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Great circle theorem and the application of the spherical cosine rule to estimate distances on a globe

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Abstract
This research paper explores my interest in the Great Circle Theorem and Non-Euclidean Geometry to apply its theory in Earth Sciences and explore navigating flight distances across the globe. I use three different equations, based on the situation of the two points on the map and their relation with each other, to obtain approximate great circle distances between the two points. The values acquired from the final equation, which is derived from the Spherical Law of Cosines, is compared with the actual great circle distances between the two points on the globe. A percentage error is calculated and the average is found, after which a correlation is drawn between the percentage error and the magnitude of the value of the great circle distance.

Keywords: Spherical law of cosines, percentage

Introduction
When I was a child, my father used to work for Etihad Airways. We as a family used to go on many trips and therefore, oftentimes I would find myself in a seat on a plane. On the plane, I would see both two-dimensional and three-dimensional maps of the globe and a small figure of a flight travelling from one point to another in an arc across the globe. Along with the displayed maps, other various information was also given, such as a compass, airspeed, ground speed, etc. One day when travelling from Madrid to New York, I noticed that, on the 2D map, the flight was travelling in a curved path, but on the 3D map, the flight was travelling in a straight line between the two points of Madrid and New York. I realised that this was due to the fact that the Earth is a sphere and therefore the closest distance between two points on a curved surface would result in an arch rather than a straight line [1].

This greatly intrigued me, and given my inquisitive nature as well as my interest in Earth Sciences and Mathematics, I set out to understand this phenomenon. I asked my father numerous questions and I found out that planes would travel through the shortest three-dimensional route possible, known as the “geodesic” or “great circle route”. I learned that when converting a flight path from a 3-D plane to a 2-D plane, it may cause the line to look more like an arc on the flat surface. So even though in actuality, the flight may be moving in a straight line, on a 2-D surface, this would result in an arc, and as I hypothesised earlier, this is due to the fact that the Earth is really a sphere, rather than a flat surface. In Euclidean geometry, this phenomenon is known as the Great Circle Theorem [2].

Key Terms
Latitude: Latitude is a way of measuring distance either north or south of the equator. Latitudes are depicted on a map of the earth with horizontal lines parallel to the equator. There are exactly 180 imaginary lines that make a circle across the globe. These lines are used to help navigate and locate points on a map. Latitude can be described as x°N or x°S of the equator. (Example: 30°N or 40°S)

