(Y,T) + g(X,T)(\nabla z\ Ric)(Y,U) + g(X,U)(\nabla_T Ric)(Y,Z) -\\ &g(X,Y)\{(\nabla_U Ric)(Z,T) + (\nabla z\ Ric)(U,T) + (\nabla_T Ric)(U,Z)\}\end{aligned} \tag{3.4}$$

The condition for Ricci tensor to be of Codazzi type, Derdzinski and Shen, 1983 [4] is given by

$$(\nabla x\ Ric)(Y,Z) = (\nabla y\ Ric)(X,Z) = (\nabla z\ Ric)(X.Y) \tag{3.5}$$

It is noticed that using (3.5), the equation (3.4) does not vanish. Hence, $W_4(X,Y,Z,T)$ does not satisfy the Bianchi differential identity, even with Ricci tensor being Codazzi type.

Einstein Field Equations
The Einstein field equations with cosmological term is given by

$$R_{\beta\gamma} - \frac{1}{2}Rg_{\beta\gamma} + \wedge g_{\beta\gamma} = kT_{\beta}, \tag{3.6}$$

where $\wedge$ is the cosmological constant, K is the non-zero gravitational constant and $T_{\beta\gamma}$ is the energy momentum tensor.
Writing W_4 tensor in the local coordinates, we have from (3.2)

$$W_{4\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{3}[g_{\alpha\gamma}R_{\beta\delta} - g_{\alpha\beta}R_{\gamma\delta}]. \tag{3.7}$$

$$W^{\alpha}_{4\beta\gamma\delta} = R^{\alpha}_{4\beta\gamma\delta} + \frac{1}{3}[g^{\alpha}_{\gamma}R_{\beta\delta} - g^{\alpha}_{\beta}R_{\gamma\delta}] \tag{3.8}$$

Contracting for $\alpha\ and\ \delta$, we get

$$W_{4\beta\gamma} = R_{\beta\gamma} + \frac{1}{3}[R_{\beta\gamma} - R_{\beta\gamma}] = R_{\beta\gamma} \tag{3.9}$$

Further contraction gives

$$W_4 = R \tag{3.10}$$

Hence, the Einstein tensor and the Einstein field equations with cosmological constant can alternatively be expressed using (3.9) and (3.10) with the help of $W_{4\beta\gamma}$ and W_4 as follows:

$$E_{\beta\gamma} = W_{4\beta\gamma} - \frac{1}{2}W_4 g_{\beta\gamma} \tag{3.11}$$

and

$$W_{4\beta\gamma} - \frac{1}{2}W_4 g_{\beta\gamma} + \wedge g_{\beta\gamma} = kT_{\beta\gamma} \tag{3.12}$$

W_4 –Flat Space Time
We now study W_4 flat Space Time and some other physical properties.

Definition: A space time is said to be $W_4 - flat$ if tensor W_4 vanishes in it.
Thus, for a W_4flat Space Time we have from (3.8)

$$R^{\alpha}_{\beta\gamma\delta} = \frac{1}{3}[g^{\alpha}_{\beta} R_{\gamma\delta} - g^{\alpha}_{\gamma} R_{\beta\delta}] \quad (3.13)$$

On contractingα *and* δ, we get

$$R_{\beta\gamma} = \frac{1}{3}\left[R_{\beta\gamma} - R_{\beta\gamma}\right] = 0 \quad (3.14)$$

Thus, a W_4flat space time signifies empty field, which may contain isolated masses as singularities. It can also be said that empty space can still be curved and gravity can still be felt.
The vector

$$\theta_i = \frac{g_{ij} \in^{jklm} R^{p}_{k} R_{pl;m}}{\sqrt{-g} R_{ab} R^{ab}}, \quad (3.15)$$

known as complexion vector of a non-null electromagnetic field with no matter (Misner and Wheeler, 1954 [5] and its vanishing implies that the field is purely electrical. It was shown [1] that the vanishing of the divergence of W^{h}_{4ijk} in an electromagnetic field implies purely electric field.
Rainich (1952) [6] has shown that the necessary and sufficient condition for the existence of the non-null electro variance are

$$R = 0 \quad (3.16)$$

$$R^{i}_{j} R^{j}_{k} = \frac{1}{4} \delta^{i}_{k} R_{ab} R^{ab} \quad (3.17)$$

$$\theta_{i;j} = \theta_{j;i} \quad (3.18)$$

From (3.9) and (3.10), it is noticed that R_{ij} can very well be replaced by W_{4ij} and R be replaced by W_4 in expressing the Rainich conditions.

Discussion

One of the advantages of W_4-Curvature tensor is that its contracted part is equal to Ricci tensor and the scalar $W_4$is equal to R, the scalar curvature. Therefore, these results can be alternatively used for getting various geometrical and physical results. Further, the W_4 −flat space time signifies empty field. W_4-Curvature tensor has additional terms as compared to Riemann curvature tensor and is likely to provide additional interpretations of geometrical and physical phenomena.

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