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Identification of irrational numbers through elementary functions

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Abstract
The paper explores on the identification of irrational numbers properly focusing on elementary function. Irrational numbers are the primary concern for rational, irrational and algebraic and can be interpreted that all rational are algebraic numbers but all irrational numbers are not algebraic. Reviewing on various books, articles, criticisms, and research on irrational numbers, what I have found that there is no proper research irritation numbers through elementary functions. So, the paper purposes to examine on how irritation number are identified through elementary functions. The paper applies descriptive method prioritizing on qualitative research design adopting data from secondary sources. The significance of the study is for determining irrational numbers via elementary function.

Keywords: Irrational and rational, elementary function, algebraic, transcendental

1. Introduction

A real number of the form $\frac{u}{v}$ where u, v are integers and $v \neq 0$ is called a rational number [5]. A real number which cannot put in that form is known as irrational number. The Set of real numbers is countable measure of that set is zero. Almost all numbers in the set of real numbers are irrational. We identify irrational numbers through rational, logarithmic and transcendental functions. Irrational numbers can be further classified into algebraic and transcendental numbers. The paper, hence, aims to explore on identifying irrational number properly tying up with its elementary function. The study is descriptive in nature and follows qualitative research design collecting data from secondary sources. The paper divides into main four sections i.e. introduction, materials and methods, results and discussion, and conclusion. The paper is significant for identifying irrational number via properly dealing with its elementary functions.

2. Materials and Methods

The paper is based on library research issuing on qualitative research design. The theories for the paper are related to the reasoning that irrational number complies two facets i.e. (i) algebraic and (ii) transcendental numbers. However, Ivan Niven's concept on irrational numbers, as he has mentioned in his book Irrational Numbers has applied as a key theory for the study.

3. Results and Discussion

3.1 Polynomials

Theorem 3.1.1 [4, 5]: If a polynomial equation with integral coefficients

$$c_n x^n + c_{n-1} x^{n-1} + \ldots + c_1 x + c_0 = 0, c_n \neq 0 \quad (1)$$

has a nonzero rational solution u/v where the integers u and v are relatively prime that is $(u, v) = 1$, then $u | c_0$ and $v | c_n$

Proof: Put $x = u/v$ in equation 1 and multiplying by v^{n-1}, we get

$$c_n u^n / v + c_{n-1} u^{n-1} + \ldots + c_1 u v^{n-2} + c_0 v^{n-1} = 0 \quad (2)$$

Since the expression $c_{n-1}u^{n-1} + ... + c_1uv^{n-2} + c_0v^{n-1}$ is integer, so the term $c_n\frac{u^n}{v}$ must be integer. Since $(u,v) = 1, v|c_n$ Again, put $x = u/v$ in equation 1 and multiplying both sides of the equation by $\frac{v^n}{u}$, we get

$$c_nu^{n-1} + c_{n-1}u^{n-2}v + \cdots + c_1v^{n-1} + c_0v^n/u = 0 \tag{3}$$

In the equation 3, the expression $c_nu^{n-1} + c_{n-1}u^{n-2}v + \cdots + c_1v^{n-1}$ is an integer, the term $c_0\frac{v^n}{u}$ must be an integer. Again as $(u,v) = 1, u|c_0$.

3.1.1 Corollary [5]
If a polynomial equation 1 with $c_n = \pm 1$ has a rational solution, that solution is an integer dividing c_0.
Proof: If $x = u/v$ is a solution of the polynomial equation 1, then $v|c_n$ but $c_n = \pm 1$ and hence $v = \pm 1$. As we consider the denominator is positive, $v = 1$. Hence the solution is $x = u$ which is an integer and that integer must divide c_0.

3.1.2 Corollary: For any integers c and $n > 0$, the only rational solution, if any, of $x^n = c$ are integers. Thus $x^n = c$ has a rational solution if and only if c is the n^{th} power of an integer.