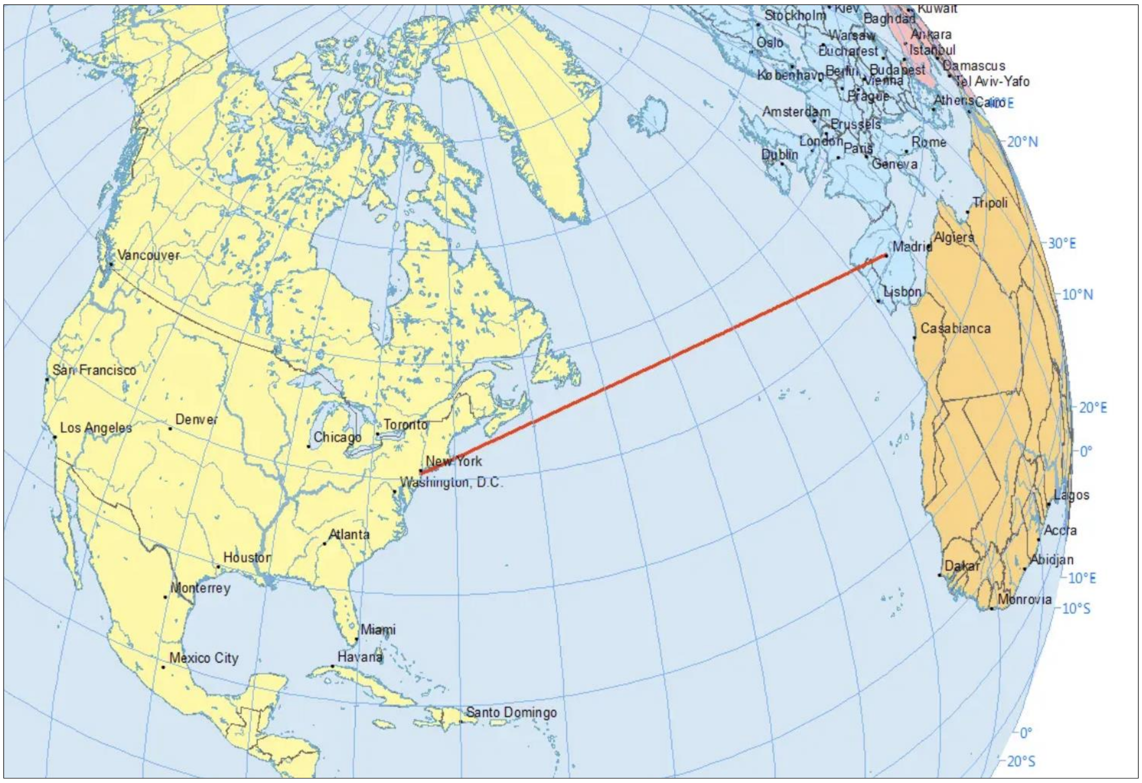

Fig 1: Flight path between Madrid and New York on 3D diagram

Fig 2: Flight path on a 2-D map between Madrid and New York

Longitude: Longitude is a way of measuring distance either East or West of the Greenwich Meridian. Longitudes are depicted on a map of the earth with vertical lines parallel to the Greenwich Meridian. There are exactly 180 imaginary lines that make a circle across the globe. These lines are used to help navigate and locate points on a map. Longitude can be described as x°E or x°W of the equator. (Example: 50°E or 20°W).

Metre: Standard metric for measuring distances in geometry. It is interesting to note that its initial definition is defined as "one ten-millionth of the distance from the North Pole to the Equator travelling along the meridian through Paris."

Great Circle Route: A circle on the surface of a sphere in a plane that passes through the centre of the sphere. The great circle is a perfect route for ships and planes because it represents the shortest distance between any two points on the sphere [3].

Small Circles: Any latitude meridian that is not passing through the largest circumference of the sphere (the equator) is called a small circle. These circles have radii smaller than that of the Earth, i.e less than approx 6400km. Any place situated in the same latitude is due exactly east or west of each other, since all small circles are parallel to each other.

Rhumb Lines (loxodromes): An arc in navigation, a path with constant bearing while passing through all latitude meridians. Rhumb lines were used extensively in the Middle Ages by ship navigators while sailing, since it was easier to navigate while travelling, requiring only the use of a compass.

Mercator Projection: A type of projection used in aeroplanes to depict a 3-D map onto a 2-D surface. Introduced in 1569 by Gerardus Mercator, it has now become a standard way of depicting the globe on a map. However, due to the fact that the globe is a spherical 3-dimensional object, when projecting the earth onto a 2-D map, areas to the northmost and southmost of the globe look disproportionately larger than they actually are. Figure 2 is an example of a Mercator projection.

Introduction to Spherical Geometry

After going through numerous research papers, I quickly came to understand that Euclidean Geometry does not apply to spherical objects. Therefore, the Spherical Geometry and the Great Circle Theorem came about. A great circle route is the straight line equivalent of spherical geometry. It represents the shortest distance between any two points on a sphere. Whilst Euclidean Geometry uses two dimensional planes to plot points and lines on, Spherical Geometry requires a sphere to plot lines, points etc. Spherical Geometry has numerous basic rules that govern the subject.

These rules are

1. A point in Spherical Geometry is defined as having no parts and lies on the surface of a sphere.
2. The shortest distance between two points on the surface of a sphere is known as “Great Circles”
3. Great Circles can be extended to circumnavigate the sphere entirely and are always finitely the length of the circumference of the sphere
4. In Spherical Geometry, the plane is the sphere itself
5. Angles are usually measured in radians and regular Euclidean Geometry laws don't apply to shapes lying on the surface of a sphere. (Examples mentioned below)
6. All right angles on the surface of a sphere are congruent
7. When lines intersect, they always do so at two points
8. Perpendicular lines can be formed by 2 intersecting great circles and form exactly 8 right angles
9. Parallel lines do not exist on a sphere, any two great circles will always intersect at two places on the surface of the sphere
10. Any great circle has the same radius of the sphere and will divide a sphere into two equal halves

As stated above, Spherical Geometry is an extreme tangent from the two- dimensional Euclidean Geometry that I was taught in my junior classes. Spherical geometry tends to reject many of the basic premises of Euclidean Geometry. Therefore, it is important to analyse and understand some of these major differences to gain a better understanding of the theory. Some major differences between Euclidean Geometry and Spherical Geometry are:

1. In Euclidean Geometry, the shortest path between any two points on a plane is known as a line segment. Line segments have a constant gradient, meaning that the rate of change of its slope is zero and the ratio between the vertical displacement and horizontal displacement remains constant. An example of a line segment between two points (A and B) can be seen below.
2.

