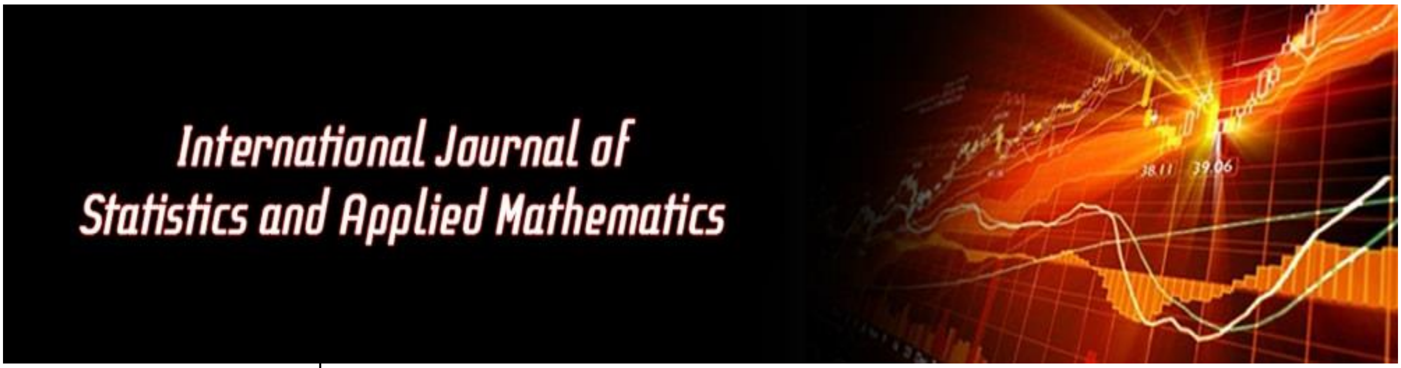

ISSN: 2456-1452
Maths 2022; 7(3): 94-106

www.mathsjournal.com
Received: 07-03-2022
Accepted: 09-04-2022

Tanmoy Kumar Debnath
Department of Business Administration, Shanto-Mariam University of Creative Technology, Dhaka-1230, Bangladesh

Dr. ABM Shahadat Hossain
Department of Applied Mathematics, University of Dhaka, Dhaka-1000, Bangladesh

Toma Debnath
Department of Applied Mathematics, University of Dhaka, Dhaka-1000, Bangladesh

Corresponding Author:
Tanmoy Kumar Debnath
Department of Business Administration, Shanto-Mariam University of Creative Technology, Dhaka-1230, Bangladesh

A numerical comparative analysis between crank-Nicolson finite difference method and binomial model for European call option price

Tanmoy Kumar Debnath, Dr. ABM Shahadat Hossain and Toma Debnath

DOI: https://doi.org/10.22271/maths.2022.v7.i3b.827

Abstract
There are several kinds of numerical techniques for solving option valuation problems. In this article Binomial model (BM) and Crank-Nicolson finite difference (CNFD) approach are applied and compared with the Black-Scholes analytic solution (BSAS) to determine the best numerical method. It has been noticed that, for the valuation of European call (EC) options at various factors, the Binomial model (BM) is found to be more accurate than the Crank-Nicolson finite difference (CNFD) technique. In addition, for the impact of high volatility, the technique of Crank-Nicolson finite difference (CNFD) approaches quicker to Black-Scholes analytic solution (BSAS) than the Binomial model (BM). Furthermore, in comparison to the Binomial model (BM), CNFD method consumes more time to determine the option price in each time step.

Keywords: Black-Scholes model, European call option, binomial model, crank-Nicolson method

1. Introduction

In modern finance, option pricing is a prime accomplishment which plays a vital role in stock market and it arrested the mind of numerous researchers. According to the nature of the option, it is a contract that provides its owner the right, but not the obligation, to purchase or sell an underlying asset or instrument at a certain strike price on or before the maturity date. The two types of options, namely, call and put options are seen in the financial stock market. There are two types of positions available for each option: long and short. European and American options are the most popular option among various styles of the option where the terms European and American do not relate to the option's geographical location. A call option provides the buyer the freedom to purchase asset at a given price by a particular timeline. Only on the expiry of the period can European options be exercised. The owner of a EC option has the right to buy the underlying asset at the date of maturity [1].

The Black-Scholes model 1973 [2] (BSM), established by Fischer Black, Myron Scholes, and Robert Merton at the beginning of the 1970s, is a well-known method for pricing various types of options. The Black-Scholes-Merton model [1] is another name for this method [1]. The theoretical approximate value of the European style option is calculated by this technique, that means, this method is widely utilized due to its ease of calculating the option price analytically. The price of options is always calculated directly using this technique.

The Finite Difference (FD) technique is a computationally effective and convenient approach for solving partial differential (PD) equations and providing an approximate numerical solution for option valuation problems and the optimal early exercise tactics. By solving the differential equation that the derivative satisfies, the FD method can determine the value of a derivative. The differential equation is transformed into a series of difference equations, which are then solved iteratively. The Explicit, Implicit, and Crank-Nicolson are the three types of FD methods. Although these techniques differ in terms of stability, precision, and computing time, they are inextricably linked, and each repetition approach is unique [3].

The BM is a fruitful and very popular technique for the option valuation problem. This approach can be used to solve some complicated option pricing models because of its simplicity and reliability.

It is notably widely used for valuing American options, which can be exercised at any moment up to and including the option's expiry date. The period till expiry is divided into many phases in this concept. This tree diagram depicts multiple probable stock price pathways during the course of an option's life. This model is identical to the BSM as the time steps are smaller. When the number of time steps approaches infinity, the popular Cox-Ross-Rubinstein binomial model [4] (CRRBM) contains the Black-Scholes analytic formula.

