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A family of r-points 1-block implicit methods with optimized region of stability for stiff initial value problems in ordinary differential equations

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Abstract
In this paper, we proposed a family of r-points 1-block implicit methods with optimized region of stability. This family of methods is derived with Mathematical 10.4 software and the stability is investigated using boundary locus techniques. The block methods are consistence, zero stable, and A-stable and satisfy other stability requirements which finds them suitable for stiff problems in ODEs. Numerical experiments are presented and results are compared with other block methods and exact solutions of some stiff ordinary differential equations. The methods have been found to show competitiveness with other numerical methods.

Keywords: Block method, stiffness, initial value problems, ordinary differential equation, collocation and interpolation, A-stability, boundary locus, linear multistep methods

Introduction

Differential equations arise in many areas of science and technology, specifically whenever a deterministic relation involving some continuously varying quantities (modeled by functions) and the rate of change in space or time (expressed as derivatives) is known. Such areas include Hamilton's equations in classical mechanics, Newton's second law in dynamics; Radioactive decay in nuclear complex analysis; the heat equation in thermodynamics; Verhulst equation in biological population growth, Lotka volterra equations in biological population dynamics; exogenous growth model etc.

The history of differential equations is traced from calculus, which was independently invented by English physicist Isaac Newton in 1676 and a German mathematician Gottfried Leibniz in 1693. Other mathematicians like Jacob Bernoulli and Leonard Euler also made remarkable contributions to the solutions of differential equations which also arise as a result of some physical phenomena in every day's life.

Most complex differential equations cannot be solved analytically, therefore approximation to the solution becomes imperative. This can be made possible by the application of suitable numerical methods. Researchers like [6, 3, 1, 13, 10, 11, 14, 20], etc have proposed different numerical methods for solving IVPs in ODEs but some of these methods are not self-starting methods and required single step methods to generate other starting values which is computationally cumbersome and prone to enormous error since the starting method and the method are not of the same order. Our interest in this work is to formulate a family of block methods with optimized region of stability suitable for stiff and non-stiff Initial Value problems in Ordinary Differential Equations of the form

$$y' = f(x), y(x_0) = y_{0,}\ x \in [a, b] \tag{1}$$

The proposed methods can also perform well for system of first order ODEs. Numerical block methods preserve the advantage of being self-starting methods and have minimal errors, it also has lesser time of computation since it solve simultaneously at all the grid points unlike some other numerical methods [11, 5, 15, 1, 7, 18, 19] etc. have worked on some numerical block methods for solution of (1) and have presented results that are competitive.

Derivation of the Methods

Our interest in this section is to construct the proposed Linear Multistep Methods that generate the block method. We consider polynomial interpolant of the form:

$p(x) = \sum_{j=0}^{2r} a_j x^j$ which also approximates the solution of (1) as

$y(x) = \sum_{j=0}^{2r} a_j x^j$, ie. $p(x) \approx y(x)$

and

$y'(x) = \sum_{j=1}^{2r} j a_j x^{j-1}$.

Collocating and interpolating at the different grid points generates the block in the form of

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m \tag{2}$$

where $A^1, B^0\ B^1$ are rXr matrices, A^0a unit diagonal matrix, h is the fixed step size, d and s are stability parameters of the methods and r = 2, 3,....

