

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2022; 7(2): 34-40
© 2022 Stats & Maths
www.mathsjournal.com
Received: 07-12-2021
Accepted: 27-01-2022

Prem Singh

Assistant Professor, Department
of Mathematics, GDC Memorial
College, Bahal, Haryana, India

New cubic b-spline origination and procedure of newton TAGE technique for reason of two-point BVP

Prem Singh

DOI: <https://doi.org/10.22271/maths.2022.v7.i2a.792>

Abstract

In this paper, we present a competent high-demand mathematical technique based on cubic spline gauge and newton TAGE methodology for the defensive schemes of two-point non-straight limit respect difficulties on a non-uniform lattice, who's convincing limits are in important construction. When the internal organization incentives for plan length are peculiar, the proposed technique is critical. Essential differential conditions with singularities are also relevant to the suggested cubic spline approach. The methodology's utility is demonstrated by computational findings.

Keywords: BVP, TAGE method

1. Introduction

Despite the fact that splines can be of any degree, cubic splines have the most norms by a long shot. A cubic spline is one that is formed up of piecewise third-degree polynomials and has a lot of control points. At the endpoints of every polynomial, the second subordinate is usually set to zero, as this provides a breaking point condition that completes the game plan of conditions. This results in a "natural" cubic spline with an acceptable tri-corner to corner structure that can be properly dealt with to obtain the polynomial coefficients. However, this option isn't the only one available and other breaking point criteria could be used as well. Rajashekhar *et al.* (2017) ^[5] studied the Efficient Numerical Solution for Seventh Order Differential Equation by Using Septic B-Spline Collocation Method with Non-Uniform Length. Busyra *et al.* (2021) ^[4] delivered a brand new cubic B-spline (CBS) approximation approach to resolve linear two-factor boundary cost problems (BVPs). This approach is primarily based totally on cubic B-spline foundation capabilities with a brand new approximation for the second-order derivative. The numerical outcomes had been as compared with the least squares approach, finite distinction approach, finite detail approach, finite extent approach, B-spline interpolation approach, prolonged cubic B-spline interpolation approach and the precise solution. Khalid *et al.* (2019) ^[2] suggested a computational model based on the cubic B-spline method for solving linear 6th order BVPs that arise in astronomy. The boundary problem is transformed into a system of linear equations using the approach specified. The technique we will build in this work is not only an approximation solution of the 6th order BVPs using cubic B-spline, but it also explains the estimated derivatives of the analytic solution from 1st order to 6th order. This unique technique is second order convergent and has a lower computational cost than many existing strategies. The findings are shown in error tables and graphs, and they are compared to previous research findings. Nazir *et al.* (2019) ^[2] Using the seventh-degree B-Spline function, we provide a numerical method for addressing the boundary value problem of The Linear Seventh Ordinary Boundary Value Problem. The formulation is based on specific terms of the seventh-order boundary value problem's order. The Collocation B-spline approach is formulated as an approximation solution, and we gain Septic B-Spline formulation. We use the proposed method to solve a seventh-order boundary value problem, and the results indicate that approximate and exact solutions agree.

Corresponding Author:

Prem Singh

Assistant Professor, Department
of Mathematics, GDC Memorial
College, Bahal, Haryana, India

