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A study of W_3 curvature tensor in LP-Sasakian manifolds

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Abstract
The object of the present paper is to study the geometrical properties of $W_3(X,Y)Z$ curvature tensor in Lorentzian Para Sasakian manifolds and prove some important results.

Keywords: LP Sasakian manifold, W_3-curvature tensor, symmetric, semi-symmetric, and W_3-flat, conservative W_3 curvature tensor

1. Introduction

Pokhariyal [3] defined the W_3-curvature tensor and studied its physical and geometrical properties in a Riemannian manifold. This tensor is defined as

$$(1.1)\ W_3(X,Y,Z,T) = R(X,Y,Z,T) + \frac{1}{n-1}[g(Y,Z)Ric(X,T) - g(Y,T)Ric(X,Z)]$$

Where R is the Riemannian curvature tensor of type (0,4), g is the Riemannian metric and Ric is the Ricci tensor of type (0,2). The tensor $W_3(X,Y,Z,T)$ is skew-symmetric in Z,T and does not satisfy the cyclic property. That is

$$(1.2)\ W_3(X,Y,Z,T) = -W_3(X,Y,T,Z)$$

And

$$(1.3)\ W_3(X,Y,Z,T) + W_3(Y,Z,X,T) + W_3(Z,X,Y,T) \neq 0$$

We can express this tensor in index notation as

$$(1.4)\ W_{3ijkl} = R_{ijkl} + \frac{1}{n-1}[g_{jk}R_{il} - g_{jl}R_{ik}]$$

In 2018, the authors S.K. Moindi, F. Njui and G.P. Pokhariyal [4] have studied the geometrical properties of $W_3(X,Y,Z,T)$ in a K-contact Riemannian manifold. On the other hand, S.O. Pambo, S.K. Moindi and B.M. Nzimbi [5] have studied η-Ricci soliton on W_3-semi symmetric LP-Sasakian manifolds. Recently, H. A. Donia, S. Shenawy and A. A. Syied [1] have considered the role of W_3-curvature tensor on relativistic space-times.
Motivated by the above results, in this paper we will investigate certain curvature properties of LP-Sasakian manifolds admitting W_3-curvature tensor.

2. Preliminaries

A manifold M^n of dimension n is called an LP-Sasakian manifold if it admits a tensor field ϕ of type (1,1), a contravariant vector field ξ, a 1-form η, a Lorentzian metric g, and satisfies the following properties [2].

(2.1) $\eta(\xi) = -1,$

(2.2) $\phi^2 X = X + \eta(X)\xi,$

(2.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$

(2.4) $g(X, \xi) = \eta(X), \nabla_X \xi = \phi X,$

(2.5) $(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi,$

Where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
Given that M^n is an LP-Sasakian manifold with the structure (ϕ, ξ, η, g), we can deduce the following [2].

(2.6) $g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$

(2.7) $R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$

(2.8) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$

(2.9) $R(\xi, X)\xi = X + \eta(X)\xi,$

(2.10) $Ric(X, \xi) = (n-1)\eta(X),$

(2.11) $Ric(\phi X, \phi Y) = Ric(X, Y) + (n-1)\eta(X)\eta(Y)$

For any vector fields X, Y, Z. We shall use the above results in the following sections.

3. A W_3-flat LP-Sasakian Manifold
Definition 3.1 An LP-Sasakian manifold is said to be flat if $R(X, Y)Z = 0$

Definition 3.2 An LP-Sasakian manifold is called W_3-flat if $W_3(X, Y)Z = 0$

Theorem 3.1 A W_3 flat LP-Sasakian manifold is an Einstein manifold.
Proof
From equation (1.1), if

(3.1) $W_3(X, Y, Z, U) = 0$

(3.2) $\Rightarrow R(X, Y, Z, U) = \dfrac{1}{n-1}[g(Y, U)Ric(X, Z) - g(Y, Z)Ric(X, U)]$

Taking contraction over X and U we have

(3.3) $Ric(Y, Z) = \dfrac{1}{n-1}[Ric(Y, Z) - g(Y, Z)r]$

(3.4) $Ric(Y, Z) = \dfrac{-r}{n-2} g(Y, Z)$

Where r denotes the scalar curvature. Hence the theorem.

4. A W_7-symmetric LP-Sasakian manifold
Definition 4.1. An LP-Sasakian space is called symmetric if $\nabla_U R(X, Y)Z = 0$.

Definition 4.2. An LP-Sasakian space is called W_3-symmetric if $\nabla_U (W_3(X, Y)Z = 0$

Theorem 4.3. A W_3-symmetric and W_3-flat LP-Sasakian manifold is a flat space.