One of the most important contributors to the field of finance was Black F, & Scholes M 1973 [2] and Merton RC 1973 [6]. Following Louis Bachelier's early work on the option, Black F, and Scholes M 1973 [2] achieved a key breakthrough by deducing a differential equation that must be satisfied by the price of any derivative and producing a closed-form solution for the theoretical price of a European option. Later, Merton RC 1973 [6] devised a jump-diffusion model. Based on risk-neutral valuation the tree methods of option pricing derived by Cox JC, Ross SA, Rubinstein M 1979 [4]. For the valuation of derivatives securities, Hull J, White A 1990 [7] employed the Explicit FD approach. A Monte Carlo method for option pricing introduced by Boyle PP 1977 [8]. Boyle P, Broadie M, Glasserman P 1997 [9] presented research improvements with enhanced efficiency and expanded the types of problems where simulation can be used after twenty years. Brennan MJ, Schwartz ES 1978 [10] explore a FD method for American option pricing. For the time derivative, Tavella D, Randall C 2000 [11] employed a slightly stabilized CNFD approach and FDs for the space derivatives. Fadugba S, Nwozo C, Babalola T 2012 [12] compared the Monte Carlo approach to the FD method for calculating the European option price. Fadugba SE, Okunlola JT, and Adeyemo AO 2013 [3] investigated the effectiveness of the BM and the FD techniques for pricing European options. For pricing some path dependent choices, Nwozo CR, Fadugba SE 2012 [13] used the Monte Carlo approach. Debnath TK, Hossain AS [14] conducted a comparison study of the Implicit and CNFD methods for option pricing.

Only numerical comparison of the BM and CNFD approach for EC options will be discussed in this study and all numerical findings and graphics will be generated using MATLAB.

Table 1: This article's glossary of abbreviations

Abbreviation	Explanation
BSM	Black-Scholes Model
BSPDE	Black-Scholes Partial Differential Equation
BSAS	Black-Scholes Analytic Solution
BM	Binomial Model
CRRBM	Cox-Ross-Rubinstein Binomial Model
FD	Finite Difference
IFD	Implicit Finite Difference
EFD	Explicit Finite Difference
CNFD	Crank-Nicolson Finite Difference
EC	European Call
EP	European Put
PD	Partial Differential

2. Methodology

The BSM and two numerical approaches for pricing EC options, the CNFD method and the BM, are presented in this section.

2.1 Black-Scholes Model

Fischer Black, Myron Scholes, and Robert Merton [2, 5, 6] made a significant milestone in the pricing of European options at the beginning of the 1970s. This renowned achievement is known as the BSM.

2.1.1 Black-Scholes Differential Equation

A riskless portfolio made up of an asset with value S and an option with value $v(t,S)$ satisfies the PD equation shown by Black, Scholes, and Merton.

$$\frac{\partial v(t,S)}{\partial t} + rS\frac{\partial v(t,S)}{\partial S} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 v(t,S)}{\partial S^2} = rv(t,S) \tag{1}$$

Here v denotes the option price as a well-established [15] function of time t and stock price S, r is the risk-free rate of interest and σ stands for the stock's volatility. This above equation is called Black-Scholes partial differential equation (BSPDE).

Equation (1) contains a large number of solutions, which correspond to all of the distinct derivatives that may be defined with S as the underlying variable. The specific derivatives can be calculated by solving the equation (1) with boundary conditions [1].

The boundary condition for a EP option is

$$v = \max(X - S, 0) \text{ when } t = T$$

The boundary condition for the EC option is

$$v = \max(S - X, 0) \text{ when } t = T$$

2.1.2 Black-Scholes option pricing Formula

For the price of EC and EP options the Black-Scholes formulas are the most well-known solutions of equation (1). These formulas are following:

$$c = N(d_1)S - N(d_2)Xe^{-rT} \tag{2}$$

$$p = N(-d_2)Xe^{-rT} - N(-d_1)S \tag{3}$$

Where

$$d_1 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + T\left(r + \frac{\sigma^2}{2}\right)\right] \tag{4}$$

$$d_2 = \frac{1}{\sigma\sqrt{T}}\left[\ln\left(\frac{S}{X}\right) + T\left(r - \frac{\sigma^2}{2}\right)\right] = d_1 - \sigma\sqrt{T} \tag{5}$$

Where c stands for call option, p stands for put option, S denotes the current stock price, t denotes the option maturity time, X stands for strike price, r stands for risk-free rate of interest, and N stands for cumulative standard normal distribution.

The processes for the implementation of the BM and the CNFD method for option pricing are presented in the following sub-sections.

2.2 Binomial Model

This model is an iterative solution for the valuation of option price throughout an option's expiry period. The iterative approach is the only choice for some options, such as the American option, because there is no known closed form solution that predicts the price over time. The derivation and implementation of the BM will be presented here.

Firstly, we split the lifetime $[0,T]$ of the option into N times subinterval of length Δt, where $\Delta t = T/N$. Suppose that at the start of a period S_0 is the stock price. Then the BM of price movements assumes that at the end of each time period, the price will either rise to uS_0 with probability p_u or down to dS_0 with probability $1 - p_u = p_d$,where u and d are the up and down factors with $0 < d < 1 < u$.

Let v represents the value of a EC option on the stock with a strike price of X. As a result, at the first time movement Δt, the call option's associated value is given by

$$v_u = \max(uS_0 - X, 0)$$
$$v_d = \max(dS_0 - X, 0)$$

Where v_u and v_d stand for the value of the call option after the upward and downward movement respectively.

At the beginning point $N = 1$, we establish a portfolio consisting of a long position in h shares and a short position in one call option to determine the value of v one period before expiry. We find the value of h that ensures the portfolio is risk-free. If the stock price rises, the portfolio value at the end of the option's lifespan is $huS_0 - v_u$, and if the stock price falls, the portfolio value is $hdS_0 - v_d$.

