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A study of W_5 curvature tensor in LP-Sasakian manifold

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Abstract

The object of the paper is to study the geometrical properties of W_5 - Curvature tensors in LP-Sasakian Manifold and prove some results.

Keywords: W_5 -flat, symmetric, semi-symmetric and recurrent

1. Introduction

A manifold M_n is said to be LP-Sasakian manifold if it admits a $(1,1)$ tensor field, a C^∞ 1 - form A and a Lorentzian metric g which satisfy the following properties Matsumoto K 1988 ^[1]

- (1.1) $\eta(\xi) = -1,$
- (1.2) $\phi^2 X = X + \eta(X)\xi,$
- (1.3) $g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$
- (1.4) $g(X, \xi) = \eta(X), \nabla_X \xi = \phi X,$
- (1.5) $(\nabla_X \phi)(Y) = g(X, Y)\xi + \eta(Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$

Where

∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g .

Given that M_n is a LP-Sasakian Manifold with the structure (ϕ, ξ, η, g) , we can deduce the following Matsumoto K 1988 ^[1]

- (1.6) $g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y),$
- (1.7) $R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$
- (1.8) $R(X, Y)\xi = \eta(Y)X - \eta(X)Y,$
- (1.9) $R(\xi, X)\xi = X + \eta(X)\xi,$
- (1.10) $Ric(X, \xi) = n - 1\eta(X),$
- (1.11) $Ric(\phi X, \phi Y) = Ric(X, Y) + (n - 1)\eta(X)\eta(Y).$

for any vector fields X, Y, Z . Pokhariyal 1988 ^[3]

Pokhariyal 1971 ^[4] defined W_5 -Curvature tensor as

$$W_5(X, Y)Z = R(X, Y)Z + \frac{1}{n-1} [g(X, Z)QY - Ric(X, Z)Y]$$

Or

$$(1.12) \quad W_5'(X, Y, Z, \xi) = R'(X, Y, Z, \xi) + \frac{1}{n-1} [g(X, Z)Ric(Y, \xi) - g(Y, \xi)Ric(X, Z)]$$

We shall use the above results in the following sections.

2. W_5 - Curvature Tensor in LP- Sasakian Manifold

Definition 2.1: An LP-Sasakian Manifold M_n is said to be W_5 -flat if $W_5(X, Y)Z = 0$

Theorem 2.1: A W_5 -flat LP-Sasakian manifold is a flat manifold.

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Proof

From (1.12) if $W'_5(X, Y, Z, \xi) = 0$

Then

$$R'(X, Y, Z, \xi) = \frac{1}{n-1} [g(Y, \xi)Ric(X, Z) - g(X, Z)Ric(Y, \xi)]$$

applying $Ric(X, Y) = (n-1)g(X, Y)$ in the above equation we have

$$R'(X, Y, Z, \xi) = \frac{1}{n-1} [g(Y, \xi)(n-1)g(X, Z) - g(X, Z)(n-1)g(Y, \xi)]$$

$$R'(X, Y, Z, \xi) = [g(Y, \xi)g(X, Z) - g(X, Z)g(Y, \xi)]$$

$$R'(X, Y, Z, \xi) = 0$$

Hence the theorem.

3. W_5 - Symmetric LP- Sasakian Manifold

Definition 3.1: An LP- Sasakian Manifold is called W_5 - symmetric if

$$\nabla_u W_5(X, Y, Z) = W'_5(X, Y, Z, U) = 0$$

Theorem 3.1: A W_5 -symmetric and W_5 - flat LP-Sasakian manifold is a flat manifold.