The numerical method given is helpful for solving high order boundary value issues since it produces low absolute errors. A broad conclusion has been included at the end. Manan *et al.* (2018) ^[8]. An algorithm to solve 3rd order boundary value problem is focused in this paper which is 8-point approximating scheme. It concludes the results with stability and convergence that is evaluated with the illustration of numerical example. This paper also contains the analysis of approximation properties for the mentioned collocation algorithm. Heilat and Hailat (2020) ^[1] prolonged cubic B-spline approach for fixing a gadget of nonlinear second-order boundary price problems. A quasi linearization method is used and the mistake estimate is acquired. The accuracy of the approach relies upon on loose parameters. We practice the approach on 4 examples and the outcomes which acquired the use of prolonged cubic B-spline suggest that the approach is green and accurate. Siraj-ul-Islam *et al.* (2006) ^[7] studied the Quadratic Non-Polynomial Spline Approach to the Solution of a System of Second-Order Boundary-Value Problems. Tahir *et al.* (2019) ^[3] proposed a new quantic B-spline approximation technique for solving the Boussinesq problem numerically. Two test examples are used to confirm the new scheme's performance and accuracy. The computational results are shown to be superior to current numerical techniques on the subject. Mustafa *et al.* (2017) ^[9] the solution of non-linear fourth order boundary value problems involving ordinary differential equations, this work provides an iterative collocation numerical technique based on interpolating subdivision schemes. The technique converges to a smooth approximate solution of the non-linear fourth order boundary value problem, according to numerical evidence. The approach's convergence is also examined. The main goal of this paper is to look into and find applications for subdivision systems in physics and engineering. Pervaiz *et al.* (2014) ^[10] developed a numerical technique for solving sixth order boundary value problems (BVPs). The sixth order BVPs were first reduced to a system of second order BVPs. The system of second order BVPs was then reduced to a system of linear algebraic equations with boundary conditions using non-polynomial cubic spline and finite difference approximations. The strategy was tested on three test issues with step sizes of $h = 1/5$ and $h = 1/10$ to see if it was useful. Tables and graphs have been used to demonstrate the method's accuracy in achieving the exact solution. Shahid S. Siddiq *et al.* (2007) ^[11]. The numerical solution of the sixth-order linear, special case boundary value problem is done using the septic spline. The definition of a septic spline is given with end conditions that are consistent with the sixth-order boundary value problem. The solution and their higher-order derivatives are approximated using the algorithm proposed. It has also been demonstrated that the approach is second-order convergent. For numerical illustrations of the developed approach, three instances are studied. The approach described in this study is compared to that produced in [M. El-Gamel, J.R. Cannon, J. Latour, A.I. Zayed, Sinc-Galerkin method for solving linear sixth order boundary-value problems, *Mathematics of Computation* 73, 247 (2003) 1325–1343], and it is found to be superior.

2. Cube Spline Formulation

We discretize arrangement locale $[0, 1]$ with non-uniform work so such an extent that $0 = x_0 < x_1 \dots < x_{N+1} = 1$. Our method includes three matrix focuses x_k, x_{k+1} & x_{k-1} where $x_k - x_{k-1} = h_k$ & $x_{k+1} - x_k = h_{k+1}$. Matrix focuses are given by $x_i = x_0 + \sum_{k=1}^i h_k, i = 1(1)N + 1$ and the work extent is $\sigma_K = (h_{k+1}/h_k) > 0$. Right when $\sigma_K = 1$, at that point it decreases to constant mesh case. Let the right estimations of $u(x)$ at matrix point x_k be implied by $U_K = u(x_k)$ and u_k be the rough estimations of U_K . At the matrix point x_k , let us show

$$P_k = \sigma_k^2 + \sigma_k - 1$$

$$Q_k = (\sigma_k + 1)(\sigma_k^2 + 3\sigma_k - 1)$$

$$R_k = \sigma_k(1 + \sigma_k - \sigma_k^2)$$

$$S_k = \sigma_k(\sigma_k + 1)$$

All through our trade, we consider that our answer contains odd number of inside grid points, for instance N as odd. We initially build up a numerical method for evaluating integral $\int_0^1 G(x)dx$.

For this we consider

$$\int_{x_{k-1}}^{x_{k+1}} G(x)dx = b_{-1}G_{k-1} + b_0G_k + b_1G_{k+1} \tag{1}$$

Where b_{-1}, b_0, b_1 are boundaries to be chose and at point x_k , we show $G_k = G(x_k)$ Further, we may compose utilizing Taylor's expansion

$$G_{k-1} = G_k - h_k G'_k + \frac{h_k^2}{2} G''_k - \frac{h_k^3}{6} G'''_k + \dots \tag{2}$$

$$G_{k+1} = G_k + h_{k+1} G'_k + \frac{h_{k+1}^2}{2} G''_k + \frac{h_{k+1}^3}{6} G'''_k + \dots \tag{3}$$

$$\begin{aligned} \int_{x_{k-1}}^{x_{k+1}} G(x)dx &= \int_{x_{k-1}}^{x_{k+1}} G(x_k + x - x_k)dx \\ &= (1 + \sigma_k)h_k G_k + (\sigma_k^2 - 1)\frac{h_k^2}{2} G'_k + (1 + \sigma_k^3)\frac{h_k^3}{6} G''_k + (\sigma_k^4 - 1)\frac{h_k^4}{24} G'''_k + \dots \end{aligned} \tag{4}$$