This two are equal when

$$huS_0 - v_u = hdS_0 - v_d$$

Or $h(uS_0 - dS_0) = v_u - v_d$

Or $h = \frac{v_u - v_d}{S_0(u-d)}$ (6)

The portfolio is riskless in the aforementioned scenario and must earn the risk-free interest rate. As we move between the nodes, equation (6) shows that h is the ratio of the change in option price to the change in stock price. The present value of the portfolio is $(huS_0 - v_u)e^{-r\Delta t}$ when the risk-free rate of interest is denoted by r . The portfolio setup cost is $hS_0 - v.$ It follows that

$$hS_0 - v = (huS_0 - v_u)e^{-r\Delta t}$$

or $v = hS_0 - (huS_0 - v_u)e^{-r\Delta t}$ (7)

Substituting the value of (6) into (7) and simplifying, then equation (7) becomes

$$v = e^{-r\Delta t}\left[\frac{e^{r\Delta t} - d}{u-d}v_u + \left(1 - \frac{e^{r\Delta t} - d}{u-d}\right)v_d\right]$$

or $$v = e^{-r\Delta t}\left(\frac{e^{r\Delta t} - d}{u-d}v_u + \frac{u - e^{r\Delta t}}{u-d}v_d\right)$$

Or $v = e^{-r\Delta t}(p_u v_u + p_d v_d)$ (8)

Where

$$p_u = \frac{e^{r\Delta t} - d}{u-d} \quad (9)$$

$$p_d = \frac{u - e^{r\Delta t}}{u-d} \quad (10)$$

For $N = 1,\ \Delta t = T$, then equation (8), (9) and (10) becomes

$$v = e^{-rT}(p_u v_u + p_d v_d)$$
(11)

Where

$$p_u = \frac{e^{rT} - d}{u-d} \quad (12)$$

$$p_d = \frac{u - e^{rT}}{u-d} \quad (13)$$

This is a one-step BM. Using a one-step BM, we determine the price of the option using equations (11), (12), and (13).
For two time step BM, at time $2\Delta t$ for two consecutive upward stock movement the stock price will be u^2S_0 where the call value will be v_{u^2} , for one upward and one downward movement the stock price will be udS_0 where the call value will be v_{ud} and for two consecutive downward movement the stock price will be d^2S_0 where the call value will be v_{d^2} .
Then we have

$$v_{u^2} = \max(u^2S_0 - X, 0)$$
$$v_{ud} = \max(udS_0 - X, 0) \quad (14)$$
$$v_{d^2} = \max(d^2S_0 - X, 0)$$

Now the repeated application of (8) gives

$$v_u = e^{-r\Delta t}(p_u v_{u^2} + p_d v_{ud}) \tag{15}$$

$$v_d = e^{-r\Delta t}(p_u v_{ud} + p_d v_{d^2}) \tag{16}$$

Substituting equation (15) and (16) into equation (8) and simplifying, then equation (8) becomes

$$v = e^{-r\Delta t}[p_u e^{-r\Delta t}(p_u v_{u^2} + p_d v_{ud}) + p_d e^{-r\Delta t}(p_u v_{ud} + p_d v_{d^2})]$$

Or $v = e^{-2r\Delta t}(p_u^2 v_{u^2} + 2 p_u p_d v_{ud} + p_d^2 v_{d^2})$ (17)

Equation (17) is called the current call value using time $2\Delta t$, the variables p_u^2, $2p_u p_d$ and p_d^2 are the probabilities that help us to obtain $u^2 S_0$, udS_0 and $d^2 S_0$ respectively.
We generalize the equation (17) at time $\Delta t = T / N$ as,

$$v = e^{-rN\Delta t}\sum_{j=0}^{N} {}_N C_j p_u^j p_d^{N-j} v_{u^j d^{N-j}}$$

Or $v = e^{-rN\Delta t}\sum_{j=0}^{N} {}_N C_j p_u^j p_d^{N-j} \max(S_0 u^j d^{N-j} - X, 0)$ (18)

Where

$$v_{u^j d^{N-j}} = \max(S_0 u^j d^{N-j} - X, 0) \text{ and } {}_N C_j = \frac{N!}{j!(N-j)!}$$

n is the smallest integer for which the option's intrinsic value in equation (18) is greater than 0. This implies that $S_0 u^n d^{N-n} \geq X$. Then equation (18) can be written as

$$v = S_0 e^{-rN\Delta t}\sum_{j=n}^{N} {}_N C_j p_u^j p_d^{N-j} u^j d^{N-j} - X e^{-rN\Delta t}\sum_{j=n}^{N} {}_N C_j p_u^j p_d^{N-j} \tag{19}$$

By equation (19) we can evaluate the current value of the call option. The term $e^{-rN\Delta t}$ is the discounting factor that reduces v to its present value. In the first term ${}_N C_j p_u^j p_d^{N-j}$ is the binomial probability of j upward movements to occur after the first N trading periods and $S_0 u^j d^{N-j}$ is the corresponding value of the asset after j upward move of the stock price. The second term is the present value of the option's strike price.
Now putting $R = e^{r\Delta t}$ in equation (19) we get,

$$v = S_0 R^{-N}\sum_{j=n}^{N} {}_N C_j p_u^j p_d^{N-j} u^j d^{N-j} - X R^{-N}\sum_{j=n}^{N} {}_N C_j p_u^j p_d^{N-j}$$

Or $v = S_0\left[\sum_{j=n}^{N} {}_N C_j (R^{-1} p_u u)^j (R^{-1} p_d d)^{N-j}\right] - XR^{-N}\left[\sum_{j=n}^{N} {}_N C_j p_u^j p_d^{N-j}\right]$ (20)

The terms in the brackets are binomial distribution functions and let $R^{-1} p_u u = p_u^*$ and $R^{-1} p_d d = p_d^*$, then equation (20) becomes

$$v = S_0 \beta(n; N, p_u^*) - XR^{-N}\beta(n; N, p_u) \tag{21}$$

The model in equation (21) was established by Cox-Ross-Rubinstein which is known as the CRRBM for the valuation of EC option. The value of a EP option can be calculated using the put-call parity formula.

$$v_c^E + Xe^{-rT} = v_p^E + S_0$$

where v_c^E represents the EC option's value and v_p^E means the value of the EP option.

2.3 Finite Difference Method

By solving PD equations with specific initial and boundary conditions, several option contract values can be determined. One of the most popular methods for solving PD equations is the FD method. The Implicit, Explicit, and Crank-Nicolson techniques are the three most used FD methods for solving the BSPDE. Only the CNFD method will be discussed in this study.