Proof
If M^n is a W_3-symmetric LP-Sasakian manifold, then we have

(4.1) $\nabla_U W_3(X, Y)Z = W_3'(X, Y, Z, U) = 0$

(4.2) $\Rightarrow R\big(X, Y, W_3(Z, U, V)\big) - W_3(R(X, Y, Z), U, V) - W_3(Z, R(X, Y, U), V) - W_3\big(Z, U, R(X, Y, V)\big) = 0$

Expanding the terms in the above expression we get

(4.3) $g\big(R\big(X,Y,W_3(Z,U,V)\big),\xi\big) = R'(X,Y,W_3(Z,U,V),\xi)$

$= g(X,\xi)g\big(Y,W_3(Z,U,V)\big) - g(Y,\xi)g\big(X,W_3(Z,U,V)\big)$

$= \eta(X)W_3'(Y,Z,U,V) - \eta(Y)W_3'(X,Z,U,V)$

(4.4) $g(W_3(R(X,Y,Z),U,V),\xi) = W_3'(R(X,Y,Z),U,V,\xi)$

$$= R(R(X,Y,Z),U,V,\xi) + \frac{1}{n-1}[g(U,V)Ric(R(X,Y,Z),\xi) - g(U,\xi)Ric(R(X,Y,Z),V)]$$

$$= R(R(X,Y,Z),U,V,\xi) + \tfrac{1}{n-1}[g(U,V)(n-1)g(R(X,Y,Z),\xi) - g(U,\xi)(n-1)g(R(X,Y,Z),V)]$$

$= R(R(X,Y,Z),U,V,\xi) + \ [g(U,V)R'(X,Y,Z),\xi) - \eta(U)R'(X,Y,Z,V)]$

$= g(R(X,Y,Z),\xi)g(U,V) - g(U,\xi)g(R(X,Y,Z),V) + g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V)$

$= 2[g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V)]$

(4.5) $g(W_3(Z,R(X,Y,U),V,\xi) = W_3'(Z,R(X,Y,U),V,\xi)$

$$\Longrightarrow R(Z,R(X,Y,U),V,T) = \frac{1}{n-1}[g(R(X,Y,U),V)Ric(Z,\xi) - g(R(X,Y,U),\xi)Ric(Z,V)]$$

$$= g(Z,\xi)g(R(X,Y,U),V) - g(Z,V)g(R(X,Y,U),\xi) + \frac{1}{n-1}[R'(X,Y,U,V)(n-1)g(Z,\xi) - R'(X,Y,U,\xi)(n-1)g(Z,V)]$$

$= g(Z,\xi)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi) + g(Z,\xi)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)$

$= \eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi) + \eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)$

$= 2[\eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)]$

(4.6) $g(W_3(Z,U,R(X,Y,V),\xi) = W_3'(Z,U,R(X,Y,V),\xi)$

$$\Rightarrow R'(Z,U,R(X,Y,V),\xi) + \tfrac{1}{n-1}\big[g\big(U,R(X,Y,V)\big)Ric(Z,\xi) - g(U,\xi)Ric\big(X,R(X,Y,V)\big)\big]$$

$$= g(Z,\xi)g\big(U,R(X,Y,V)\big) - g\big(Z,R(X,Y,V)\big)g(U,\xi) + \tfrac{1}{n-1}\big[R'(X,Y,V,U)(n-1)g(Z,\xi) - g(U,\xi)(n-1)g\big(Z,R(X,Y,V)\big)\big]$$

$= g(Z,\xi)R'(X,Y,V,U) - g(U,\xi)R'(X,Y,V,) + g(Z,\xi)R'(X,Y,V,U) - g(U,\xi)R'(X,Y,V,Z)$

$= \eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z) + \eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z)$

$= 2[\eta(Z)R'(X,Y,V,U) - \eta(U)R'(X,Y,V,Z)]$

Using equations (4.3), (4.4), (4.5), and (4.6) in equation (4.2) we have

(4.7) $\eta(X)W_3'(Y,Z,U,V) - \eta(Y)W_3'(X,Z,U,V) - 2[g(U,V)R'(X,Y,Z,\xi) - \eta(U)R'(X,Y,Z,V)] - 2[\eta(Z)R'(X,Y,U,V) - g(Z,V)R'(X,Y,U,\xi)] - 2[\eta(Z)R'(X,Y,V,U) - \ \eta(U)R'(X,Y,V,Z)] = 0$

In a W_3-flat manifold, $W_3' = 0$, hence the first two terms vanish. Coefficients of $\eta(Z)$ and $\eta(U)$ vanish due to R' being skew-symmetric with respect to the last two variables. We thus have

(4.8) $2[g(Z,V)R'(X,Y,U,\xi) - g(U,V)R'(X,Y,Z,\xi)] = 0$

Since $g(U,V) \neq g(Z,V) \neq 0$ for arbitrary vectors U,V,Z, this implies that if W_3 is symmetric then

(4.9) $R'(X,Y,Z,\xi) = 0$

This completes the proof.

5. W_3-semi symmetric LP-Sasakian manifold

Definition 5.1. An LP-Sasakian manifold is called semi-symmetric if $R(X,Y)R(U,V)Z = 0$.