Proof

From the symmetric property we have

$$\nabla_u W_5(X, Y, Z) = R(X, Y, W_5(Z, U, V)) - W_5(R(X, Y, Z), U, V) -$$

$$W_5(Z, R(X, Y, U), V) - W_5(Z, U, R(X, Y, V)) = 0$$

Expanding the above expression

$$(3.1) R(X, Y, W_5(Z, U, V)) = R'(X, Y, W_5(Z, U, V), \xi)$$

$$R'(X, Y, W_5(Z, U, V), \xi) = g(X, \xi)g(Y, W_5(Z, U, V)) - g(Y, \xi)g(X, W_5(Z, U, V)) \\ = \eta(X)W'_5(Y, Z, U, V) - \eta(Y)W'_5(X, Z, U, V)$$

$$(3.2) W_5(R(X, Y, Z), U, V) = W'_5(R(X, Y, Z), U, V, \xi)$$

$$W'_5(R(X, Y, Z), U, V, \xi) = R'(R(X, Y, Z), U, V, \xi) + \frac{1}{n-1} [g(R(X, Y, Z), V)Ric(U, \xi) \\ - g(U, \xi)Ric(R(X, Y, Z), V)] \text{ on using } Ric(X, Y) = (n-1)g(X, Y) \text{ we have}$$

$$W'_5(R(X, Y, Z), U, V, \xi) = R'(R(X, Y, Z), U, V, \xi) + \frac{n-1}{n-1} [g(R(X, Y, Z), V)g(U, \xi) \\ - g(U, \xi)g(R(X, Y, Z), V)]$$

$$W'_5(R(X, Y, Z), U, V, \xi) = R'(R(X, Y, Z), U, V, \xi)$$

$$R'(R(X, Y, Z), U, V, \xi) = g(U, V)g(R(X, Y, Z), \xi) - g(R(X, Y, Z), V)g(U, \xi) \\ = g(U, V)R'(X, Y, Z, \xi) - \eta(U)R'(X, Y, Z, V)$$

$$(3.3) W_5(Z, R(X, Y, U), V) = W'_5(Z, R(X, Y, U), V, \xi)$$

$$W'_5(Z, R(X, Y, U), V, \xi) = R'(Z, R(X, Y, U), V, \xi) + \frac{1}{n-1} [g(Z, V)Ric(R(X, Y, U), \xi) \\ - g(R(X, Y, U), \xi)Ric(Z, V)]$$

on using $Ric(X, Y) = (n-1)g(X, Y)$ we have

$$W'_5(Z, R(X, Y, U), V, \xi) = R'(Z, R(X, Y, U), V, \xi) + \frac{n-1}{n-1} [g(Z, V)g(R(X, Y, U), \xi) \\ - g(R(X, Y, U), \xi)g(Z, V)]$$

$$W'_5(Z, R(X, Y, U), V, \xi) = R'(Z, R(X, Y, U), V, \xi)$$

$$R'(Z, R(X, Y, U), V, \xi) = g(R(X, Y, U), V)g(Z, \xi) - g(Z, V)g(R(X, Y, U), \xi) \\ = R'(X, Y, U, V)\eta(Z) - g(Z, V)R'(X, Y, U, \xi)$$

$$(3.4) W'_5(Z, U, R(X, Y, V)) = W'_5(Z, U, R(X, Y, V), \xi)$$

$$W'_5(Z, U, R(X, Y, V), \xi) = R'(Z, U, R(X, Y, V), \xi) + \frac{1}{n-1} [g(Z, R(X, Y, V))\text{Ric}(U, \xi) \\ - g(U, \xi)\text{Ric}(Z, R(X, Y, V))]$$

on using $\text{Ric}(X, Y) = (n - 1)g(X, Y)$ we have

$$W'_5(Z, U, R(X, Y, V), \xi) = R'(Z, U, R(X, Y, V), \xi) + \frac{n-1}{n-1} [g(Z, R(X, Y, V))g(U, \xi) \\ - g(U, \xi)g(Z, R(X, Y, V))]$$

$$W'_5(Z, U, R(X, Y, V), \xi) = R'(Z, U, R(X, Y, V), \xi) \\ = g(U, R(X, Y, V))g(Z, \xi) - g(Z, R(X, Y, V))g(U, \xi) \\ = R'(X, Y, V, U)\eta(Z) - R'(X, Y, V, Z)\eta(U)$$

putting equations (3.1), (3.2), (3.3) and (3.4) together

$$\eta(X) W'_5(Y, Z, U, V) - \eta(Y) W'_5(X, Z, U, V) - g(U, V)R'(X, Y, Z, \xi) + \eta(U)R'(X, Y, Z, V) - R'(X, Y, U, V)\eta(Z) + \\ g(Z, V)R'(X, Y, U, \xi) - R'(X, Y, V, U)\eta(Z) + R'(X, Y, V, Z)\eta(U) = 0$$

Because $\eta(X)$ and $\eta(Y)$ are non-zero, from the definition of symmetry $W'_5(Y, Z, U, V)$ and $W'_5(X, Z, U, V)$ are zero. Also $\eta(U)$ and $\eta(Z)$ are skew-symmetric with respect to the last two terms.