Fig 3: A line segment

Although, on a 2-dimensional plane, a line segment is the shortest path between two points, on any curved surface, such as the Earth, Euclidean Geometry rules don't apply. If we were to pick any two points on the surface of a sphere and draw a straight line between those two points, we would end up with a great circle route.

Euclidean Geometry also states that any finite line connecting two points on a plane can be extended to infinity in both directions and will never meet. However this is not the case in Spherical Geometry. If a line connecting two points on a sphere is stretched out, it would travel along the sphere and reach its original starting point in a loop. If stretched to infinity, the line will wrap around the sphere an infinite amount of times over.

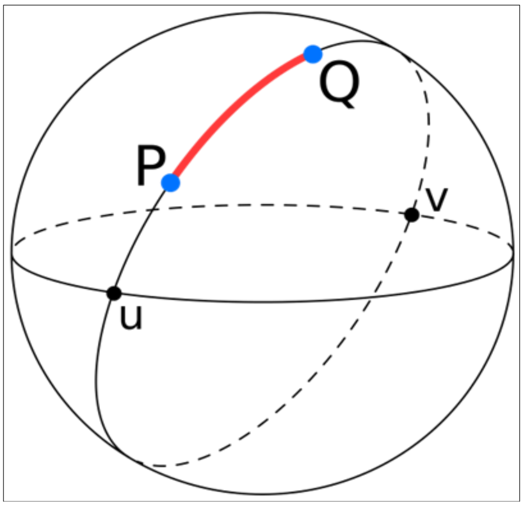

Fig 4: Great Circle Routes stretched to infinity

As seen in figure 4 above, the great circle joining points P to Q is stretched to infinity. As a result, the line travels around the sphere reaching point v and u on the equator of the sphere and then circumnavigating back to point P. As a result, a circle has been formed across the sphere, with a radius of the sphere itself and a circumference of the equator of the sphere.

3. Another major difference between Euclidean Geometry and Spherical Geometry is regarding the statement that “parallel lines never intersect”. Although this statement stands true in Euclidean Geometry on a two- dimensional plane, it doesn't apply for a three dimensional sphere. There are no parallel lines on a sphere. Any two great circles drawn on the surface of a sphere will always intersect at two places on the sphere. On Earth, although at the equator all latitude lines are parallel to each other at

90 degrees, they eventually converge at the poles of the planet. The same can be seen in figure 5 below.

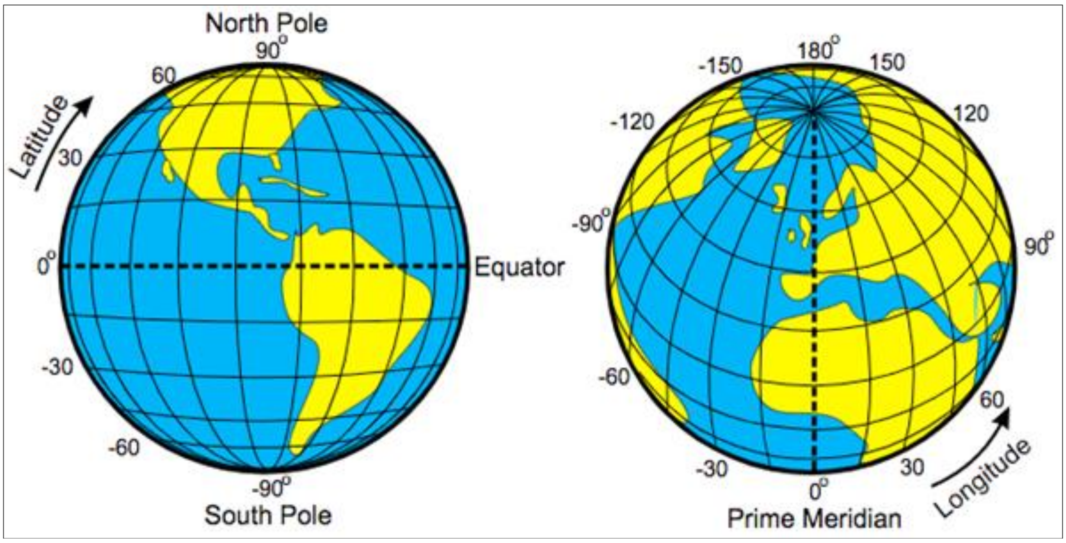

Fig 5: Longitude and Latitude Lines

4. The way that angles are measured in Spherical Geometry is also vastly different from that of Euclidean Geometry. For example, In Spherical Geometry, angles are calculated between 2 or more great circles, resulting in differences in theory when compared to Euclidean trigonometry. In Euclidean Geometry, we are taught that all angles in a triangle must sum up to 180 degrees. However, this does not stand true in the case of Spherical Geometry. On a spherical plane, the sum of all interior angles of a triangle will always exceed 180 degrees.