Proof: Suppose the number $x = u/v$ with $v > 0$ is a solution of the equation $x^n - c = 0$. Then $v|1$ that is $v = 1$. Hence the only possible rational solution of the equation $x^n - c = 0$ is integer $x = u$. This also shows that c is the nth power of u.
From the corollary 3.1.2, we can say that $\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{8}, \cdots$ are irrational numbers because there are no integral solutions of $x^2 = 2, x^2 = 3, x^2 = 5, x^2 = 8, ...$

The Theorem 3.1.1 can be used for certain values of Trigonometric functions as well.
From above theorem and corollaries, we can deduce that

3.1.3 Corollary [5]
If m is a positive integer which is not the n^{th} power of an integer, then $\sqrt[n]{m}$ is irrational.
Proof: It is obvious.

3.2 Trigonometric function
In this section, we discuss how to identify irrational numbers through trigonometric functions and later one we show that π is irrational number.

3.2.1 Lemma [9]
If n is a positive integer, and $g(x)$ any polynomial with integral coefficients, then $h(x) = \frac{x^n g(x)}{n!}$ and all its derivatives, evaluated at $x = 0$ are integers. Moreover, with the possible exception of the case $j = n$, the integer $h^n(0)$ is divisible by $n + 1$. If $g(x)$ is a factor of x, then no exception is needed in the case $j = n$.
Proof: Since $h(x) = \frac{x^n g(x)}{n!}$where $g(x) = c_0x^m +,\cdots, +c_{(m-1)}x + c_m$ and c_j are integers, we have $f^{(j)}(0) = c_j(j!)/n!$ ...
If $j < n$, then $h^{(j)}(0) = 0$ which is divisible by $n + 1$. If $j = n$, then $h^{(n)} = c_m$. If $g(x)$ is a factor of x, then $g(0) = 0$. In that case $h^{(n)}(0)$ is divisible by $n + 1$.

3.2.2 Lemma
If $f(x)$ is a polynomial in $(r - x)^2$, then $f^{(j)}(r) = 0$ for any odd integer j.
Proof: For any odd $j, f^{(j)}(x)$ becomes a polynomial in odd powers only of $(r - x)$. Hence $f^j(r - x) = 0$.

3.2.1 Theorem [9]
Let r be a non-zero rational number. Then cos r, sin r and tan r are irrational numbers.
Proof: First we prove the case of $\cos r$. Since $cos(-r) = cos\, r$, it is sufficient to prove the case of positive rational number r. Let $r = u/v$ where u and v are positive integers. Define a function $f(x)$ by

$$f(x) = \frac{x^{(p-1)}(u-vx)^{2p}(2u-vx)^{p-1}}{(p-1)!}$$

$$= \frac{(r-x)^{2p}\{r^2-(r-x)^2\}^{p-1}v^{3p-1}}{(p-1)!} \tag{4}$$

where p is a fixed odd prime. Now, for any real x with $0 < x < r$,

$$0 < f(x) = \frac{r^{2p}\{r^2\}^{p-1}v^{3p-1}}{(p-1)!} = \frac{r^{4p-2}v^{3p-1}}{(p-1)!} \tag{5}$$

By using all even derivatives of $f(x)$, we define a function $F(x)$ as

$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - f^{(6)}(x) + \ldots - f^{(4p-2)}(x) \quad (6)$$

Then we have

$$\frac{d}{dx}\{F'(x)sinx - F(x)cosx\} = F^{(2)}(x)sinx + F(x)sinx = f(x)sinx,$$

and

$$\int_0^r f(x)\sin x dx = F'(r)\sin r - F(r)\cos r + F(0). \quad (7)$$

By the definition, the function $f(x)$ is a polynomial in $(r-x)^2$. Also by definition of the function $F(x)$ and the Lemma 3.2.2 we have $F'(r) = 0$. But also $f(x)$ has the form of $h(x)$ in Lemma 3.2.1, with n is replaced by $p-1$. We observe that $f^{(j)}(0)$ is an integer for every value of j and moreover that $f^{(j)}(0)$ is a multiple of p unless $j = p-1$. With calculation from equation 4 we have