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ 0 & 0 & 1 & 0 & . & 0 \\ . & . & 0 & 1 & 0 & 0 \\ . & . & . & 0 & 1 & 0 \\ 0 & 0 & 0 & . & . & 1 \end{bmatrix}, \quad A^1 = \begin{bmatrix} 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & . & . & 1 \\ 0 & 0 & 0 & 0 & . & 1 \\ . & . & 0 & 0 & 0 & 1 \\ . & . & . & 0 & 0 & 1 \\ 0 & 0 & 0 & . & . & 1 \end{bmatrix}, \quad B^1 = \begin{bmatrix} a_1 & a_2 & . & . & . & a_r \\ a_{r+1} & a_{r+2} & . & . & . & a_{2r} \\ a_{2r+1} & a_{2r+2} & . & . & . & a_{3r} \\ a_{3r+1} & a_{3r+2} & . & . & . & a_{4r} \\ . & . & . & . & . & . \\ a_{nr+1} & a_{nr+2} & . & . & . & a_{(n+1)r} \end{bmatrix} \quad B^0 = \begin{bmatrix} c_1 & c_2 & . & . & . & c_r \\ c_{r+1} & c_{r+2} & . & . & . & c_{2r} \\ c_{2r+1} & c_{2r+2} & . & . & . & c_{3r} \\ c_{3r+1} & c_{3r+2} & . & . & . & c_{4r} \\ . & . & . & . & . & . \\ c_{nr+1} & c_{nr+2} & . & . & . & c_{(n+1)r} \end{bmatrix},$$

where n = 0, 1, 2,…

Y_m and F_m are defined as $Y_m = (y_{m+1}, y_{m+2}, \ldots y_{m+r})^T$

$F_m = (f_{m+1}, f_{m+2}, \ldots f_{m+r})^T$.

For r=2 we have

$$Y_m = (y_{m+1}, y_{m+2})^T,\ F_m = (f_{m+1}, f_{m+2})^T,\ Y_{m-1} = (y_{m-1}, y_m)^T,\ F_{m-1} = (f_{m-1}, f_m)^T$$

And (2) above becomes

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m+1} \\ y_{m+2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_{m-1} \\ y_m \end{bmatrix} + \frac{h}{s} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} f_{m-1} \\ f_m \end{bmatrix} + \frac{h}{d} \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} f_{m+1} \\ f_{m+2} \end{bmatrix} \tag{3}$$

Note that $A^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B^0 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}.$

Expand (3) we obtain

$$\left.\begin{aligned} y_{m+1} &= y_m + \frac{h}{s}a_1 f_{m-1} + \frac{h}{s}a_2 f_m + \frac{h}{d}c_1 f_{m+1} + \frac{h}{d}c_2 f_{m+2} \\ y_{m+2} &= y_m + \frac{h}{s}a_3 f_{m-1} + \frac{h}{s}a_4 f_m + \frac{h}{d}c_3 f_{m+1} + \frac{h}{d}c_4 f_{m+2} \end{aligned}\right\} \quad (4)$$

Obtaining the unknown parameters in (4) with Mathematica (10.4) we obtain

$$a_1 = -\frac{1}{24},\ a_2 = \frac{13}{24},\ a_3 = 0,\ a_4 = \frac{1}{3},$$

$$B^1 = \begin{bmatrix} -\frac{1}{24} & \frac{13}{24} \\ 0 & \frac{1}{3} \end{bmatrix},$$

$$c_1 = \frac{13}{24},\ c_2 = -\frac{1}{24},\ c_3 = \frac{4}{3},\ c_4 = \frac{1}{3}$$

$$B^0 = \begin{bmatrix} \frac{13}{24} & -\frac{1}{24} \\ \frac{4}{3} & \frac{1}{3} \end{bmatrix}$$

We have

$$\left.\begin{aligned} y_{m+1} &= y_m + -\frac{h}{24s} f_{m-1} + \frac{13h}{24s} f_m + \frac{13h}{24d} f_{m+1} - \frac{h}{24d} f_{m+2} \\ y_{m+2} &= y_m + \frac{h}{3s} f_m + \frac{4h}{3d} f_{m+1} + \frac{h}{3d} f_{m+2} \end{aligned}\right\} \quad (5)$$

for r = 3 we have

$$Y_m = (y_{m+1}, y_{m+2}\ , y_{m+3}\ )^T,\ F_m = (f_{m+1}, f_{m+2}\ , f_{m+3})^T,\ Y_{m-1} = (y_{m-2}, y_{m-1}\ , y_m)^T,\ F_{m-1} = (f_{m-2}, f_{m-1}, f_m\ )^T$$