Likewise right hand side of (1), utilizing (2), (3) as

$$b_{-1} \left[G_K - h_K G'_K + \frac{h_K^2}{2} G''_K + \frac{h_K^4}{24} G^{IV}_K - \dots \right] + b_0 G_K + b_1 \left[G_K + \sigma_k h_K G'_K + \frac{\sigma_k^2 h_K^2}{2} G''_K + \frac{\sigma_k^3 h_K^3}{6} G'''_K + \frac{\sigma_k^4 h_K^4}{24} G^{IV}_K + \dots \right] \tag{5}$$

Using (4)-(5), comparing both sides of (1), we get

$$b_0 + b_1 + b_{-1} = (1 + \sigma_k) h_k \tag{6}$$

$$(b_1 \sigma_k - b_{-1}) h_k = (\sigma_k^2 - 1) \frac{h_k^2}{2} \text{ It implies } b_1 \sigma_k - b_{-1} = (\sigma_k^2 - 1) \frac{h_k}{2}. \tag{7}$$

$$b_1 \sigma_k^2 b_{-1} = (\sigma_k^3 - 1) \frac{h_k}{3} \tag{8}$$

Solving (6)-(8), we get

$$b_0 = \frac{(1 + \sigma_k)^3}{6\sigma_k^k} h^k, b_1 = \frac{(1 + \sigma_k)(2\sigma_k - 1)}{6\sigma_k^k} h^k, b_{-1} = \frac{(1 + \sigma_k)(2 - \sigma_k)}{6} h_k$$

Accordingly one might be compose third request integral formula (2) in variable mesh frame as

$$\int_{x_{k-1}}^{x_{k+1}} G(x) dx = \frac{(1+\sigma_k)h_k}{6\sigma_k} [\sigma_k(2 - \sigma_k)G_{k-1} + (1 + \sigma_k)^2 G_k + (2\sigma_k - 1)G_{k+1}], k = 1(1)N \tag{9}$$

Note that coefficients related with right hand side of (9) are sure, assuming likewise, for example, $2 - \sigma_k > 0$ & $2\sigma_k - 1 > 0$ i.e $\frac{1}{2} < \sigma_k < 2$. Regardless, for constant work case, integral formulas (9) diminish to Simpson's 1/3rd Rule. By then the estimation of integral

$$\int_0^1 G(x) dx = \int_{x_0}^{x_2} G(x) dx + \int_{x_2}^{x_4} G(x) dx + \dots + \int_{x_{N-1}}^{x_{N+1}} G(x) dx \tag{10}$$

Can be found by the reiterated use of formula (2), when N is odd. Directly we analyzed the variable work cubic spline strategy for differential con. (3). Cubic spline interjecting polynomial of degree three then $[x_{k-1}, x_k]$ can be made as

$$S(x) = \frac{(x_k-x)^3}{6h_k} M_{k-1} + \frac{(-x_{k-1}+x)^3}{6h_k} M_k + \left(u_{k-1} - \frac{h_k^2}{6} M_{k-1}\right) \frac{(x_k-x)}{h_k} + \left(u_k - \frac{h_k^2}{6} M_k\right) \frac{(-x_{k-1}+x)}{h_k}, x_{k-1} \leq x \leq x_k, k = 1(1)N + 1 \tag{11}$$