2.3.1 Discretization of the Equation

With respect to time, t, and the underlying asset price, S, we discretize the equation (1). Divide the (t, S) plane into a sufficiently thick grid by using tiny increments ΔS and Δt. We define an array of $N+1$ evenly spaced grid points $t_0, \ldots, t_N$ to discretize the time derivative with $t_{i+1} - t_i = \Delta t$ and $\Delta t = \frac{T}{N}$.

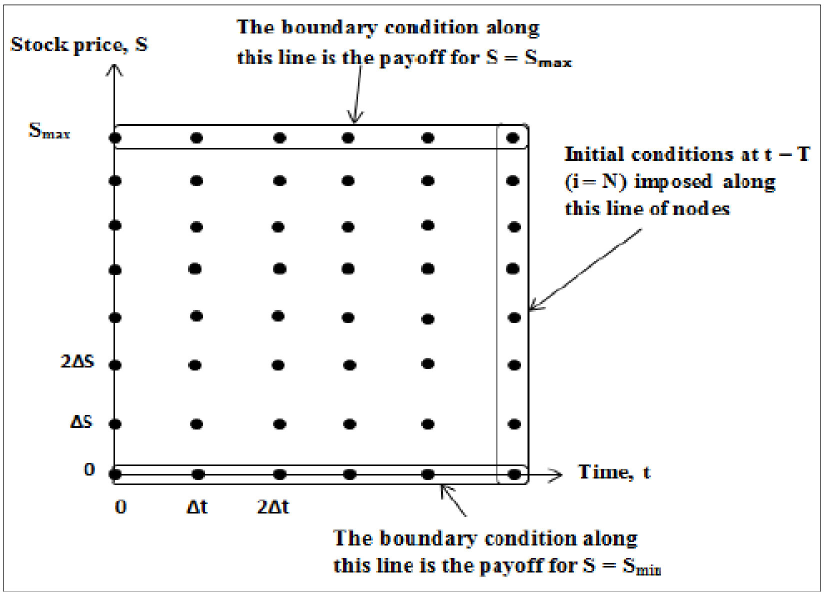

Fig 1: The grid points for the FD approximations

We know that the stock value cannot fall below 0 and highest stock price can be represented by $S_{\max}$. We have $M+1$ equally spaced grid points $S_0, \ldots, S_M$ to discretize the stock price derivative with $S_{j+1} - S_j = \Delta S$ and $\Delta S = \frac{S_{\max}}{M}$.

When the underlying asset has value S_j, we indicate the value of the derivative at time step t_i as

$$v_{i,j} = v(i\Delta t, j\Delta S) = v(t_i, S_j) = v(t, S)$$

where i and j denote the number of discrete holding period and stock value increments, respectively, Δt and ΔS denote the discrete time to expiration and stock market price increments. The quantities $v_{N,j}$ for $j = 0, 1, 2, \ldots, M$, $v_{i,0}$ and $v_{i,M}$ for $i = 0, 1, \ldots, N$ are referred to as the boundary values which may or may not be known ahead of time but in our PD equation they are known. The quantities $v_{i,j}$ for $i = 0, 1, \ldots, N-1$ and $j = 1, 2, \ldots, M-1$ are referred to as interior points or values.

2.3.2 Boundary Conditions

At time instant $i\Delta t$, in order to solve equation (1), we need the option values at the following marginal boundary.

1. The asset value at the upper level
2. The asset value at the lowest level
3. At option expiration date, the defined initial values.

In Fig 1, the value of a EC option at the upper asset boundary is:

$$v_{i,M} = M\Delta S - Xe^{-r(N-i)\Delta t}, \qquad i = 0,1,\ldots,N \tag{22}$$

The price of EC options at the lower asset boundary is:

$$v_{i,0} = 0, \qquad i = 0,1,\ldots,N \tag{23}$$

Now at the option expiration date, the preliminary EC option price is:

$$v_{N,j} = \max(j\Delta S - X, 0), \qquad j = 0,1,\ldots,M \tag{24}$$

Equations (22), (23) and (24) define the prices of the EC option along the three borders of the grid in Fig 1, where $S = S_{\max}$, $S = 0$ and $t = T$.

2.3.3 Finite Difference Approximations

In the FD method, the partial derivative occurring in the PD equation is replaced by approximations based on Taylor series expansions of function near the points of interest.

Consider the option price $v(t,S)$ for an inner grid point (i, j). In Taylor's series, the expansion of $v(t, S+\Delta S)$ can be written as

$$\frac{\partial v}{\partial S} = \frac{v_{i,j+1} - v_{i,j}}{\Delta S}. \tag{25}$$

The above equation is called first forward difference approximation. The Taylor's series expansion of $v(t, S-\Delta S)$ can be expressed as

$$\frac{\partial v}{\partial S} = \frac{v_{i,j} - v_{i,j-1}}{\Delta S}. \tag{26}$$

Which is called first backward difference approximation. The first order partial derivative results in the central difference given by

$$\frac{\partial v}{\partial S} = \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta S}. \tag{27}$$

The symmetric central difference approximation can be used to estimate second order partial derivatives. which can be written as

$$\frac{\partial^2 v}{\partial S^2} = \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta S^2}. \tag{28}$$

The Taylor's series expansion of $f(t+\Delta t, S)$ and $f(t-\Delta t, S)$ is called the forward and backward difference approximation for the maturity time respectively.

$$\frac{\partial v}{\partial t} = \frac{v_{i+1,j} - v_{i,j}}{\Delta t}. \tag{29}$$

$$\frac{\partial v}{\partial t} = \frac{v_{i,j} - v_{i-1,j}}{\Delta t}. \tag{30}$$

Now through the above equations, equation (1) will be transformed into difference equation which will be solved iteratively to figure out the approximate value of $v(t,S)$.

