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Proposed tests for location and scale in a nondecreasing ordered alternative

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Abstract
Six nonparametric tests are proposed for the nondecreasing ordered alternative when testing for a difference in either location or scale. The six tests are various combinations of a well-known ordered alternative test for location and a technique that Moses used to convert a scale testing problem into a location testing problem by transforming the data. A simulation study is conducted to determine how well the proposed tests maintain their significance levels. Powers are estimated for the proposed tests under a variety of conditions for three, four and five populations. Several types of variable parameters are considered: when the location parameters are different and the scale parameters are equal; when the location parameters are equal and the scale parameters are different; when the location and scale parameters are both different. Equal and unequal sample sizes of 18 and 30 are considered. Subgroup sizes of 3 and 6 are both used when applying the Moses technique. Recommendations are given.

Keywords: Nonparametric, completely randomized design, scale and location parameters

Introduction

Ordered alternatives tests are sometimes used in life-testing experiments and drug-screening studies. An ordered alternative test is used to gain power if the researcher thinks parameters will be ordered in a certain way if they are different. Let $X_{i1}, X_{i2}, \ldots, X_{in_i}, i = 1,2,\ldots,k\ (k \geq 3)$ be random independent samples each of size n_i from k populations, where $-\infty < \mu_i < +\infty$ and $\sigma_i > 0$ are location and scale parameters, respectively. We are interested in testing the hypotheses in (1)

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k \ and\ \sigma_1 = \sigma_2 = \cdots = \sigma_k. \quad (1)$$

Versus

$H_a: \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \ and\ \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_k$, at least one of the inequalities is strict

Background

Lepage [1] proposed a two-sample nonparametric test for location and scale. The Lepage test is a combination of the Wilcoxon test (W) 1945 [2] for location and the Ansari-Bradley test (AB) 1960 [3] for scale alternatives.
Let m and n be the size of two random samples

$X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$, N=m+n.

$$H_0: \mu_1 = \mu_2 \ and\ \sigma_1 = \sigma_2$$

$$H_a: \mu_1 \neq \mu_2 \ and\ \sigma_1 \neq \sigma_2$$

The Lepage test is defined as

$$LP = \frac{[W - E(W)]^2}{Var(W)} + \frac{[AB - E(AB)]^2}{Var(AB)}$$

LP has a chi-square distribution with two degrees of freedom when the null hypothesis is true. Several other researchers designed nonparametric tests similar to Lepage 1971 [1]. Marozzi 2013 [4] has compared these tests and found that there is no clear test to recommend for all situations.
Alsubie and Magel 2020a [5] proposed two tests L_1 and L_2 for the simple tree alternative for testing both location and scale parameters. These tests are a combination of the Fligner-Wolfe test 1982 [6] for detecting location changes and the modified Ansari-Bradley test 1960 [3] for detecting scale changes. The hypotheses are

$$H_0: \mu_1 = \mu_2 = \cdots = \mu_k, \text{ and } \sigma_1 = \sigma_2 = \cdots = \sigma_k \quad (2)$$

$$H_\alpha: \mu_1 \le [\mu_2, \dots, \mu_k] \text{ and } \sigma_1 \le [\sigma_2, \dots, \sigma_k]$$

(At least one inequality is strict)

The L_1 test is given by

$$L_1 = \frac{FW^* + AB^*}{\sqrt{2}}$$

where the FW^* represents the standardized test statistic for Fligner-Wolfe test statistic and AB^* represents the standardized test statistic for Ansari-Bradley test statistic.
The second test, L_2, is given by:

$$L_2 = \frac{FW + AB - E(FW + AB)}{\sqrt{var(FW) + var(AB)}}$$

When the null hypothesis is true, the asymptotic distributions of L_1 and L_2 are both standard normal distributions.
Alsubie *et al*. 2020a [5] found that when underlying distributions were approximately symmetric, L_2 has the highest powers when the change is only in location parameters. In all other cases L_1 had the highest powers.
In addition to the two tests previously mentioned, Alsubie and Magel 2020b [7] proposed three other tests M_1, M_2 and M_3 for the simple tree alternative for location and scale testing in (2). These tests are a combination of applying the Fligner-Wolfe test for detecting location changes to the original data and using the Moses [8] technique to transform data and apply the Fligner-Wolfe test to the transformed data.
The first test, M_1, is

$$M_1 = \frac{FW^* + M^*}{\sqrt{2}}$$

Where the FW^* represents the standardized test statistic for Fligner-Wolfe test statistic and M^* represents the standardized test statistic for Moses test statistic.
The second test is given by:

$$M_2 = \frac{FW + \mathrm{M} - E(\mathrm{FW} + \mathrm{M})}{\sqrt{var(\mathrm{FW}) + var(\mathrm{M})}}$$

The third test is given by:

$$M_3 = \frac{FW + 3\mathrm{M} - E(\mathrm{FW} + 3\mathrm{M})}{\sqrt{var(\mathrm{FW} + 3\mathrm{M})}}$$

When the null hypothesis is true, the asymptotic distributions of M_1, M_2 and M_3 are standard normal distributions.
Alsubie and Magel 2020b [7] found that M_2 has the highest powers when the change is only in location parameters. When the underlying distributions were approximately symmetric, L_1 has the highest powers when both the location and scale parameters are different or when just the scale parameters are different.

Materials and Methods
We developed 6 tests to test the hypothesis in (1). All 6 tests use the Moses technique (Moses, 1963) [8] to transform data so that tests for scale parameters can be done by using tests for location parameters on the transformed data.

Jonckheere-Terpstra (JT) Test + Moses test
The first two tests, JM_1 and JM_2, use Jonckheere-Terpstra (JT) test (1954, 1952) [9, 10] on both the original data and the data transformed by the Moses technique. For $k (k \ge 3)$ populations , the JT test is defined by

$$JT = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} U_{ij}.$$

U_{ij} is the number of pairs of observations (a, b) for which x_{ia} is less than x_{jb}, where $x_{ia} (or\ x_{jb})$ denotes the $ath (or\ bth)$ sample observation for $ith (or\ jth)$ population $i (or\ j)$. The *JT* test is first used on the original data to test for location and denoted by JT_1. Using the Moses technique, the observations from each sample are randomly divided into sub-samples of equal size. For each sub-sample, the sum of the squared deviations of observations from their sample mean is obtained. Let $m_1, m_2, \dots, m_k$ be the number of sup-groups based on the initial samples from each population. The *JT* test is applied on this transformed data consisting of the sum of squares associated with each of the sub-samples. This is denoted as JT_2. It is noted that the sample sizes from each population are smaller than the initial sample sizes from each population when calculating JT_2.
The first proposed test statistic is denoted JM_1 where JT_1^* and JT_2^* are the standardized versions of the two previous test statistics mentioned.

$$JM_1 = \frac{JT_1^* + JT_2^*}{\sqrt{2}}$$

The expected value and the variance for JT_1 are given by Daniel [11].

$$E_0(JT_1) = \frac{N^2 - \sum_{i=1}^{k} n_i^2}{4}$$

and

$$var_0(JT_1) = \frac{N^2(2N+3) - \sum_{i=1}^{k} n_i^2(2n_i+3)}{72}$$

$N = n_1 + n_2 + \cdots + n_k$ ($n_1, n_2, \dots, n_k$ are the sample sizes for each population). The mean and variance for JT_2 may be found by replacing the original sample sizes $n_1, n_2, \dots, n_k$ by $m_1, m_2, \dots, m_k$ ($m_1, m_2, \dots, m_k$ are the number of sup-groups based on using Moses technique on the original samples from the populations).
The second proposed test statistic JM_2 is given below

$$JM_2 = \frac{JT_1 + JT_2 - [E(JT_1) + E(JT_2)]}{\sqrt{var(JT_1) + var(JT_2)}}$$

Both JT_1 and JT_2 will have an asymptotic normal distributions under the null hypothesis [10]. Therefore, both JM_1 and JM_2 will have asymptotic standard normal distributions. The null hypothesis in (1) will be rejected for large values.