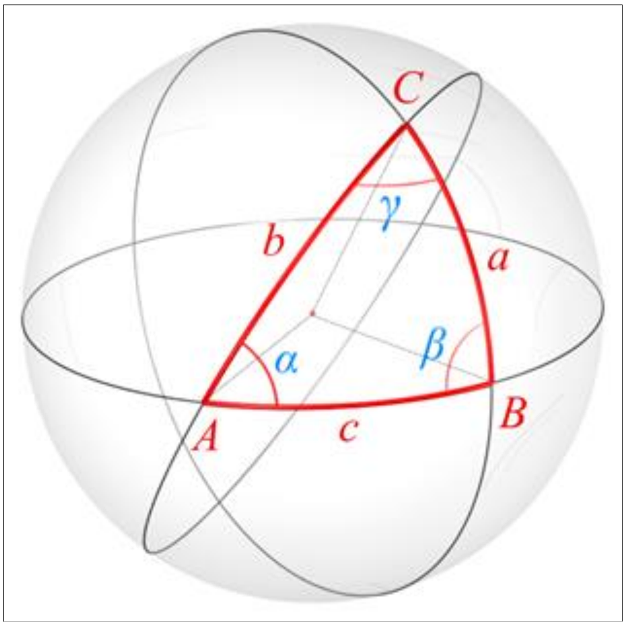

Fig 6: Angles of a Triangle on a Sphere

Formula Derivation and Calculations

Now that I have a considerable amount of knowledge and understanding of the key concepts regarding Spherical Geometry and the Great Circle Theorem, I may now apply them in calculating distances across a sphere. There are three ways in which this can be done, depending on the location of the two points between which distance must be calculated [4].

The three situations are

- Both points on the globe lie on the equator or same meridian of longitude
- Both points lie on parallel latitudes
- Points lie on random parts of the globe

For the first situation, the calculation is straightforward. Since both points lie on the same meridian of longitude, or on the equator, all we have to do is calculate the angle between the two points on the globe in relation to each other. If we assume that the equator is the circumference of a circle where the radius is the same as that of the Earth, then we can easily find the distance between two points on the circumference of that circle by deriving its formula from the basics of circle geometry [5].

For this situation, let's take two points that lie on the equator. As we already know, the Equator is a great circle with radius 6400 km, we can easily calculate the distance between any two points on the equator using the same equation of the length of an arc. A city that lies on the equator is Libreville in Gabon. Singapore lies on the equator as well. To find the distance between the two cities along the equator, we first require their exact location on the globe. Libreville is 9.27◦E and Singapore is 103.81◦E

First, we must calculate the difference in angle between the two points [6].

$$= 103.81^\circ - 9.27^\circ$$

$$= 94.54^\circ$$

$$\therefore,\ \theta = 94.54^\circ$$

Let "l" be the length of the arc:

$$l = \frac{r\pi}{180^\circ} \times \theta$$

$$l = \frac{6400\pi}{180°} \times 94.54°$$

$$= 10560.219$$

$$\approx 10560\ km$$

Paris lies at approx 48.67◦ N and 2.33◦ E. We already know that each meridian is a great circle of approx 6400km radius. The angle between the Equator and the Latitude of Paris is already given to be 48.67◦. To find the angle between the Latitude of Paris and the North Pole, we must subtract the angle between the latitude of Paris and the Equator from 90◦.

$$90° - 48.67° = 41.33°$$

Using the equation for the length of an arc of a circle with a given radius, we can calculate the distance between the North Pole and Paris.