And (2) above becomes

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{m-2} \\ y_{m-1} \\ y_m \end{bmatrix} + \frac{h}{s}\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}\begin{bmatrix} f_{m-2} \\ f_{m-1} \\ f_m \end{bmatrix} + \frac{h}{d}\begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}\begin{bmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \end{bmatrix} \quad (6)$$

Note $A^0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix}, B^0 = \begin{bmatrix} c_1 & c_2 & c_3 \\ c_4 & c_5 & c_6 \\ c_7 & c_8 & c_9 \end{bmatrix}$

Expand (6) we obtain

$$\left.\begin{aligned} y_{m+1} &= y_m + \frac{h}{s}a_1 f_{m-2} + \frac{h}{s}a_2 f_{m-1} + \frac{h}{s}a_3 f_m + \frac{h}{d}c_1 f_{m+1} + \frac{h}{d}c_2 f_{m+2} + \frac{h}{d}c_3 f_{m+3} \\ y_{m+2} &= y_m + \frac{h}{s}a_4 f_{m-2} + \frac{h}{s}a_5 f_{m-1} + \frac{h}{s}a_6 f_m + \frac{h}{d}c_4 f_{m+1} + \frac{h}{d}c_5 f_{m+2} + \frac{h}{d}c_6 f_{m+3} \\ y_{m+3} &= y_m + \frac{h}{s}a_7 f_{m-2} + \frac{h}{s}a_8 f_{m-1} + \frac{h}{s}a_9 f_m + \frac{h}{d}c_7 f_{m+1} + \frac{h}{d}c_8 f_{m+2} + \frac{h}{d}c_9 f_{m+3} \end{aligned}\right\} \tag{7}$$

Obtaining the unknown parameters in (7) with Mathematica (10.4) we have

$$a_1 = \frac{11}{1440},\ a_2 = -\frac{31}{480},\ a_3 = \frac{401}{720},\ a_4 = 0,\ a_5 = -\frac{1}{90},\ a_6 = \frac{17}{45},\ a_7 = \frac{3}{160},\ a_8 = -\frac{21}{160},\ a_9 = \frac{57}{80}$$

$$B^1 = \begin{bmatrix} \frac{11}{1440} & -\frac{31}{480} & \frac{401}{720} \\ 0 & -\frac{1}{90} & \frac{17}{45} \\ \frac{3}{160} & -\frac{21}{160} & \frac{57}{80} \end{bmatrix}$$

$$c_1 = \frac{401}{720},\ c_2 = -\frac{31}{480},\ c_3 = \frac{11}{1440},\ c_4 = \frac{19}{15},\ c_5 = \frac{17}{45},\ c_6 = -\frac{17}{45},\ c_7 = \frac{59}{80},\ c_8 = \frac{219}{160},\ c_9 = \frac{51}{160}$$

$$B^0 = \begin{bmatrix} \frac{401}{720} & -\frac{31}{480} & \frac{11}{1440} \\ \frac{19}{15} & \frac{17}{45} & -\frac{17}{45} \\ \frac{59}{80} & -\frac{219}{160} & \frac{51}{160} \end{bmatrix}$$

we have

$$\left.\begin{aligned} y_{m+1} &= y_m + \frac{11}{1440}\frac{h}{s} f_{m-2} - \frac{31}{480}\frac{h}{s} f_{m-1} + \frac{401}{720}\frac{h}{s} f_m + \frac{401}{720}\frac{h}{d} f_{m+1} - \frac{31}{480}\frac{h}{d} f_{m+2} + \frac{11}{1440}\frac{h}{d} f_{m+3} \\ y_{m+2} &= y_m - \frac{h}{90s} f_{m-1} + \frac{17h}{45s} f_m + \frac{19h}{15d}c_4 f_{m+1} + \frac{17h}{45d} f_{m+2} - \frac{17h}{45d} f_{m+3} \\ y_{m+3} &= y_m + \frac{3h}{160s} f_{m-2} - \frac{21h}{160s} f_{m-1} + \frac{57h}{80s} f_m + \frac{59h}{80d} f_{m+1} + \frac{219h}{160d} f_{m+2} + \frac{51h}{160d} f_{m+3} \end{aligned}\right\} \tag{8}$$