For Implicit finite difference (IFD) method, substituting equations (27), (28) and (29) into the equation (1) and replace S by $j\Delta S$ we obtain,

$$a_j v_{i,j-1} + b_j v_{i,j} + c_j v_{i,j+1} = v_{i+1,j} \qquad (31)$$

for $j = 1, 2, ..., M-1$ and $i = 0, 1, ..., N-1$

Where

$$a_j = \frac{1}{2} rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t;$$
$$b_j = 1 + \sigma^2 j^2 \Delta t + r\Delta t;$$
$$c_j = -\frac{1}{2} rj\Delta t - \frac{1}{2}\sigma^2 j^2 \Delta t.$$

Here we express $v_{i+1,j}$ implicitly in-terms of the unknowns $v_{i,j-1}$, $v_{i,j}$ and $v_{i,j+1}$. This method has accuracy up to $O(\Delta t, \Delta S^2)$.

For Explicit finite difference (EFD) method substituting Equations (27), (28) and (30) into the equation (1) and noting that $S = j\Delta S$ and rearranging terms we obtain,

$$a_j^* v_{i,j-1} + b_j^* v_{i,j} + c_j^* v_{i,j+1} = v_{i-1,j}$$

For $j = 1, 2, ..., M-1$ and $i = N, ..., 1$

Where

$$a_j^* = -\frac{1}{2} rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t;$$
$$b_j^* = 1 - \sigma^2 j^2 \Delta t - r\Delta t;$$
$$c_j^* = \frac{1}{2} rj\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t.$$

2.3.4 Crank-Nicolson Finite Difference Method

The CNFD method represents an average of the IFD and the EFD method. To discretize the equation (1), At the point $v_{i-1/2,j}$, we apply a central approximation for $\frac{\partial v}{\partial t}$:

$$\frac{\partial v_{i-1/2,j}}{\partial t} = \frac{v_{i,j} - v_{i-1,j}}{\Delta t} + O(\Delta t^2)$$

At the point $v_{i-1/2,j}$, we apply a central approximation for $\frac{\partial v}{\partial S}$:

$$\frac{\partial v_{i-1/2,j}}{\partial S} = \frac{1}{2}\left[\frac{\partial v_{i-1,j}}{\partial S} + \frac{\partial v_{i,j}}{\partial S}\right] = \frac{1}{2}\left[\frac{v_{i-1,j+1} - v_{i-1,j-1}}{2\Delta S} + \frac{v_{i,j+1} - v_{i,j-1}}{2\Delta S}\right] + O(\Delta S^2)$$

We use a standard approximation for $\frac{\partial^2 v}{\partial S^2}$ at the point $v_{i-1/2,j}$:

$$\frac{\partial^2 v_{i-1/2,j}}{\partial S^2} = \frac{1}{2}\left[\frac{\partial^2 v_{i-1,j}}{\partial S^2} + \frac{\partial^2 v_{i,j}}{\partial S^2}\right]$$

$$= \frac{1}{2}\left[\frac{v_{i-1,j+1} - 2v_{i-1,j} + v_{i-1,j-1}}{\Delta S^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{\Delta S^2}\right] + O(\Delta S^2).$$

When these approximations are substituted into the BS PDE (1), we obtain,

$$-a_j^{'} v_{i-1,j-1} + (1 - b_j^{'})v_{i-1,j} - c_j^{'} v_{i-1,j+1} = a_j^{'} v_{i,j-1} + (1 + b_j^{'})v_{i,j} + c_j^{'} v_{i,j+1} \tag{32}$$

Where

$$a_j^{'} = \frac{\Delta t}{4}(\sigma^2 j^2 - rj)$$

$$b_j^{'} = -\frac{\Delta t}{2}(\sigma^2 j^2 + r)$$

$$c_j^{'} = \frac{\Delta t}{4}(\sigma^2 j^2 + rj)$$

This method has accuracy up to $O(\Delta t^2, \Delta S^2)$.

3. Numerical Experiments and Results

This section demonstrates numerical implementations of CNFD method and BM for EC option price. Numerical example and analysis of results generated as follows:

3.1 Numerical Example

In case of EC option, we will compare the correctness of the BM and CNFD methods to the BSAS for the following factors.

$$S = 42, X = 40, r = 0.10, \sigma = 0.2, T = 0.5, 3$$

The results obtained are shown in Table 1 for $T = 0.5$ and Table 2 for $T = 3$.

For different rate interest, in case of EC option, we will compare the robustness of the BM and CNFD methods to the BSAS for the following parameters.

$$S = 42, X = 40, r = 0.15, 0.20, \sigma = 0.2, T = 0.5$$

The results obtained are shown in Table 3 for $r = 0.15$ and Table 4 for $r = 0.20$.

For different value of volatility, in case of EC option, we will compare the effectiveness of the BM and CNFD methods to the BSAS for the following variables.

$$S = 42, X = 40, r = 0.10, \sigma = 0.25, 0.30, 0.45, T = 0.5$$

The results obtained are shown in Table 5 for $\sigma = 0.25$, Table 6 for $\sigma = 0.30$ and Table 7 for $\sigma = 0.45$. The results generated from the BM and CNFD method are represented in the following tables and figures by using MATLAB.