Which satisfies the accompanying with conditions in each subinterval $[x_{k-1}, x_k], k = 1(1)N + 1, S(x)$, coincides with a polynomial of degree three,

$$S(x) \in C^2[0,1] \text{ and } S(x_k) = u_k, k = 0(1)N + 1$$

Where $M_k = u''(x_k) = \psi(x_k, u(x_k), u'(x_k)) \equiv \text{say}$

And $m_k = u'(x_k), k = 0N + 1$

Now,

$$u(x) \cong s(x) = \frac{(x_{k+1}-x)^3}{6h_{k+1}} M_k + \frac{(-x_k+x)^3}{6h_{k+1}} M_{k+1} + \left(u_k - \frac{h_{k+1}^2}{6} M_k\right) \frac{(x_{k+1}-x)}{h_{k+1}} + \left(u_{k+1} - \frac{h_{k+1}^2}{6} M_{k+1}\right) \frac{(-x_k+x)}{h_{k+1}}; x \in [x_k, x_{k+1}]$$

$$\hat{S}(x) = -\frac{(x_{k+1}-x)^2}{2h_{k+1}} M_k + \frac{(-x_k+x)^2}{2h_{k+1}} M_{k+1} + \frac{u_{k+1}-u_k}{h_{k+1}} - \frac{h_{k+1}}{6} [M_{k+1} - M_k], x \in [x_k, x_{k+1}]$$

From continuity con. $\hat{S}(x_{k-}) = \hat{S}(x_{k+})$

i.e. $\lim_{\epsilon \rightarrow 0} S'(x_k - \epsilon) = \lim_{\epsilon \rightarrow 0} S'(x_k + \epsilon)$ one obtain

$$\frac{h_k}{6} M_{k-1} + \frac{h_{k+1}+h_k}{3} M_k + \frac{h_{k+1}}{6} M_{k+1} = \frac{u_{k+1}-u_k}{h_{k+1}} - \frac{-u_{k-1}+u_k}{h_k}, k = 1(1)N$$

Where $M_k = u'' x_k, M_{k+1} = u'' x_{k+1} M_{k-1} = u'' x_{k-1}$,

Here one have equations in questions $N + 2$ unknowns $M_0, M_1, M_2, \dots, M_N, M_{N+1}$

If $M_0 = u'' x_0$ and $M_{N+1} = u'' x_{N+1}$ are use then one can calculate $M_1, M_2 \dots, M_N$,

$$\bar{M}_k = \psi(x_k, U_k, \bar{m}_k), \quad (12)$$

$$\bar{M}_{k+1} = \psi(x_{k+1}, U_{k+1}, \bar{m}_{k+1}) \quad (13)$$

$$\bar{M}_{k-1} = \psi(x_{k-1}, U_{k-1}, \bar{m}_{k-1}) \quad (14)$$

$$S(x_{k+1}) = U_{k+1}$$

And

$$S'(x_{k+1}) \cong m_{k+1} = \frac{U_{k+1} - U_k}{h_{k+1}} + \frac{h_{k+1}}{6} [M_k + 2M_{k+1}]$$

$$S'(x_{k-1}) \cong m_{k-1} = \frac{U_k - U_{k-1}}{h_k} + \frac{h_k}{6} [M_{k-1} + M_k]$$

Define $G = \frac{\partial \psi}{\partial u}$, $G' = \frac{dG}{dx}$ etc.

We require accompanying approximations

$$M_k \cong \bar{M}_k = \psi(x_k, U_k, \bar{m}_k) = \psi_k + \frac{1}{6} \sigma_k h_k^2 U_k'' G_k + O(h_k^3)$$

$$M_{k+1} \cong \bar{M}_{k+1} = \psi(x_{k+1}, U_{k+1}, \bar{m}_{k+1}) = \psi_{k+1} - \frac{1}{6} \sigma_k (\sigma_k + 1) h_k^2 U_k'' G_k + O(h_k^3)$$

$$M_{k-1} \cong \bar{M}_{k-1} = \psi(x_{k-1}, U_{k-1}, \bar{m}_{k-1}) = \psi_{k-1} - \frac{1}{6} \sigma_k (\sigma_k + 1) h_k^2 U_k'' G_k + O(h_k^3)$$

One need $O(h_k^3)$ approximation for \hat{U}_k^l and $\hat{U}_{k\pm 1}^l$.