Let "l" be the length of the arc

$$l = \frac{r\pi}{180°} \times \theta$$

Inserting the values for r and θ:

$$= \frac{6400\pi}{180°} \times 41.33°$$

$$= 4616.6053$$

$$\approx 4617\ km$$

We can use the same method to estimate the distance between Paris and the Equator

$$l = \frac{r\pi}{180°} \times \theta$$

$$\frac{6400\pi}{180°} \times 48.67°$$

$$= 5436.4912$$

$$\approx 5436\ km$$

Let us use another example for the same situation. Melbourne is at 37.82◦ S and 144.97◦ E. Let's try to calculate the distance between Melbourne and the North Pole, The South Pole and the Equator, assuming that we are travelling along the meridian line.

The angle between the Equator and Melbourne is already given to be 37.82◦. We already know that each meridian line is a great circle with the radius of Earth, approx 6400km.

Let "l" be the length of the arc:

$$l = \frac{r\pi}{180°} \times \theta$$

Inserting the values for r and θ:

$$= \frac{6400\pi}{180°} \times 37.82°$$

$$= 4224.53°$$

$$\approx 4225\ km$$

To find the angle between Melbourne and the North Pole, we must add 90◦. The reason for adding 90◦ instead of subtracting is due to the fact that Melbourne is located in the Southern Hemisphere.

$$\theta = 90° + 37.82°$$

$$\theta = 127.82°$$

Let "l" be the length of the arc:

$$l = \frac{r\pi}{180°} \times \theta$$

$$l = \frac{6400\pi}{180°} \times 127.82°$$

$$= 14277.63$$

$$\approx 14278\ km$$

To find the angle between Melbourne and the South Pole, we must subtract 90◦.

$$\theta = 90° - 37.82°$$

$$\theta = 52.18°$$

Let "l" be the length of the arc:

$$l = \frac{r\pi}{180°} \times \theta$$

$$l = \frac{6400\pi}{180°} \times 52.18°$$

$$= 5828.56$$

$$\approx 5829\ km$$

Now, let's try to determine the distance between two points on the globe, which share the same latitude. Sydney City and Margaret River both lie on the 33.5◦S parallel latitude. Sydney lies on 151.13◦E and Margaret River lies on 115.04◦E. To calculate the distance between these two points, we would have to calculate the length of the small circle travelling along the 33.5◦S latitude.

First we require to calculate the radius of the small circle connecting the two points:

$$r = 6400 \times \cos 33.5°$$

$$= 5336.869$$

$$\approx 5337\ km$$

The longitudinal angle difference between the two points is:

$$\theta = 151.13° - 115.04°$$

$$= 36.09°$$

Let "l" be the length of the arc:

$$l = \frac{r\pi}{180^\circ} \times \theta$$
$$= \frac{5337\pi}{180^\circ} \times 36.09^\circ$$
$$= 3361.719$$
$$\approx 3362\ km$$

Finally, let's find the distance between any two points on the globe, using only their coordinates. To do this, there are three different techniques, each of varying difficulties. The easiest method is by using the "Spherical Law of Cosines". This method is often used to calculate distances on a perfect sphere, therefore, it tends to be slightly inaccurate while calculating distances on the globe, since the Earth is slightly ellipsoid in shape. The second method, the most commonly used by geoscientists, is the "haversine" calculation method. It is slightly tedious and taxing to calculate using pen and paper, so scientists use a computer to insert the values and calculate the distance. It is usually precise and can be used to find distances between two points on a sphere fairly accurately, however, it has issues finding any two points at each end of a diameter, known as "antipodal points". The third and most complex method is finding the distance between two points on an ellipsoid using the Vincenty Formula. In this case, the sphere is an ellipsoid with equal major and minor axes. However, this is a very complex formula and is not frequently used to calculate distances on the globe. Most geoscientists prefer to use the "Spherical Law of Cosines" equation for its relative simplicity and accuracy.

The Spherical Law of Cosines is given by the equation:

$$D = ((\sin\theta \times \sin\phi) + (\cos\theta \times \cos\phi \times \cos|c|))$$

Where:
"D" Input of arccos in distance equation
"d" is the length of the great circle connecting Points A and B
"θ" is the Latitude of Point A
"Φ" is the Latitude of Point B
"c" is the absolute difference in Longitude between Points A and B

Using this equation, let's find the distance between any two points on the map. Initially, at the start of this paper, I mentioned my flight journey from Madrid to New York. Let's calculate the distance between Adolfo Suárez Madrid–Barajas Airport in Madrid, and John F. Kennedy International Airport in New York. To start off, we require the coordinates of both these airports. Let Madrid Airport be Point A and New York Airport be Point B. Point A has coordinates of (40.4983° N, 3.5676° W) and Point B has coordinates (40.6413° N, 73.7781° W).