$$f^{(p-1)}(0) = u^{2p}(2u)^{p-1} \quad (8)$$

Choose the odd prime to be $p > u$. Since u is a positive integer, it follows that $f^{(p-1)}(0)$ is not divisible by p. Thus we conclude that $F(0)$ is an integer not divisible by p; say $F(0) = q$ with $(p,q) = 1$.
From equation 7, we observe $F(r)$. Now from the equation 4, we have

$$f(r-x) = \frac{x^{2p}\{r^2-x^2\}^{p-1}v^{3p-1}}{(p-1)!}$$

$$= \frac{x^{2p}\{u^2-v^2x^2\}^{p-1}v^{p+1}}{(p-1)!}$$

Thus $f(r-x)$ has the form of $h(x)$ in Lemma 3.2.1 with n replace by $p-1$ and $g(x)$ replaced by

$$x^{p+1}\{u^2 - v^2x^2\}^{p-1}v^{p+1}$$

Hence by Lemma 3.2.1, for every j, the number $f^{(j)}(r)$ is an integer divisible by p and so there exist an integer m such that $F(r) = pm$. Assume that $cos\, r$ is rational. So there exist two integers d and $k > 0$ such that $cos\, r = d/k$. Thus from the equation 6, we get

$$k\int_0^r f(x)\sin x dx = -pmd + kq \quad (9)$$

Choose the prime p sufficiently large so such that $p > k$. Thus p is not a divisor of kq, it follows that $-pmd + kq$ is a non-zero integer. On the other hand, the left hand side of the equation 9 can be kept

$$\left|k\int_0^r f(x) sin\, xdx\right| < kr\frac{r^{4p-2}v^{3p-1}}{(p-1)!} = kr^3v^2\frac{\{r^4v^3\}^{p-1}}{(p-1)!} = \frac{c_1c_2^{p-1}}{(p-1)!} \quad (10)$$

where $c_1 = kr^3v^2$ and $c_2 = r^4v^3$ are independent of p. Now, as p tends to infinity, $\frac{c_1c_2^{p-1}}{(p-1)!}$ tends to zero, and so we can choose p sufficiently large so that the left side number of equation 9 lies between -1 and 1. This leads to a Contradiction. Hence $cos\, r$ must be irrational number.
If sin r were rational number for a rational number $r \neq 0$, then $cos\, 2r = 1 - sin^2\, r$ would be rational number. But by theorem 1, $cos\, 2r$ is irrational number. Hence sin r must be irrational number.
Similarly, if tan r were rational, so would

$$\cos 2r = \frac{1-tan^2r}{1+tan^2r}$$

be rational, again in contradiction to the theorem. Also $sec\, r, cosec\, r, cot\, r$ are irrational number, since they are the reciprocals of irrational numbers.

3.2.1 Corollary [12]

$$cos^{-1}r, sin^{-1}r, tan^{-r}r$$

are irrational numbers for any rational number r.

Proof: Let $cos^{-1}r = p$. We assume that p is a non-zero rational number. Then $cos\, p = r$, contrary to Theorem 3.2.1. By using similar arguments, we can show for other two functions as well.

3.2.2 Corollary [7]
π is irrational number.

Proof: If π were rational, then by theorem 3.2.1 $cos\, r$ would be irrational number, where as $cos\, \pi = -1$. Thus π must be irrational number.

3.3 Hyperbolic function
3.3.1 Theorem [9].
For any rational number $r \neq 0$, $cosh\, r, sinh\, r, tanh\, r$ are irrational numbers.