Let, $\hat{U}_k^l = \bar{m}_k + a_k h_k [\bar{M}_{k+1} - \bar{M}_{k-1}]$ where “ a_k ” is a parameter to be considered.

$$= m_k + \left(\frac{1}{6} \sigma_k + a_k (\sigma_k + 1)\right) h_k^2 U_k'' + O(h_k^3) \quad (15)$$

The approximation (15) to be of, coefficient of must be zero;

$$a_k = -\frac{\sigma_k}{6(1 + \sigma_k)}$$

Hence one obtain

$$\hat{U}_k^l = \bar{m}_k - \frac{\sigma_k}{6(1 + \sigma_k)} h_k [\bar{M}_{k+1} - \bar{M}_{k-1}] = m_k + O(h_k^3) \quad (16)$$

Similarly,

$$\hat{U}_{k+1}^l = \frac{U_{k+1} - U_k}{\sigma_k h_k} + \frac{\sigma_k h_k}{6} [\bar{M}_k + 2\bar{M}_{k+1}] = m_{k+1} + O(h_k^3) \quad (17)$$

$$\hat{U}_{k-1}^l = \frac{U_{k-1} + U_k}{h_k} - \frac{h_k}{6} [\bar{M}_k + 2\bar{M}_{k-1}] = m_{k-1} + O(h_k^3) \quad (18)$$

Finally, by assistance of (16) - (18), one can assess

$$\hat{\psi}_{k+1} = \psi(x_{k+1}, U_{k+1}, \hat{U}_{k+1}^l) = \psi_{k+1} + O(h_k^3)$$

$$\hat{\psi}_{k-1} = \psi(x_{k-1}, U_{k-1}, \hat{U}_{k-1}^l) = \psi_{k-1} + O(h_k^3)$$

$$\hat{\psi}_k = \psi(x_k, U_k, \hat{U}_k^l) = \psi_k + O(h_k^3)$$

By then the cubic spline approximation of precision of $O(h_k^3)$ for given integral differential conditions (1) may be created as

$$U_{k+1} - (1 + \sigma_k)U_k + \sigma_k U_{k-1} = \frac{h_k^2}{12} [P_k \hat{\Psi}_{k+1} + Q_k \hat{\Psi}_k + R_k \hat{\Psi}_{k-1}] + \hat{T}_k, k = 1(1)N \quad (19)$$

Where $\hat{T}_k = O(h_k^5)$

For convergence of the distinction method, coefficients on right hand side of conditions (19) must be sure, i.e., one has the condition $\frac{\sqrt{5}-1}{2} < \sigma_k < \frac{\sqrt{5}+1}{2}$ forced on our choice of mesh proportion parameter.

Note that the limit esteems are given by $u_0 = U_0 = a_0$ & $u_{N+1} = U_{N+1} = a_1$. Joining the limit esteems in numerical approximation (19), we can procure a tri-diagonal framework of equations. If the differential equation is linear, one can comprehend the framework by AGE iterative strategy and for nonlinear case, one can compared the framework by Newton-AGE iterative method.

3. Application of Newton TAGE Method

Next we talk about the utilization of Newton-TAGE iterative strategy for nonlinear difference conditions (19). Dismissing the mistake term, one may re-compose nonlinear difference equation (19) as

$$\phi_k(u_{k-1}, u_k, u_{k+1}) = -U_{k+1} + (1 + \sigma_k)u_k - \sigma_k u_{k-1} = \frac{h_k^2}{12} [P_k \hat{\Psi}_{k+1} + Q_k \hat{\Psi}_k + R_k \hat{\Psi}_{k-1}], k = 1(1)N, \sigma_k \neq 1 \tag{20}$$

Let us denote,

$$\phi(u) = \begin{bmatrix} \phi_1(u) \\ \phi_2(u) \\ \vdots \\ \phi_N(u) \end{bmatrix}_{N \times 1}$$

And,

$$a_k(u) = \frac{\partial \phi_k}{\partial u_{k-1}} \quad k = 2(1)N,$$

$$2b_k(u) = \frac{\partial \phi_k}{\partial u_k} \quad k = 1(1)N$$

$$c_k(u) = \frac{\partial \phi_k}{\partial u_{k+1}} \quad k = 1(1)N - 1$$

The Jacobian of $\phi(u)$ may be spoken as

$$(T_1 + T_2) \equiv \frac{\partial \phi(u)}{\partial u} = \begin{bmatrix} 2b_1(u) & c_1(u) & & & 0 \\ a_2(u) & 2b_2(u) & c_2(u) & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \cdot \\ 0 & & a_{N-1}(u) & 2b_{N-1}(u) & c_{N-1}(u) \\ & & & a_N(u) & 2b_N(u) \end{bmatrix}_{N \times N}$$