when $r = 4$ we have

$$Y_m = \left(y_{m+1}, y_{m+2}, y_{m+3}, y_{m+4}\right)^T,\ F_m = \left(f_{m+1}, f_{m+2}, f_{m+3}, f_{m+4}\right)^T,\ Y_{m-1} = \left(y_{m-3}, y_{m-2}, y_{m-1}, y_m\right)^T,$$

$$F_{m-1} = \left(f_{m-3}, f_{m-2}, f_{m-1}, f_m\right)^T$$

And (2) above becomes

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{m+1} \\ y_{m+2} \\ y_{m+3} \\ y_{m+4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} y_{m-3} \\ y_{m-2} \\ y_{m-1} \\ y_{m} \end{bmatrix} + \mathrm{h/s}\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\begin{bmatrix} f_{m-3} \\ f_{m-2} \\ f_{m-1} \\ f_{m} \end{bmatrix} + \mathrm{h/d}\begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \\ c_9 & c_{10} & c_{11} & c_{12} \\ c_{13} & c_{14} & c_{15} & c_{16} \end{bmatrix}\begin{bmatrix} f_{m+1} \\ f_{m+2} \\ f_{m+3} \\ f_{m+4} \end{bmatrix} \tag{9}$$

Note

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A^1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, B^1 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix},$$

$$B^0 = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \\ c_5 & c_6 & c_7 & c_8 \\ c_9 & c_{10} & c_{11} & c_{12} \\ c_{13} & c_{14} & c_{15} & c_{16} \end{bmatrix}$$

Expanding (9) we have

$$\left.\begin{aligned}
&y_{m+1} = y_m + \frac{h}{s}a_1 f_{m-3} + \frac{h}{s}a_2 f_{m-2} + \frac{h}{s}a_3 f_{m-1} + \frac{h}{s}a_4 f_m + \frac{h}{d}c_1 f_{m+1} + \frac{h}{d}c_2 f_{m+2} + \\
&\frac{h}{d}c_3 f_{m+3} + \frac{h}{d}c_4 f_{m+4} \\
&y_{m+2} = y_m + \frac{h}{s}a_5 f_{m-3} + \frac{h}{s}a_6 f_{m-2} + \frac{h}{s}a_7 f_{m-1} + \frac{h}{s}a_8 f_m + \frac{h}{d}c_5 f_{m+1} + \frac{h}{d}c_6 f_{m+2} + \\
&\frac{h}{d}c_7 f_{m+3} + \frac{h}{d}c_8 f_{m+4} \\
&y_{m+3} = y_m + \frac{h}{s}a_9 f_{m-3} + \frac{h}{s}a_{10} f_{m-2} + \frac{h}{s}a_{11} f_{m-1} + \frac{h}{s}a_{12} f_m + \frac{h}{d}c_9 f_{m+1} + \frac{h}{d}c_{10} f_{m+2} + \\
&\frac{h}{d}c_{11} f_{m+3} + \frac{h}{d}c_{12} f_{m+4} \\
&y_{m+4} = y_m + \frac{h}{s}a_{13} f_{m-3} + \frac{h}{s}a_{14} f_{m-2} + \frac{h}{s}a_{15} f_{m-1} + \frac{h}{s}a_{16} f_m + \frac{h}{d}c_{13} f_{m+1} + \frac{h}{d}c_{14} f_{m+2} + \\
&\frac{h}{d}c_{15} f_{m+3} + \frac{h}{d}c_{16} f_{m+4}
\end{aligned}\right\} \tag{10}$$