Proof: Suppose that $r > 0$ is a rational number. Then there exist two positive integers u, v such that $r = u/v$. We define a function $f(x)$ as in the equation 4. Now from equations 7 and 9, we get

$$F(x) = f(x) + f^{(2)}(x) + f^{(4)}(x) + f^{(6)}(x) \cdots + f^{(4p-2)}(x)$$

and

$$\int_0^r f(x) \sinh x dx = [F(x) \cosh x - F'(x) \sinh x]_0^r = F(r) \cosh r - F'(r) \sinh r - F(0). \quad (11)$$

The left hand side of the equation 11 positive because $f(x) > 0$ and $sinhx$ is
positive for $0 < x < r$. Applying Lemma 4.1 to functions $f(x)$ and $f(r - x)$
we get that $f^j(0)$ and $f^j(r)$ are integers for every j. Thus $F(0)$ and $F(r)$ are integers. Also by Lemma 4.2, $F'(r) = 0$.
Suppose that $cosh\, r = d/k$ with $k > 0$. Then the equation 11 can be written as

$$k \int_0^r f(x) \sinh x dx = dF(r) - kF(0) \quad (12)$$

The right hand side of the equation 12 is an integer where as the left hand side we see that

$$0 < k \int_0^r f(x) \sinh x dx < kr \frac{r^{4p-2}v^{3p-1}}{(p-1)!}.\frac{e^r e^{-r}}{2} = \frac{kr^3v^2(e^r - e^{-r})}{2}.\frac{\{r^4v^3\}^{p-1}}{(p-1)!} < 1.$$

for sufficiently large value of p. Thus we have a contradiction. Hence $cosh\, r$ must be irrational.
Since $cosh\, 2r = 2\, sinh^2 r + 1$, $sinh\, r$ is also irrational number. Also, as

$$cosh\, 2r = \frac{1 + tanh^2 r}{1 - tanh^2 r}, \tanh r$$

is irrational number.

3.4 Exponential function
3.4.1 Lemma: A number α is rational if and only if there is a positive integer k such that $[k\alpha] = k\alpha$.
Proof: Suppose that α is a rational number. Then there exist two integers u and $v > 0$ such that $\alpha = u/v$. Choose an integer k such that $v|k$. Then $k\alpha = u$ this implies $[k\alpha] = u$ by the definition of greatest integer function. Hence $[k\alpha] = u = k\alpha$. Converse can be done.

3.4.2 Lemma: A number α is rational if and only if there is a positive integer k such that $[(k!)\alpha] = (k!)\alpha$.
Proof: By applying similar arguments as Lemma 3.4.1. we get the result.

3.4.1 Theorem: e is irrational number.
Proof: We know that $e = \sum_{j=0}^{\infty} \frac{1}{j!}$. Suppose that e is a rational number.
Then there exist a positive integer k such that

$$[(k!)e] = k! \sum_{j=0}^{k} \frac{1}{j!} < k!\, e. \quad (13)$$

Contradiction. Hence e must be irrational.

From Theorem 3.4.1, we get that the number e is irrational number. Now we show that the value of e^r is irrational for every rational number $r \neq 0$.

3.4.2 Theorem [8, 1].
If $r \neq 0$ is any rational number, then e^r is irrational number.

Proof. If e^r were rational, so would be its reciprocal e^{-r}. This implies that $\frac{e^r+e^{-1}}{2} = \cosh r$ is rational number. But by Theorem 5.1, $\cosh r$ is irrational for every rational $r \neq 0$. This leads to a Contradiction. Hence e^r must be irrational number for every rational number $r \neq 0$.

3.5 Logarithmic function
3.5.1 Theorem [7]
Let p and q be any two distinct prime numbers. Then $log_p q$ is irrational number.
Proof: On the contrary, suppose that $log_p q$ is a rational number, say equal to u/v with $(u, v) = 1$. That is

$$log_p q = \text{u/v}$$

That is $q = p^{u/v}$ this implies $p^v = p^u$ that is $q|p^u$ contradiction as $q \nmid p$.

Theorem 3.5.2 [6]. Let p and q be any two distinct prime numbers. Then $\frac{log\ p}{\log\ q}$ are irrational number.
Proof. Suppose that

$$\frac{log\ p}{\log\ q} = \frac{u}{v} \tag{14}$$

This implies that $log\ p^v = log\ q^u$ that is $p^v = q^u$ contradiction.