3.2 Table of Results

Table 1: For $T = 0.5$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 4.7594					
	Numerical Methods		Absolute Error		Computation Time	
	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson
25	4.7721	4.3200	0.0127	0.4394	0.000031	0.005166
50	4.7615	4.6981	0.0021	0.0613	0.000062	0.004855
75	4.7534	4.7426	0.0060	0.0168	0.000166	0.025588
100	4.7618	4.7381	0.0024	0.0213	0.000183	0.008955
150	4.7585	4.7486	0.0009	0.0108	0.000337	0.053620
200	4.7614	4.7570	0.0020	0.0024	0.000560	0.055422
250	4.7603	4.7518	0.0009	0.0076	0.000869	0.068822
300	4.7580	4.7582	0.0014	0.0012	0.001872	0.101147
350	4.7598	4.7576	0.0004	0.0018	0.001670	0.173381
400	4.7604	4.7580	0.0010	0.0014	0.002171	0.347074
450	4.7601	4.7589	0.0007	0.0005	0.002756	0.429753
500	4.7593	4.7575	0.0001	0.0019	0.003331	0.570848
1000	4.7598	4.7589	0.0004	0.0005	0.013263	4.401673
2000	4.7595	4.7593	0.0001	0.0001	0.054732	31.954687
3000	4.7596	4.7594	0.0002	0.0000	0.131168	105.625431
4000	4.7594	4.7594	0.0000	0.0000	0.267547	260.038199
5000	4.7594	4.7594	0.0000	0.0000	0.457616	540.597542

Table 2: For $T = 3$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 13.3627					
	Numerical Methods		Absolute Error		Computation Time	
	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson
50	13.3671	13.3187	0.0044	0.0440	0.00006	0.83765
100	13.3628	13.3493	0.0001	0.0134	0.00018	0.04477
200	13.3567	13.3601	0.0060	0.0026	0.00061	0.06844
500	13.3631	13.3618	0.0004	0.0009	0.00340	0.59343
1000	13.3625	13.3624	0.0002	0.0003	0.01329	4.53413
2000	13.3624	13.3626	0.0003	0.0001	0.05749	31.84802
4000	13.3627	13.3626	0.0000	0.0001	0.26609	260.84349

Table 3: For $r = 0.15$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 5.4759					
	Numerical Methods		Absolute Error		Computation Time	
	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson
50	5.4767	5.3867	0.0008	0.0892	0.00006	0.60356
100	5.4776	5.4508	0.0017	0.0251	0.00023	0.04681
200	5.4773	5.4721	0.0014	0.0038	0.00059	0.06447
500	5.4758	5.4740	0.0001	0.0019	0.00342	0.58952
1000	5.4762	5.4754	0.0003	0.0005	0.01441	4.37571
2000	5.4760	5.4758	0.0001	0.0001	0.06154	32.78744
4000	5.4759	5.4759	0.0000	0.0000	0.26092	281.14032

Table 4: For $r = 0.20$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 6.2214					
	Numerical Methods		Absolute Error		Computation Time	
	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson	Binomial	Crank-Nicolson
50	6.2209	6.1043	0.0005	0.1171	0.00006	0.655384
100	6.2223	6.1928	0.0009	0.0286	0.00017	0.01446
200	6.2223	6.2163	0.0009	0.0051	0.00066	0.03786
500	6.2212	6.2195	0.0002	0.0019	0.00358	0.61897
1000	6.2216	6.2209	0.0002	0.0005	0.01323	4.39779
2000	6.2215	6.2213	0.0001	0.0001	0.05508	31.86016
4000	6.2214	6.2214	0.0000	0.0000	0.26191	270.87098

Table 5: For $\sigma = 0.25$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 5.2220					
	Numerical Methods		**Absolute Error**		**Computation Time**	
	Binomial	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**
50	5.2117	5.1840	0.0103	0.0380	0.00006	0.00870
100	5.2268	5.2053	0.0048	0.0167	0.00017	0.00918
200	5.2199	5.2204	0.0021	0.0016	0.00060	0.03373
500	5.2214	5.2204	0.0006	0.0016	0.00488	0.59969
1000	5.2224	5.2216	0.0004	0.0004	0.01963	4.33869
2000	5.2220	5.2219	0.0000	0.0001	0.05807	32.94561
4000	5.2220	5.2219	0.0000	0.0001	0.26252	281.28027

Table 6: For $\sigma = 0.30$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 5.7147					
	Numerical Methods		**Absolute Error**		**Computation Time**	
	Binomial	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**
50	5.7179	5.6896	0.0032	0.0251	0.000087	0.758767
100	5.7144	5.7012	0.0003	0.0135	0.000327	0.040250
200	5.7179	5.7136	0.0032	0.0011	0.000575	0.082516
500	5.7161	5.7134	0.0014	0.0013	0.003367	0.616354
1000	5.7153	5.7144	0.0006	0.0003	0.013318	4.451367
2000	5.7147	5.7146	0.0000	0.0001	0.057662	32.568759
4000	5.7148	5.7147	0.0001	0.0000	0.266701	274.103842

Table 7: For $\sigma = 0.45$, the efficiency of the BM and CNFD method in comparison to BSAS for the EC option

N	Black-Scholes European Call Price, E_c = 7.2745					
	Numerical Methods		**Absolute Error**		**Computation Time**	
	Binomial	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**	**Binomial**	**Crank-Nicolson**
50	7.2976	7.2641	0.0231	0.01040	0.000284	0.005271
100	7.2796	7.2661	0.0051	0.00840	0.000535	0.009443
200	7.2720	7.2740	0.0025	0.00040	0.000934	0.040073
500	7.2760	7.2736	0.0015	0.00087	0.003571	0.585527
1000	7.2756	7.2743	0.0011	0.00021	0.013462	4.389413
2000	7.2751	7.2745	0.0006	0.00005	0.052841	32.606984
4000	7.2745	7.2745	0.0000	0.00000	0.230885	278.254408

3.3 Discussion of Results

The comparative numerical option values of BM and CNFD method against Black-Scholes analytic option price for EC option are presented in the Table 1 help to illustrate the variation of option price with different time step size N. Table 1 demonstrates that the number of time step size N increases then Δt decreases (i.e., $N \to \infty, \Delta t \to 0$) and the option values for both numerical methods converge to BSAS.