Obtaining the unknown parameters in (10) with Mathematical (10.4) we have

$$B^1=\begin{bmatrix}-\frac{191}{120960} & \frac{1879}{120960} & -\frac{353}{4480} & \frac{68323}{120960}\\ 0 & \frac{1}{756} & -\frac{2}{105} & \frac{167}{420}\\ -\frac{13}{4480} & \frac{117}{4480} & -\frac{513}{4480} & \frac{2777}{4480}\\ \frac{8}{945} & -\frac{64}{945} & \frac{8}{35} & -\frac{106}{945}\end{bmatrix},\quad B^0=\begin{bmatrix}\frac{68323}{120960} & -\frac{353}{4480} & \frac{1879}{120960} & -\frac{191}{120960}\\ \frac{1172}{945} & \frac{167}{420} & -\frac{2}{105} & \frac{1}{756}\\ \frac{3897}{4480} & \frac{1107}{896} & \frac{337}{896} & -\frac{9}{896}\\ \frac{1784}{945} & \frac{8}{35} & \frac{1448}{945} & \frac{278}{945}\end{bmatrix}$$

For r = 5 we have

$$Y_m=\left(y_{m+1},y_{m+2},y_{m+3},y_{m+4},y_{m+5}\right)^T,$$

$$F_m=\left(f_{m+1},f_{m+2},f_{m+3},f_{m+4},f_{m+5}\right)^T,$$

$$Y_{m-1}=\left(y_{m-4},y_{m-3},y_{m-2},y_{m-1},y_m\right)^T,$$

$$F_{m-1}=\left(f_{m-4},f_{m-3},f_{m-2},f_{m-1},f_m\right)^T$$

And (2) above becomes

$$A^0Y_m=A^1Y_{m-1}+\frac{h}{s}B^1F_{m-1}+\frac{h}{d}B^0F_m$$

$$\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{m+1}\\y_{m+2}\\y_{m+3}\\y_{m+4}\\y_{m+5}\end{bmatrix}=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{m-4}\\y_{m-3}\\y_{m-2}\\y_{m-1}\\y_m\end{bmatrix}+\mathrm{h/s}\begin{bmatrix}a_1&a_2&a_3&a_4&a_5\\a_6&a_7&a_8&a_9&a_{10}\\a_{11}&a_{12}&a_{13}&a_{14}&a_{15}\\a_{16}&a_{17}&a_{18}&a_{19}&a_{20}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}\end{bmatrix}\begin{bmatrix}f_{m-4}\\f_{m-3}\\f_{m-2}\\f_{m-1}\\f_m\end{bmatrix}+\mathrm{h/d}\begin{bmatrix}c_1&c_2&c_3&c_4&c_5\\c_6&c_7&c_8&c_9&c_{10}\\c_{11}&c_{12}&c_{13}&c_{14}&c_{15}\\c_{16}&c_{17}&c_{18}&c_{19}&c_{20}\\c_{21}&c_{22}&c_{23}&c_{24}&c_{25}\end{bmatrix}\begin{bmatrix}f_{m+1}\\f_{m+2}\\f_{m+3}\\f_{m+4}\\f_{m+5}\end{bmatrix}\qquad(11)$$

Note

$$A^0=\begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix},\ A^1=\begin{bmatrix}0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\\0&0&0&0&1\end{bmatrix},\ B^1=\begin{bmatrix}a_1&a_2&a_3&a_4&a_5\\a_6&a_7&a_8&a_9&a_{10}\\a_{11}&a_{12}&a_{13}&a_{14}&a_{15}\\a_{16}&a_{17}&a_{18}&a_{19}&a_{20}\\a_{21}&a_{22}&a_{23}&a_{24}&a_{25}\end{bmatrix},\ B^0=\begin{bmatrix}c_1&c_2&c_3&c_4&c_5\\c_6&c_7&c_8&c_9&c_{10}\\c_{11}&c_{12}&c_{13}&c_{14}&c_{15}\\c_{16}&c_{17}&c_{18}&c_{19}&c_{20}\\c_{21}&c_{22}&c_{23}&c_{24}&c_{25}\end{bmatrix}$$