The Spherical Law of Cosines is given by the equation:

$$D = ((\sin\theta \times \sin\phi) + (\cos\theta \times \cos\phi \times \cos|c|))$$

$$d = \frac{6400\pi}{180^\circ} \times \cos^{-1}(D)$$

Where:
"D" Input of arccos in distance equation
"d" is the length of the great circle connecting Points A and B
"θ" is the Latitude of Point A
"Φ" is the Latitude of Point B
"c" is the absolute difference in Longitude between Points A and B

Here:
θ = 40.50°
Φ = 40.64°
c = 73.7781° - 3.5676°
= 70.21°

So:

$$D = ((\sin 40.50 \times \sin 40.64) + (\cos 40.50 \times \cos 40.64 \times \cos|70.21|))$$
$$= (0.649 \times 0.651) + (0.760 \times 0.759 \times 0.339)$$
$$= 0.422 + 0.196$$
$$= 0.618$$
$$d = \frac{6400\pi}{180^\circ} \times \cos^{-1}(0.618)$$
$$= 5789.44$$
$$\approx 5790\ km$$

While working on my research paper, my father recommended a website which is used to track and plot the great circle distances between any two points on the globe. Using this website, I found out the actual great circle distance between Madrid Airport and JFK Airport to be approx. 5775 km. My calculations were just 15km off with approximately 0.26% error.

I calculated the percentage error using this formula:

$$\frac{\textit{Actual Great Circle Distance} - \textit{Calculated Great Circle Distance}}{\textit{Actual Great Circle Distance}} \times 100$$

To further my research and for my own practice, I will be calculating the distance between random points on the globe and calculate the percentage error. I will do this for 10 great circle distances and create a spreadsheet displaying my values and the percentage errors.

Table 1: Percentage Error Calculations

Point A	Point B	Great Circle Distance using Spherical Law of Cosines (km)	Actual Great Circle Distance (km)	Difference in Values (km)	Percentage Error (%)
Indira Gandhi International Airport, Delhi	Soekarno–Hatta International Airport, Jakarta	4865.56	4990.54	124.98	2.50%
Athens International Airport, Athens	O. R Tambo International Airport, Johannesburg	7011.16	7107.38	96.22	1.35%
Los Angeles International Airport, Los Angeles	Hong Kong International Airport, Hong Kong	11664.34	11684.87	20.53	0.17%
Hamad International Airport, Doha	John F. Kennedy International Airport, New York	10766.99	10771.08	4.82	0.04%
Heathrow Airport, London	Indira Gandhi International Airport, Delhi	6698. 40	6744.21	45.81	0.68%
Adolfo Suárez Madrid–Barajas Airport, Madrid	John F. Kennedy International Airport, New York	5789.44	5775	14.44	0.26%
Abu Dhabi International Airport, Abu Dhabi	Chhatrapati Shivaji Maharaj International Airport, Mumbai	2009.21	1973.69	35.52	1.81%
Frankfurt Airport, Frankfurt	Dubai International Airport, Dubai	4902.82	4849.96	52.86	1.09%
Paris Charles de Gaulle Airport, Paris	International Airport of Brasilia, Brasilia	8742.17	8729.08	13.09	0.15%
Singapore Changi Airport, Singapore	Abu Dhabi International Airport, Abu Dhabi	5980.52	5887.31	93.21	1.58%

Through my calculations, I found out that the average percentage error was 0.96%. I also noticed a trend with the percentage errors. It was noticed that, as the distance calculated increased, the percentage error fell. This trend can be best noticed by the difference in percentage error values of the great circle distance calculated between Delhi and Jakarta and the great circle distance calculated between Doha and New York. Delhi and Jakarta are close by, so the percentage error was greater, at 2.50%, whilst Doha and New York are quite far away, therefore resulting in a smaller percentage error of 0.04%.

Conclusion

Through my research, I learned a variety of new concepts. Initially, I started off by understanding the key concepts of Non-Euclidean Geometry, and its comparisons and contrasts between Euclidean Geometry. Next, I learned about the Great Circle Theorem and its applications in the real world. I then applied these concepts to calculate the distance between two points on Earth, using only location coordinates. I further applied my knowledge to find the percentage error that I incurred while using the Spherical Law of Cosines. Through my research, I concluded that the Spherical Law of Cosines achieves a higher level of accuracy while being used to calculate larger distances across the Globe.

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