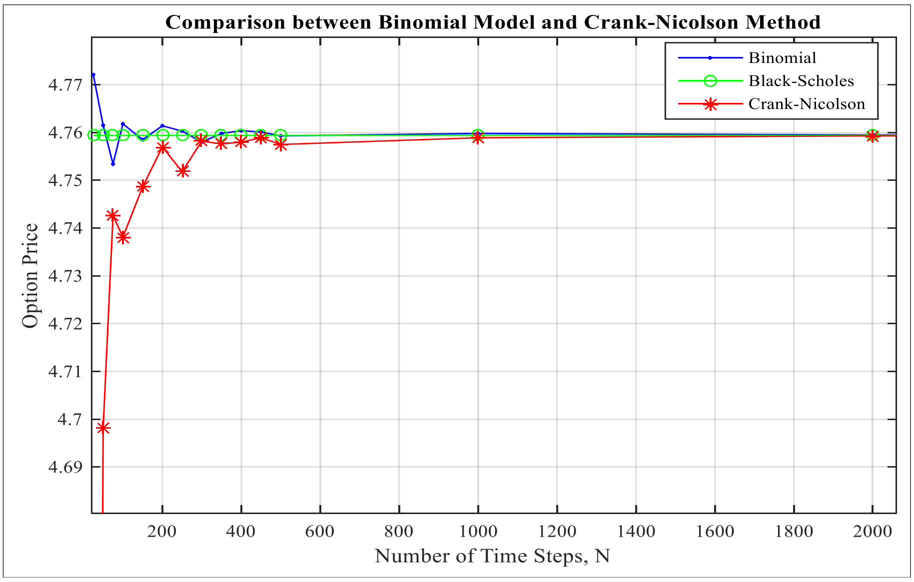

Fig 2: EC option prices against the number of time steps, N for BM and CNFD method.

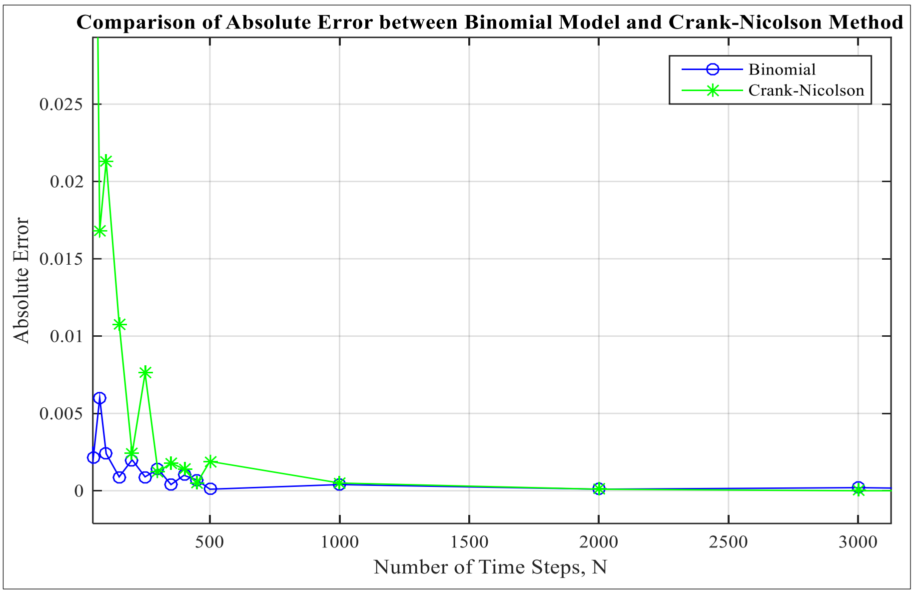

Fig 3: Absolute error of EC option prices against the number of time steps, N for BM and CNFD method

That means, when the time step size N is increasing the obtained absolute error is declining which is clearly shown by the graph in Fig 1 and 2. Fig 1 shows that, at the start both method have fluctuations but as the number of time steps rises, the irregular movement of the prices of option decreases and the numerical results using BM gives better results than that of using CNFD technique as well as converges to the BSAS faster than CNFD method.

For the maturity time $T = 3$ years the option value has increased over the option value of maturity time $T = 0.5$ years and the results using BM is still better than that of using CNFD method as shown at Table 2.

Among the six factors which affect the option price, when the value of rate of interest r increases, the value of the option price increases. Based on this concepts for $r = 0.15$ and 0.20, Table 3 and 4 shows that the option prices are increased and the BM moves faster to the BSAS than CNFD technique as the number of time steps N increases which is clearly shown in Fig 4.

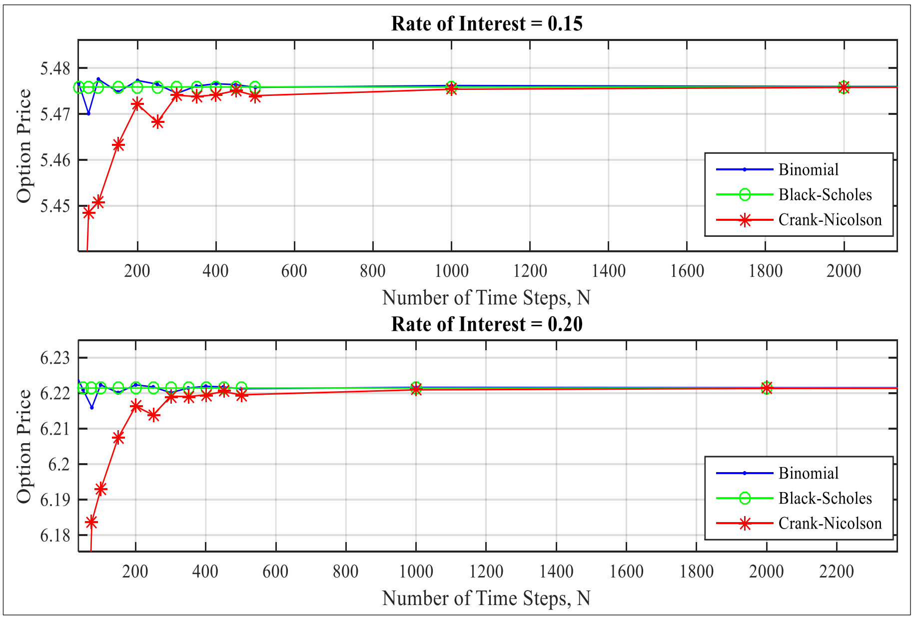

Fig 4: EC option prices against the number of time steps, N for BM and CNFD method (volatility, $r = 0.15$ and 0.20)

For $\sigma = 0.20$, Table 1 shows that BM works very well for the valuation of EC option comparative to CNFD approach. But when volatility increases, i.e., for $\sigma = 0.25, 0.30$ and 0.45, from Table 5, 6 and 7 we figure out less absolute error for CNFD method comparative to BM as the number of time steps increases. That means, CNFD method gives better option price results than that of using BM which is clearly shown in Fig 5.