the expansion of (11) *gives*

$$\left.\begin{aligned}
y_{m+1} &= y_m+\frac{h}{s}a_1f_{m-4}+\frac{h}{s}a_2f_{m-3}+\frac{h}{s}a_3f_{m-2}+\frac{h}{s}a_4f_{m-1}+\frac{h}{s}a_5f_m+\frac{h}{d}c_1f_{m+1}+\frac{h}{d}c_2f_{m+2}+\\
&\quad\frac{h}{d}c_3f_{m+3}+\frac{h}{d}c_4f_{m+4}+\frac{h}{d}c_5f_{m+5}\\
y_{m+5} &= y_m+\frac{h}{s}a_{21}f_{m-4}+\frac{h}{s}a_{22}f_{m-3}+\frac{h}{s}a_{23}f_{m-2}+\frac{h}{s}a_{24}f_{m-1}+\frac{h}{s}a_{25}f_m+\frac{h}{d}c_{21}f_{m+1}+\frac{h}{d}c_{22}f_{m+2}+\\
&\quad\frac{h}{d}c_{23}f_{m+3}+\frac{h}{d}c_{24}f_{m+4}+\frac{h}{d}c_{25}f_{m+5}\\
y_{m+2} &= y_m+\frac{h}{s}a_6f_{m-4}+\frac{h}{s}a_7f_{m-3}+\frac{h}{s}a_8f_{m-2}+\frac{h}{s}a_9f_{m-1}+\frac{h}{s}a_{10}f_m+\frac{h}{d}c_6f_{m+1}+\frac{h}{d}c_7f_{m+2}+\\
&\quad\frac{h}{d}c_8f_{m+3}+\frac{h}{d}c_9f_{m+4}+\frac{h}{d}c_{10}f_{m+5}\\
y_{m+4} &= y_m+\frac{h}{s}a_{16}f_{m-4}+\frac{h}{s}a_{17}f_{m-3}+\frac{h}{s}a_{18}f_{m-2}+\frac{h}{s}a_{19}f_{m-1}+\frac{h}{s}a_{20}f_m+\frac{h}{d}c_{16}f_{m+1}+\frac{h}{d}c_{17}f_{m+2}+\\
&\quad\frac{h}{d}c_{18}f_{m+3}+\frac{h}{d}c_{19}f_{m+4}+\frac{h}{d}c_{20}f_{m+5}
\end{aligned}\right\}$$

To obtain unknown parameters in (12) with Mathematical (10.4) we have

$$B^1=\begin{bmatrix}\frac{2497}{7257600}&-\frac{28939}{7257600}&\frac{581}{25920}&-\frac{40111}{453600}&\frac{2067169}{3628800}\\0&-\frac{23}{113400}&\frac{167}{56700}&-\frac{701}{28350}&\frac{23189}{56700}\\\frac{7}{12800}&-\frac{77}{12800}&\frac{351}{11200}&-\frac{123}{1120}&\frac{26321}{44800}\\-\frac{13}{14175}&\frac{26}{2835}&-\frac{572}{14175}&\frac{1336}{14175}&\frac{2764}{14175}\\\frac{25}{3584}&-\frac{20675}{290304}&\frac{11975}{36288}&-\frac{16775}{18144}&\frac{298505}{145152}\end{bmatrix},$$

$$B^0=\begin{bmatrix}\frac{2067169}{3628800}&-\frac{40111}{453600}&\frac{581}{25920}&-\frac{28939}{7257600}&\frac{2497}{7257600}\\\frac{13903}{11340}&\frac{23189}{56700}&-\frac{701}{28350}&\frac{167}{56700}&-\frac{23}{113400}\\\frac{41553}{44800}&\frac{6609}{5600}&\frac{4559}{11200}&-\frac{351}{17920}&\frac{113}{89600}\\\frac{20908}{14175}&\frac{1628}{2835}&\frac{19192}{14175}&\frac{4909}{14175}&-\frac{94}{14175}\\\frac{131575}{145152}&\frac{50425}{18144}&-\frac{7225}{36288}&\frac{478525}{290304}&\frac{81385}{290304}\end{bmatrix}$$

Consistency of the Method: The LMM method can be proved to be consistent if and only if the following conditions are satisfied [9].

$$\sum_{i=0}^{k} \alpha_i = 0, \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k} \beta_i$$

Definition (1.1): Block method is consistent if it has order at least one [4].