Newton method is given by

$$(T_1 + T_2)\Delta u^{(s)} = \Delta(u^{(s)}), s = 0, 1, 2 \dots \tag{21}$$

Where T_1 & T_2 are of indistinct casing from A_1, A_2 , & $u^{(0)}$ is hidden estimation for u & $\Delta u^{(s)}$ is any halfway vector. We characterize

$$u^{(s+1)} = u^{(s)} + \Delta u^{(s)}, s = 0, 1, 2 \dots \tag{22}$$

By then a two-advance explicit Newton-TAGE strategy may be formed as

$$(T_1 + \rho_1 I)\Delta v = -\phi(u^{(s)}) - (T_2 + \rho_1 I)\Delta v^{(s)}, s \geq 0 \tag{23}$$

$$(T_2 + \rho_2 I)\Delta v^{(s+1)} = -\phi(u^{(s)}) - (T_1 + \rho_2 I)\Delta v^{(s)}, s \geq 0 \tag{24}$$

Where Δv is any moderate vector. Since the matrices $(T_1 + \rho_1 I)$ & $(T_2 + \rho_2 I)$ comprise of (2×2) sub-matrices, hereafter they are successfully invertible and from (23) & (24), we get

$$\Delta v = (T_1 + \rho_1 I)^{-1} [-\phi(u^{(s)}) - (T_2 + \rho_1 I)\Delta v^{(s)}], s \geq 0 \tag{25}$$

$$\Delta v^{(s+1)} = (T_2 + \rho_2 I)^{-1} [-\phi(u^{(s)}) - (T_1 + \rho_2 I)\Delta v^{(s)}], s \geq 0 \tag{26}$$

Finally, by the assistance of (25)-(26), from (23) one can secure numerical solution at $(s + 1)$ th iteration. For intermingling, it is required that one pick $u^{(0)}$ "close" to solution u .

4. Results and Conclusions

In this section, we dealt with the following with two benchmark issues on both uniform and non-uniform work whose correct arrangement is known to us, and we differentiated the outcomes and the corresponding successive over relaxation and Newton-SOR approach. For uniform work, one take $\sigma_k = \sigma = 1/7$ for non-uniform mesh, one take $\sigma_k = \sigma = a$ constant $\neq 1, k = 1(1)N + 1$. By then estimation of first mesh isolating on left is given by

$$h_1 = \frac{(1-\sigma)}{(1-\sigma^{N+1})}, \sigma \neq 1 \tag{27}$$

Thusly, for non-uniform mesh, given the assessment of N , we can figure initial step length h_1 from above associated and rest of the work is likewise controlled by $h_{k+1} = \sigma h_k, k = 1(1)N$. Such a constrigent isn't relevant to uniform work case. By taking $\sigma_k = \sigma = 1$, one can apply the strategies clearly to uniform mesh. All calculations were finished utilizing MATLAB. In all cases, one has considered $u^{(0)} = 0$ and the iterations were stopped when the preeminent blunder $|u^{(s+1)} - u^{(s)}| \leq 10^{-10}$ resistance was refined. Relationships of plots of right and mathematical answers for picked boundaries for all of the issues analyzed in points of reference are given.

Example 4.1: (Linear Singular Problem).

$$u'' + \frac{\alpha}{x}u' = 12x^2 + 4 \left[\alpha x^2 + 4x^6 - \frac{\alpha}{x^2} \right] e^{x^4} + 48 \int_0^1 [s^3 x^6 e^{x^4 s^4}] ds, \tag{28}$$

$$0 < x < 1, 0 < s < 1$$

The limit esteems are given by $u(0) = 1, u(1) = e$. The right arrangement of problem is given by $(x) = e^{x^4}$. The root mean square (RMS) mistakes, optimal estimations $(\rho_{opt}, \rho_{1opt}, \rho_{2opt})$ of relaxation parameters and number of iterations both for TAGE & SOR methods for non-uniform & uniform mesh case are ordered in Table 1 for $(\alpha\sigma)$.