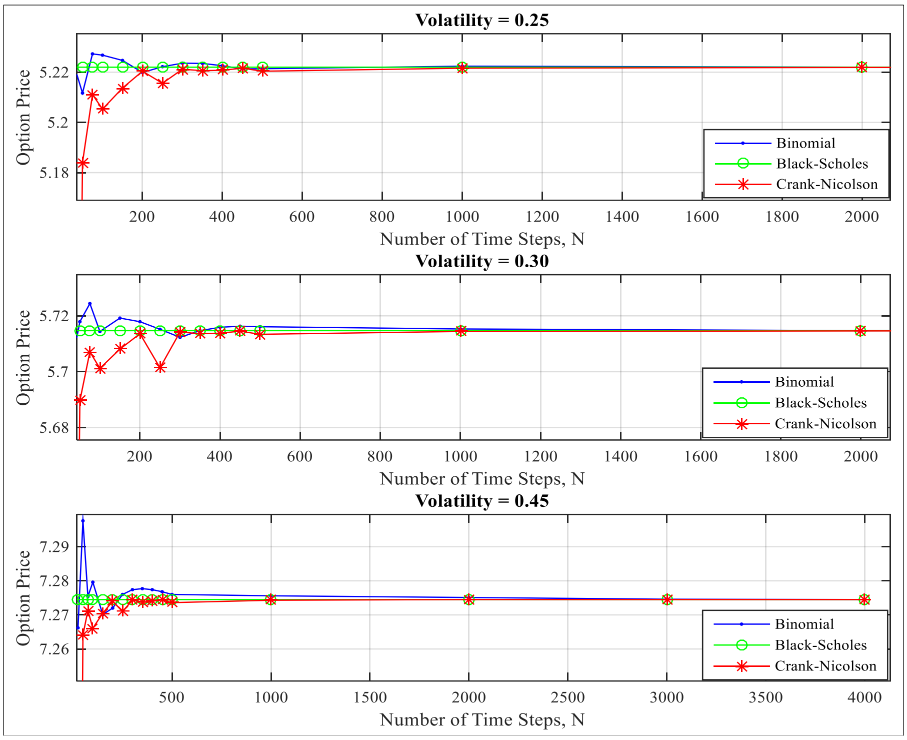

Fig 5: EC option prices against the number of time steps, N for BM and CNFD method (volatility, $\sigma = 0.25, 0.30$ and 0.45)

Besides the comparative option prices and the absolute error, the total execution time to evaluate the EC option price (for the factors maturity time $T = 0.5$ and 3, rate of interest $r = 0.10, 0.15$ and 0.20, volatility $\sigma = 0.20, 0.25, 0.30$ and 0.45) is presented in the last two columns of all tables. From tables we can see that the number of time steps has an influence on execution time. The results show that when the time steps N go up, the computation time also increases for the both numerical method but the total elapsed time to run CNFD method is more than that of BM. That means, the BM is faster than the CNFD method.

4. Conclusions

Based on the result and discussion above, it can be concluded that as the number of time steps increase, a numerical solution using BM is more accurate than that of using CNFD method for the valuation of EC option. But when volatility increases, CNFD method converges more quickly to the BSAS than the BM. That means, for higher volatility, the option price solution using CNFD techniques is better than that of using BM. As the number of time steps increase, the computation time in both methods also goes up where the total computation time to run CNFD method is always higher than that of BM.

5. References

1. Hull JC. Option, Futures and Other Derivatives. 8th edition, Pearson Education; c2012.
2. Black F, Scholes M. The valuation of options and corporate liabilities. Journal of Political Economy. 1973;81(3):637-54.
3. Fadugba SE, Okunlola JT, Adeyemo AO. On the robustness of binomial model and finite difference method for pricing European options. International Journal of IT, Engineering and Applied Sciences Research. 2013;2(2):5-11.
4. Cox JC, Ross SA, Rubinstein M. Option pricing: A simplified approach. Journal of financial Economics. 1979;7(3):229-63.
5. Merton RC. Option pricing when underlying stock returns are discontinuous. Journal of financial economics. 1976;3(1-2):125-44.
6. Merton RC. Theory of rational option pricing. The Bell Journal of economics and management science. 1973;4(1):141-83.
7. Hull J, White A. Valuing derivative securities using the explicit finite difference method. Journal of Financial and Quantitative Analysis. 1990;25(1):87-100.
8. Boyle PP. Options: A monte carlo approach. Journal of financial economics. 1977;4(3):323-38.
9. Boyle P, Broadie M, Glasserman P. Monte Carlo methods for security pricing. Journal of economic dynamics and control. 1997;21(8-9):1267-321.
10. Brennan MJ, Schwartz ES. Finite difference methods and jump processes arising in the pricing of contingent claims: A synthesis. Journal of Financial and Quantitative Analysis. 1978;13(3):461-74.
11. Tavella D, Randall C. Pricing financial instruments: The finite difference method. John Wiley & Sons; c2000, p 13.
12. Fadugba S, Nwozo C, Babalola T. The comparative study of finite difference method and Monte Carlo method for pricing European option. Mathematical Theory and Modeling. 2012;2(4):60-7.
13. Nwozo CR, Fadugba SE. Monte Carlo method for pricing some path dependent options. International Journal of Applied Mathematics. 2012;25(6):763-78.
14. Debnath TK, Hossain AS. A Comparative Study between Implicit and Crank-Nicolson Finite Difference Method for Option Pricing. GANIT: Journal of Bangladesh Mathematical Society. 2020;40(1):13-27.
15. Joshi M.S. The Concepts and Practice of Mathematical Finance. 2nd Edition, Cambridge University Press; c2008, p116pp.