Zero Stability: Block method is said to be zero stable if the roots r of the characteristics polynomial $\rho(r)$ defined by $\rho(r)$ =det[$rA^0 - A^1$] satisfies $|r| \le 1$ and every root with $|r_0| = 1$ has multiplicity not exceeding one in the limit as $h \to 0$

$$A^0 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \quad A^1 = \begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$\rho(r) = \begin{vmatrix} r & 0 & 0 & -1 \\ 0 & r & 0 & -1 \\ 0 & 0 & r & -1 \\ 0 & 0 & 0 & r-1 \end{vmatrix}$$

$$r^3(r-1) = 0$$

$$r = 0, 0, 0, 1$$

Therefore, the block method is zero stable.

Theorem (1.1): The necessary conditions for LMM to converge are; it must be consistent and zero stable [8].
Since the method satisfies these conditions, therefore it is convergent.

Stability of the method: Consider the block method below (2) above

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} B^1 F_{m-1} + \frac{h}{d} B^0 F_m$$

$$y'_m = \lambda y_m \;\&\; y'_{m-1} = \lambda y_{m-1} \tag{13}$$

Applying the test equation (13) on (2) we have

$$A^0 Y_m = A^1 Y_{m-1} + \frac{h}{s} \lambda B^1 y_{m-1} + \frac{h}{d} \lambda B^0 y_m$$

$$A^0 Y_m - \frac{h}{d} \lambda B^0 y_m = A^1 Y_{m-1} + \frac{h}{s} \lambda B^1 y_{m-1}$$

$$y_m (A^0 - \tfrac{h}{d} \lambda B^0) = Y_{m-1} (A^1 + \tfrac{h}{s} \lambda B^1)$$

$$y_m = \frac{Y_{m-1} (A^1 + \frac{h}{s} \lambda B^1)}{(A^0 - \frac{h}{d} \lambda B^0)}$$

$$y_m = \left(A^0 - \tfrac{h}{d} \lambda B^0\right)^{-1} \left(A^1 + \tfrac{h}{s} \lambda B^1\right) Y_{m-1} \tag{14}$$

$let\ z = h\lambda$

$$y_m = \left(A^0 - \tfrac{z}{d} B^0\right)^{-1} \left(A^1 + \tfrac{z}{s} B^1\right) Y_{m-1} \tag{15}$$

Where $\left(A^0 - \frac{z}{d}B^0\right)^{-1}\left(A^1 + \frac{z}{s}B^1\right)$ is called amplification matrix of the method?

Order and Error Constant

Definition (1.2): The order of LMM and its associated linear operator given by

$$L[y(x);h] = \sum_{j=0}^{k}[\alpha_j y(x+jh) - h\beta_j y'(x+jh)]$$

Is defined as a unique integer p such that $C_p = 0, q = 0(1)p \; and \; C_{p+1} \neq 0$
Where the $C_q$are constant defined as

$$C_0 = \alpha_0 + \alpha_1 + \cdots + \alpha_k$$

$$C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_1 + \cdots + \beta_k)$$

$$C_q = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \cdots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \cdots + k^{q-1}\beta_k)$$

$q = 2,3, \ldots . k$ [7].