Modified Jonckheere-Terpstra (MJT) Test + Moses test
The third and fourth proposed tests $MJM_1$and MJM_2both use the modified Jonckheere-Terpstra (MJT) Test [12] on both the original data and the data transformed by the Moses technique. For $k (k \geq 3)$ populations, the MJT test is defined by

$$MJT = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (j - i) U_{ij}$$

U_{ij} is the number of pairs of observations (a, b) for which x_{ia} is less than $x_{jb}, where\ x_{ia} (or\ x_{jb})$ denotes the $ath (or\ bth)$ sample observation for $ith (or\ jth)$ population. We denote MJT_1 and MJT_2 as the MJT test performed on the original and transformed data, respectively.
The third proposed test statistic MJM_1 is given below where MJT_1^* and MJT_2^* denote the standardized versions of the previously mentioned test statistics.

$$MJM_1 = \frac{MJT_1^* + MJT_2^*}{\sqrt{2}}$$

The fourth proposed test statistic MJM_2 is given below

$$MJM_2 = \frac{MJT_1 + MJT_2 - [E(MJT_1) + E(MJT_2)]}{\sqrt{var(MJT_1) + var(MJT_2)}}$$

The expected value and the variance for MJT_1 are given by [13]

$$E(MJT_1) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (j - i) E(U_{ij})$$

$$var(MJT_1) = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} (j - i)^2 var(U_{ij}) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} \sum_{l=j+1}^{k} Cov(U_{ij}, U_{il})$$

$$E_0(U_{ij}) = \frac{1}{2} n_i n_j, \forall i \neq j$$

$var_0(U_{ij}) = \frac{1}{12} n_i n_j (n_i + n_j + 1), \forall i \neq j$

$Cov(U_{ij}, U_{il}) = Cov(U_{ji}, U_{li}) = \frac{1}{12} n_i n_j n_l, if\ all\ i, j, and\ l\ are\ different$

$Cov(U_{ij}, U_{li}) = Cov(U_{ji}, U_{il}) = -\frac{1}{12} n_i n_j n_l, if\ all\ i, j, and\ l\ are\ different$

$Cov(U_{ij}, U_{lm}) = 0, if\ all\ i, j, and\ l\ are\ different$

where n_i, n_j, n_l are the sample sizes for population i, j, and l. To find the expected value and variance of MJT_2, replace the original sample sizes by m_i, m_j, m_l.
Both MJT_1 and MJT_2 have asymptotic normal distributions under H_0 [12]. Both MJM_1 and MJM_2 will have asymptotic standard normal distributions under the null hypothesis. The null hypothesis in (1) will be rejected for large values.

Shan Test + Moses test
The fifth and sixth proposed tests, $SM_1$and SM_2, both use the Shan Test [14] on the original data and then on the data transformed by the Moses technique.
For $k(k \geq 3)$ populations, the samples from all the populations are first combined together and the observations are ranked. The Shan test is given by

$$S = \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} D_{ij},$$

Where

$$D_{ij} = \sum_{l=1}^{n_i} \sum_{m=1}^{n_j} Z_{ijlm}, Z_{ijlm} = (R_{jm} - R_{il}) I(x_{jm} > x_{il}), x_{il} (or\ x_{jm})$$

Denotes the $lth(or\ mth)$ sample observation for $ith(or\ jth)$ population$i(or\ j)$, and $R_{il}(or\ R_{jm})$ denotes the rank of the observation $x_{il}(or\ x_{jm})$ in the combined data. S_1 and S_2 denote the Shan test statistics applied to the original data and the transformed data, respectively.
The fifth proposed test statistic SM_1 is given below where S_1^* and S_2^*, denote the standardized versions of the previously mentioned test statistics.