Definition 5.2. An LP-Sasakian manifold is called W_3-semi symmetric if $R(X,Y)W_3(U,V)Z = 0$.

Theorem 5.1. A W_3-semi symmetric LP-Sasakian manifold is a W_3-symmetric manifold.

Proof

Taking the inner product

(5.1) $g(R(X,Y)W_3(U,V)Z,\xi) = R'(X,Y,W_3(U,V,Z),\xi) = 0$

$g(X,\xi)g\big(Y,W_3(U,V,Z)\big) - g\big(X,W_3(U,V,Z)\big)g(Y,\xi) = 0$

$\eta(X)W_3'(U,V,Z,Y) - \eta(Y)W_3'(U,V,Z,X) = 0$

Since $\eta(X)$ and $\eta(Y)$ are non-zero, $\Rightarrow W_3' = 0$ i.e.

$\nabla_U W_3(X,Y)Z = W_3'(X,Y,Z,U) = 0$

Hence a W_3-semi symmetric LP-Sasakian manifold is a W_3-symmetric manifold.

6. LP-Sasakian manifold satisfying $(div\, W_3)(X,Y)Z = 0$

Given that

(6.1) $W_3(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}[g(Y,Z)QX - S(X,Z)Y]$

Where Q is the symmetric endomorphism of the tangent space at every point and S is the Ricci tensor of type (0,2).

Differentiation (6.1) covariantly, we have

(6.2) $(\nabla_U W_3)(X,Y)Z = (\nabla_U R)(X,Y)Z + \frac{1}{n-1}[g(Y,Z)(\nabla_U Q)(X) - (\nabla_U S)(X,Z)Y]$.

Contracting equation (6.2), we obtain

(6.3) $(div\, W_3)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}[g(Y,Z)dr(X) - (\nabla_X S)(Y,Z)]$

$(div\, W_3)(X,Y)Z = \frac{n-2}{n-1}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) + \frac{1}{n-1}g(Y,Z)dr(X)$

Let us suppose that $(div\, W_3)(X,Y)Z = 0$, then equation (6.3) becomes

(6.4) $\frac{n-2}{n-1}(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = -\frac{1}{n-1}g(Y,Z)dr(X)$

Putting $X = \xi$ in (6.4), we get

(6.5) $\frac{n-2}{n-1}\big(\nabla_\xi S\big)(Y,Z) - (\nabla_Y S)(\xi,Z) = -\frac{1}{n-1}g(Y,Z)dr(\xi)$

Using equation (2.4) and the fact that $L_\xi S = 0$, the first term in the left hand side of equation (6.5) can be expressed as

(6.6) $\big(\nabla_\xi S\big)(Y,Z) = \nabla_\xi S(Y,Z) - S\big(\nabla_\xi Y,Z\big) - S\big(Y,\nabla_\xi Z\big)$

$= \nabla_\xi S(Y,Z) - S([\xi,Y] + \nabla_Y \xi, Z) + S(Y,[\xi,Z] + \nabla_Z \xi)$

$= \nabla_\xi S(Y,Z) - S([\xi,Y],Z) - S(\nabla_Y \xi, Z) - S(Y,[\xi,Z]) - S(Y,\nabla_Z \xi)$

$= \big(L_\xi S\big)(Y,Z) - S(\nabla_Y \xi, Z) - S(Y,\nabla_Z \xi)$

$= -S(\phi Y,Z) - S(Y,\phi Z)$

$= -S(\phi Y, Z) + S(\phi Y, Z), \phi\ is\ skew\ symmetric$

$= 0$

Expanding the second term of (6.5) and using (2.10), we get

(6.7) $(\nabla_Y S)(\xi, Z) = \nabla_Y S(\xi, Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z)$

$= \nabla_Y[(n-1)g(\xi, Z)] - S(\phi Y, Z) - (n-1)g(\xi, \nabla_Y Z)$

$= (n-1)[g(\nabla_Y \xi, Z) + g(\xi, \nabla_Y Z)] - S(\phi Y, Z) - (n-1)g(\xi, \nabla_Y Z)$

$= (n-1)g(\phi Y, Z) - S(\phi Y, Z)$

Lastly

(6.8) $g(Y, Z)dr(\xi) = 0, since\ dr(\xi) = 0$

Now using equation (6.6), (6.7), and (6.8) in equation (6.5) we obtain

(6.9) $S(\phi Y, Z) = (n-1)g(\phi Y, Z)$

Replacing Z with ϕZ in (6.9), leads to

(6.10) $S(\phi Y, \phi Z) = (n-1)g(\phi Y, \phi Z)$

$\Rightarrow S(Y, Z) = (n-1)g(Y, Z)$

Which on contracting yields $r = n(n-1)$.

This leads to the following theorem

Theorem 6.1. If $(div\ W_3)(X, Y)Z = 0$ in an LP-Sasakian manifold, then the manifold is Einstein and it is of constant scalar curvature n(n-1).

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