Example 4.2: (Nonlinear Singular Problem)

$$u'' + \frac{\alpha}{x}u' - \frac{\alpha}{x^2}u = uu' + (2 + \alpha + x^2)coshx - x^3(xsinha + 2coshx)coshx + (4 + \alpha) \int_0^1 x^2 cosh(xs) ds, 0 < x < 1, 0 < s < 1. \tag{29}$$

The boundary estimations are given by $u(0) = 0, u(1) = cosh(1)$.

The solution of problem is given by $(x) = x^2 cosh x$. The RMS errors, optimal values $(\rho_{opt}, \rho_{1opt}, \rho_{2opt})$ of relaxation parameters & number of iterations both for Newton- TAGE & Newton-SOR methods for non-uniform and uniform mesh case are tabulated in Table 2(A) for $(\alpha, \sigma) = (0, 0.8)$.

Example-1: The RMS Errors (Non uniform mesh case)

Table 1: Non uniform mesh case

	SOR			TAGE				RMS errors
	ρ_{opt}	ITER	cputime	ρ_{1opt}	ρ_{2opt}	ITER	cputime	
11	1.496	39	0.004082 s	0.566	0.569	29	0.003195 s	0.1503(-03)
21	1.666	65	0.006457 s	0.336	0.340	48	0.004028 s	0.3964(-04)
31	1.740	81	0.008218 s	0.261	0.264	61	0.006792 s	0.2694(-04)
41	1.766	91	0.011755 s	0.201	0.206	71	0.008318 s	0.2175(-04)
51	1.787	110	0.014496 s	0.170	0.174	78	0.009766 s	0.1895(-04)
61	1.794	128	0.019992 s	0.164	0.168	86	0.010686 s	0.1715(-04)
71	1.805	132	0.022071 s	0.154	0.158	88	0.011101 s	0.1583(-04)
81	1.828	149	0.032575 s	0.134	0.139	92	0.014352 s	0.1479(-04)

Example-2: The RMS Errors (uniform mesh case)**Table 2:** Uniform mesh case

	SOR			Tage				RMS errors
	ρ_{opt}	Iter	CPU time	ρ_{1opt}	ρ_{2opt}	Iter	CPU time	
11	1.527	39	0.004332 s	0.525	0.530	31	0.002968 s	0.1220(-03)
15	1.621	53	0.005787 s	0.400	0.405	43	0.003516 s	0.5182(-04)
21	1.707	71	0.006912 s	0.277	0.283	60	0.003960 s	0.1927(-04)
25	1.746	83	0.007802 s	0.262	0.265	67	0.004512 s	0.1132(-04)
31	1.789	101	0.012164 s	0.205	0.210	82	0.005284 s	0.5793(-05)
41	1.835	131	0.013998 s	0.157	0.164	107	0.006848 s	0.2377(-05)
51	1.869	161	0.027139 s	0.119	0.123	137	0.007131 s	0.1171(-05)
63	1.888	197	0.051308 s	0.103	0.106	158	0.015108 s	0.5853(-06)
71	1.903	221	0.046796 s	0.086	0.092	187	0.021915 s	0.3938(-06)
81	1.915	251	0.059709 s	0.081	0.084	203	0.032910 s	0.2536(-06)

5. Conclusion

Another three-point variable work approach based on cubic spline surmise for the game plan of second request nonlinear two-point limit respect issue with absolutely healthy driving cutoff points has been demonstrated. All things considered, the recommended system for $\sigma_K = 1$ is monitored to see whether it reduces to an anticipated work approach for precision of request four. When the number of internal framework explanations behind the course of action space is unusual, the recommended technique is applicable. Both immediate and nonlinear difficulties with lone coefficients, as well as driving cutoff points in critical packaging are well-suited to the approach. The application of TAGE and Newton-TAGE techniques demonstrates the power of differentiating SOR and Newton-SOR procedures in terms of cycle count. The offered solutions could be applied to multi-dimensional situations.

6. References

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