$$SM_1 = \frac{S_1^* + S_2^*}{\sqrt{2}}$$

The sixth proposed test statistic SM_2 is given below

$$SM_2 = \frac{S_1 + S_2 - [E(S_1) + E(S_2)]}{\sqrt{var(S_1) + var(S_2)}}$$

The expected value and the variance for S_1 are given by

$$E_0(S_1) = \frac{N+1}{6} \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j,$$

and

$$var_0(S_1) = \left(\frac{N^2+N}{12} - \frac{(N+1)^2}{36}\right) \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} n_i n_j + 2\left[\sum_{i=1}^{k-1} n_i \binom{\sum_{j=i+1}^{k} n_j}{2} + \sum_{i=2}^{k} n_i \binom{\sum_{j=1}^{i-1} n_j}{2}\right] CovA + 2\left(\sum_{i=1}^{k-2} \sum_{j=i+1}^{k-1} \sum_{l=j+1}^{k} n_i n_j n_l\right) CovB,$$

where $CovA = \frac{2N^2+N-1}{90}$, and $CovB = \frac{-7N^2-11N-4}{360}$, n_i, n_j are the sample sizes for population i and j, $N = n_1 + n_2 + \cdots + n_k$. For S_2, replace the sample sizes by $m_1, m_2, \ldots, m_k$ instead of the $n_1, n_2, \ldots, n_k$ ($m_1, m_2, \ldots, m_k$ are the number of sup-groups for each population).
Both S_1 and S_2 have asymptotic normal distributions under H_0 [14]. When H_0 is true, the asymptotic distributions of SM_1 and SM_2 are standard normal distributions. The null hypothesis in (1) is rejected for large values.

Simulation Study
A simulation study was conducted to compare the six proposed tests and was implemented in SAS version 9.4. The proposed test statistics were compared assuming random samples from normal distributions, t-distributions with 3 degrees of freedom, and exponential distributions. The function RAND was used to generate random samples from a specific distribution in SAS. The six proposed tests were compared in two steps. The first step was to estimate the alpha values of the proposed test statistics. Replications of 5000 samples were used for all simulations. The stated alpha values for the proposed test statistics were 0.05. The alpha values were estimated by the total number of times the null hypothesis was rejected when the null hypothesis was true and then divided by 5000. The second step was to compare the powers of the test statistics under various conditions. Powers were

estimated by counting the total number of times the proposed tests were rejected under a given condition divided by 5000. Three, four, and five populations were considered.

When estimating the powers, three varying conditions were assumed. First, the location parameters were different, and the scale parameters were equal. Second, the location parameters were equal, and the scale parameters were different. Lastly, both the location parameters and the scale parameters were different. Equal and nonequal sample sizes of 18 and 30 were used for all populations. Two different subgroup sample sizes (3 and 6) were used for each scenario when applying the Moses technique.

Results and Discussions

Tables 1-4 present the results of the simulation study for three treatments under normal distributions. All of the proposed tests maintained their estimated alpha values (Table 1). When the populations have unequal location parameters and equal scale parameters, the standardize last tests (all tests with a subscript of 2) have higher estimated powers than all the standardize first tests (all tests with a subscript of 1) (Table 1). When the populations have equal location parameters and unequal scale parameters, the standardize first tests have higher estimated powers than all the standardize last tests (Table 2). When the populations have unequal location parameters and unequal scale parameters, the standardize first tests generally have the higher estimated powers (Tables 3 and 4).