But since $g(Z, V) \neq 0$ and $g(U, V) \neq 0$ we have $R'(X, Y, U, \xi) = 0$ and $R'(X, Y, Z, \xi) = 0$

Hence the theorem.

4. W_5 -Semisymmetric Lorentzian Para- Sasakian Manifold

Definition 4.1: A LP-Sasakian manifold is said to be W_5 -semi-symmetric if

$$R(X, Y)W_5(U, V)Z = 0.$$

Theorem 4.1: A W_5 -semi-symmetric LP-Sasakian manifold is W_5 -symmetric LP-Sasakian manifold.

Proof:

Suppose

$$R(X, Y)W_5(U, V)Z = 0$$

this implies

$$g(R(X, Y)W_5(U, V)Z, \xi) = R'(X, Y, W_5(U, V)Z, \xi) \\ = g(X, \xi)g(Y, W_5(U, V)Z) - g(Y, \xi)g(X, W_5(U, V)Z) \\ = \eta(X)W'_5(Y, U, V)Z - \eta(Y)W'_5(X, U, V)Z$$

Because $\eta(X)$ and $\eta(Y)$ are non-zero, it means $W'_5(Y, U, V)Z = W'_5(X, U, V)Z = 0$ but $\nabla_u W_5(X, Y, Z) = W'_5(X, Y, Z, U) = 0$
Hence the theorem.

5. W_5 - Recurrent LP - Sasakian Manifold

Definition 5.1: A LP - Sasakian Manifold is said to be W_5 - recurrent if De UC 1992 ^[5]

$$\nabla_u W_5(X, Y)Z = B(U)W_5(X, Y)Z$$

Theorem 5.2: If a LP -Sasakian Manifold is W_5 - Recurrent and Ricci -Recurrent, then for the same recurrence parameter its recurrent.

Proof

Given that

$$W'_5(X, Y, Z, \xi) = R'(X, Y, Z, \xi) + \frac{1}{n-1} [g(X, Z)\text{Ric}(Y, \xi) - g(Y, \xi)\text{Ric}(X, Z)]$$

$$\nabla_u W'_5(X, Y, Z, \xi) = B(U) W'_5(X, Y, Z, \xi)$$

$$\nabla_u W'_5(X, Y, Z, \xi) = \nabla_u R'(X, Y, Z, \xi) + \frac{1}{n-1} [g(X, Z)(\nabla_u \text{Ric})(Y, \xi) - g(Y, \xi)(\nabla_u \text{Ric})(X, Z)]$$

$$\text{But } (\nabla_u \text{Ric})(Y, \xi) = B(U)\text{Ric}(Y, \xi) \text{ and } (\nabla_u \text{Ric})(X, Z) = B(U)\text{Ric}(X, Z)$$

$$\nabla_u W'_5(X, Y, Z, \xi) = \nabla_u R'(X, Y, Z, \xi) + \frac{1}{n-1} [g(X, Z) B(U)\text{Ric}(Y, \xi) - g(Y, \xi) B(U)\text{Ric}(X, Z)] = B(U) W'_5(X, Y, Z, \xi)$$

Then using $\text{Ric}(Y, \xi) = (n-1)g(Y, \xi)$ and $\text{Ric}(X, Z) = (n-1)g(X, Z)$ we get

$$\nabla_u W'_5(X, Y, Z, \xi) = \nabla_u R'(X, Y, Z, \xi) + \frac{n-1}{n-1} [g(X, Z) B(U)g(Y, \xi) - g(Y, \xi) B(U)g(X, Z)] = B(U) W'_5(X, Y, Z, \xi)$$

$$\nabla_u R'(X, Y, Z, \xi) = B(U) W'_5(X, Y, Z, \xi)$$

Hence the theorem.

6. References

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