Theorem 3.5.3 [1]. If u, v are distinct non negative integers and p, q are distinct primes, then

$$log_{10}(p^u q^v) \tag{15}$$

is an irrational number

Proof. Suppose that the number $log_{10}(p^u q^v)$ is rational. Then there exist two integers c, d such that

$$log_{10}(p^u q^v) = c/d$$

$$\Longrightarrow p^u q^v = 10^{c/d}$$

The equality holds if $p = 2\ or\ 5$, and $q = 2\ or\ 5$ and $ud = c$, and $vd = c$ that is $ud = vd$. Since u and v are different, so are ud and vd. Hence the number is irrational.
Now we prove more general result as follows

3.5.4 Theorem [2, 3]. For any positive rational $r, log_{10}\ r$ is irrational unless $r = 10^n$ for some integer m.
Proof. If $log_{10} r$ is rational, then so is $log_{10} r^{-1}$. Hence we take $r > 1$. Since $r > 1$ is rational number, there exist two positive integers u and v with $(u, v) = 1$ such that $r = u/v$.
Suppose that $log_{10}\ r = c/d$ where c and d are positive integers with $(c, d) = 1$. This implies that $r = 10^{c/d}$ that is

$$\frac{u}{v} = 10^{c/d}$$

$$\Longrightarrow \frac{u^d}{v^d} = 10^c$$

$$\Longrightarrow u^d = 10^c.v^d$$

Which implies that $v = 1$ and $u^d = 10^c$. Thus a must be of the form $2^j 5^j$ for some positive integers i and j and consequently $c = id$. But $(c, d) = 1$, so that $d = 1$ and $r = u/v = u = u^d = 10^c$. Hence $log_{10} r$ is irrational except for the case $r = 10^n$ for some positive integer n.
This theorem can be generalized from logarithmic base 10 to any rational base $v \neq 1$.

3.5.5 Theorem [10]
If r and $v \neq 1$ are positive rational numbers, then $log_v\ r$ is irrational number unless there exist integers in and m such that $r^m = v^n$.
Proof. The proof of the theorem is similar as the proof of the Theorem 3.5.4.

3.5.6 Theorem [10]. If $r \neq 1$ a positive rational number, then $log_e r = \log r$ is irrational.
Proof. We know that $r = e^{log r} = \log e^r$. Suppose that $log\, r$ is rational. Then by Theorem 3.5.2, $e^{\log r} = r$ is irrational. Contradiction since r is rational. Hence $log\, r$ is irrational for every positive rational $r \neq 1$.

3.6 Open Problem

So many numbers are discussed whether it is rational or it is irrational but there so many number whether they are rational or irrational and algebraic or transcendental. The following problems as still open [7, 13].

1. Euler's constant e and π are irrational. Are $e \pm \pi$ irrationals? Furthermore, there is no two integers m, n such that $em \pm n\pi$.
2. $e\pi, \frac{\pi}{e}, 2^e, \pi^e, \pi^{\sqrt{2}}, log\, \pi$ are irrational.
3. Calalan's Constant or Euler- Mascheroni's Constant γ is irrational.

$$\gamma = \lim_{n\to\infty}\left(-\ln n + \sum_{k=1}^{n}\frac{1}{k}\right)$$

Are Tetrations ${}^{n}\pi$ and ${}^{n}e$ rational number for some integer $n > 1$?

4. Conclusion

The study has dealt with that a real number of the form $\frac{u}{v}$ where u, v are integers and $v \neq 0$ is called a rational number. To be more specific, it has concretized the sense of rational number highlighting the aforesaid formulaic form. Similarly, the study has focused that almost all numbers in the set of real numbers are irrational. So many numbers are discussed whether it is rational or it is irrational but there so many number whether they are rational or irrational and algebraic or transcendental. The paper has also minutely examined and met a conclusion that irrational numbers include both algebraic and transcendental. The finding of the study is fairly facilitated with the discussion matters as: polynomial, trigonometric function, hyperbolic function, exponential, logarithmic function and open problem related to rational or irrational number. Moreover, the paper has found out that irrational numbers are completely determined as with the elementary function. Therefore, the paper becomes a base for new research in the field of irrational numbers.

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