Table 1: Percentage of Rejection for k=3 Populations; Normal Distribution with different means and equal standard deviations ($n_1 = n_2 = n_3 = 30$; subgroup sample size=3)

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	JM_1	JM_2	MJM_1	MJM_2	SM_1	SM_2
0	1	0	1	0	1	0.0508	0.0536	0.0490	0.0506	0.0476	0.0492
0	1	0.5	1	1	1	0.8294	0.9814	0.8236	0.9766	0.8318	0.9815
0	1	0.5	1	0.5	1	0.3628	0.5816	0.3756	0.5734	0.3610	0.5812
0	1	0	1	1	1	0.8130	0.9746	0.8258	0.9748	0.8420	0.9794
0	1	0.75	1	1	1	0.8168	0.9750	0.8168	0.9780	0.8260	0.9778
0	1	0.2	1	1	1	0.8266	0.9780	0.8140	0.9780	0.8282	0.9786

When the subgroup sample size increases from 3 to 6, the estimated powers for testing location and scale parameters become lower (compare Tables 3 and 4). For the situations of unequal location parameters and equal scale parameters, the SM_2 test tends to have the highest estimated powers (Table 1). For the situations of equal location parameters and unequal scale parameters, the SM_1 test generally has the highest estimated powers (Table 2). For the situations of unequal location parameters and unequal scale parameters, the SM_1 test has the highest estimated powers (Tables 3 and 4).

Table 2: Percentage of Rejection for k=3 Populations; Normal Distribution with equal means and different standard deviations ($n_1 = n_2 = n_3 = 30$; subgroup sample size=3)

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	JM_1	JM_2	MJM_1	MJM_2	SM_1	SM_2
0	1	0	2	0	3	0.8060	0.1660	0.7844	0.1604	0.8476	0.0962
0	1	0	2	0	2	0.4886	0.0954	0.4876	0.1018	0.5504	0.0592
0	1	0	1	0	2	0.5116	0.1352	0.5058	0.1294	0.5564	0.0878
0	1	0	2.5	0	3	0.7818	0.1470	0.7732	0.1504	0.8450	0.0880
0	1	0	1.5	0	3	0.8046	0.1850	0.7860	0.1728	0.8380	0.1018

Table 3: Percentage of Rejection for k=3 Populations; Normal Distribution with different means and different standard deviations ($n_1 = n_2 = n_3 = 30$; subgroup sample size=3)

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	JM_1	JM_2	MJM_1	MJM_2	SM_1	SM_2
0	1	0.5	2	1	3	0.9872	0.7718	0.9846	0.7576	0.9902	0.6348
0	1	0.5	2	0.5	2	0.8264	0.4836	0.8404	0.4866	0.8608	0.3842
0	1	0	1	1	2	0.9560	0.8462	0.9612	0.8642	0.9624	0.7990
0	1	0.75	2.5	1	3	0.9852	0.7638	0.9860	0.7464	0.9902	0.6348
0	1	0.2	1.5	1	3	0.9848	0.7592	0.9832	0.7522	0.9884	0.6310

Table 4: Percentage of Rejection for k=3 Populations; Normal Distribution with different means and different standard deviations ($n_1 = n_2 = n_3 = 30$; subgroup sample size=6)

μ_1	σ_1	μ_2	σ_2	μ_3	σ_3	JM_1	JM_2	MJM_1	MJM_2	SM_1	SM_2
0	1	0.5	2	1	3	0.9830	0.6390	0.9782	0.6142	0.9832	0.5896
0	1	0.5	2	0.5	2	0.7850	0.3586	0.8380	0.3658	0.7744	0.3358
0	1	0	1	1	2	0.9488	0.7792	0.9524	0.7908	0.9526	0.7568
0	1	0.75	2.5	1	3	0.9774	0.6226	0.9806	0.5908	0.9806	0.5610
0	1	0.2	1.5	1	3	0.9752	0.6344	0.9784	0.6090	0.9772	0.5718

Results of the simulation study for three treatments under the T distribution were similar to the results for the normal. All of the proposed tests maintained their alpha values. When the populations have unequal location parameters and equal scale parameters, the standardize last tests have higher estimated powers than all the standardize first tests. When the populations have equal location parameters and unequal scale parameters, the standardize first tests have higher estimated powers than all standardize last tests. When the populations have unequal location parameters and unequal scale parameters, the standardize first tests tend to have the higher estimated powers.

For the situations of unequal location parameters and equal scale parameters, the SM_2 test tends to have the highest estimated powers. For the situations of unequal location parameters and unequal scale parameters, the SM_1 test has the highest estimated powers. This was also true when just the scale parameters were different.

For the non-symmetric distribution (exponential), none of the tests maintained their alpha values. The estimated alpha values were greater than the stated alpha value. The JT test is an extension of the Mann-Whitney test, and for the non-symmetrical population, the Mann-Whitney and Moses tests are not independent (Hollander, 2013) [13]. The same situation happens with the Shan test. The tests which standardize last have lower alpha values than the tests which standardize the individual tests first. The difference is around 0.02. The estimated alpha values for the tests that standardized last were around 0.06 to 0.08. For the exponential distribution, when the difference was only in treatment scale parameters, the powers of all tests became too low to compare. These tests are not recommended for heavily skewed distributions.

Conclusions

Overall, we recommend keeping the subgroup sample size small to allow for larger sample sizes after using the Moses technique. If the distribution that one is sampling from is assumed to be approximately symmetric, and only the location parameters are different, SM_2 has the highest powers. When the scale parameters were not equal, regardless of whether location parameters were equal or not, SM_1 has the highest powers. So, if the researcher wants to test for a difference in either location or scale, we recommend using SM_1.

References

1. Lepage Y. A combination of Wilcoxon's and Ansari-Bradley's statistics. Biometrika. 1971;58(1):213-217.
2. Wilcoxon, Frank. Individual Comparisons by Ranking Methods. Biometrics Bull. 1945;1(6):80-83.
3. Ansari AR, Bradley RA. Rank-Sum Tests for Dispersion. Annals of Mathematical Statistics. 1960;31:1174-1189.
4. Marozzi M. Nonparametric Simultaneous Tests for Location and Scale Testing: A Comparison of Several Methods. Communications in Statistics – Simulation and Computation. 2013;42(6):1298-1317.
5. Alsubie A, Magel R. Proposed nonparametric tests for the simple tree alternative for location and scale testing. International Journal of Statistics and Applied Mathematics. 2020a;5:26-32.
6. Fligner M, Wolfe D. Distribution-free tests for comparing several treatments with a control. Statistica Neerlandica. 1982;36(3):119-127.
7. Alsubie A, Magel R. Proposed nonparametric tests using Moses test for location and scale testing. Journal of Progressive Research in Mathematics. 2020b;16:2877-2887.
8. Moses LE. Rank Tests of Dispersion. Annals of Mathematical statistics. 1963;34(3):973-983.
9. Jonckheere AR. A distribution-free k-sample test against ordered alternatives. Biometrika. 1954;41(1/2):133-45.
10. Terpstra TJ. The asymptotic normality and consistency of Kendall's test against trend, when ties are present in one ranking. Indag. Math. 1952;14(3):327-33.
11. Daniel WW. Applied Nonparametric Statistics. 2nd Edition, PWS - Kent Publishing Company, Boston; c1990, p. 237.
12. Neuhaueser Markus, Liu Ping-Yu, Hothorn Ludwig A. Nonparametric Tests for Trend: Jonckheere's Test, a Modification and a Maximum Test. Biometrical Journal. 1998;40(8):899-909.
13. Hollander M, Wolfe D. Nonparametric Statistical Methods, 2nd Edition (Wiley Series in Probability and Statistics). John Wiley & Sons, Inc., New York; c2013, p. 203-209.
14. Shan Guogen, Young Daniel, Kang LE. A New Powerful Nonparametric Rank Test for Ordered Alternative Problem. PLoS One. 2014;9(11):e112924.