The associated linear operator (2) is

$$L[y(x);h] = A^0Y_m - A^1Y_{m-1} - \frac{h}{s}B^1F_{m-1} - \frac{h}{d}B^0F_m \tag{16}$$

Expanding (16) we obtain

$$L[y(x);h] = c_0y(x_n) + c_1hy'(x_n) + \cdots + c_ph^py^p(x_n) + \cdots$$

The methods (2) is said to be of order p if $c_0 = c_1 = \cdots = c_p = 0 \; and \; c_{p+1} \neq 0$ where c_{p+1} is the error constant of the block method. Therefore the order p and the associated error constant for each r-point is given as follows.

for r =2 is of order p = 4 with error constant $\begin{bmatrix} \frac{11}{720} & -\frac{1}{90} \end{bmatrix}^T$

for r = 3 is of order p = 6 with error constant, $\begin{bmatrix} -\frac{191}{60480} & \frac{1}{756} & -\frac{29}{2240} \end{bmatrix}^T$

for r = 4 is of order p = 8 with error constant, $\begin{bmatrix} \frac{2497}{3628800} & -\frac{23}{113400} & \frac{81}{44800} & -\frac{107}{14175} \end{bmatrix}^T$

for r = 5 is of order p = 10 with error constant, $\begin{bmatrix} -\frac{14797}{95800320} & \frac{263}{7484400} & -\frac{127}{394240} & \frac{367}{467775} & -\frac{114985}{19160064} \end{bmatrix}^T$

Definition (1.3): A numerical method is said to be A-Stable if the stability region contain the entire left hand side of the complex plane [4].
One of the basic requirements for every numerical method to solve stiff IVPs in ODEs is that the method must be A-stable.
Applying the boundary locus techniques on the amplification matrix of (2) we have

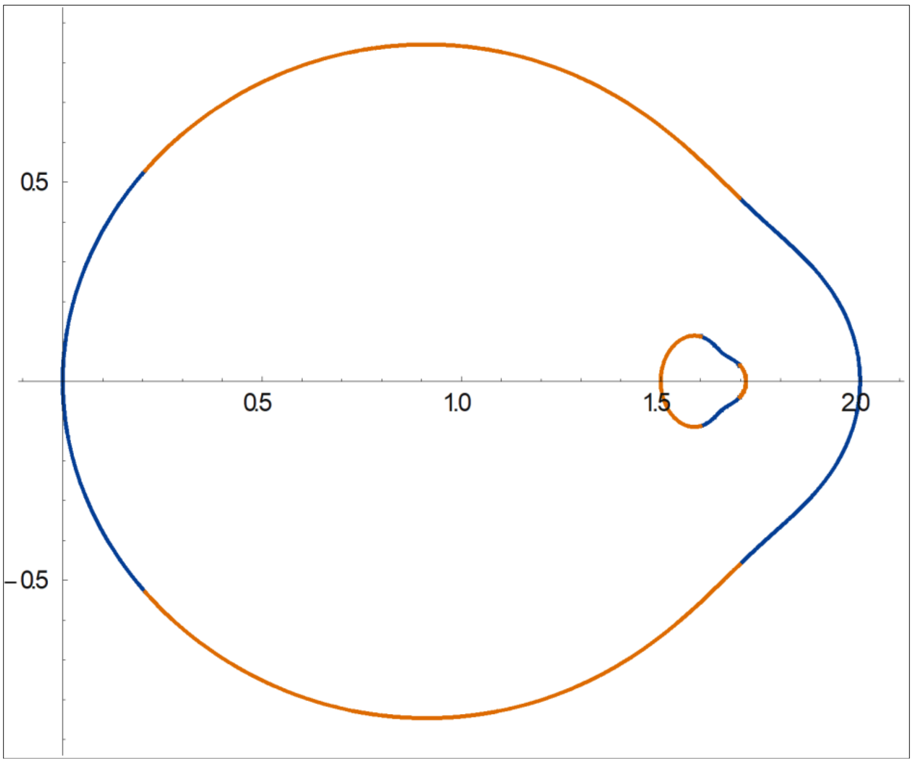

Fig 1: A-stable method for r =2, d= -1 and s= -1.

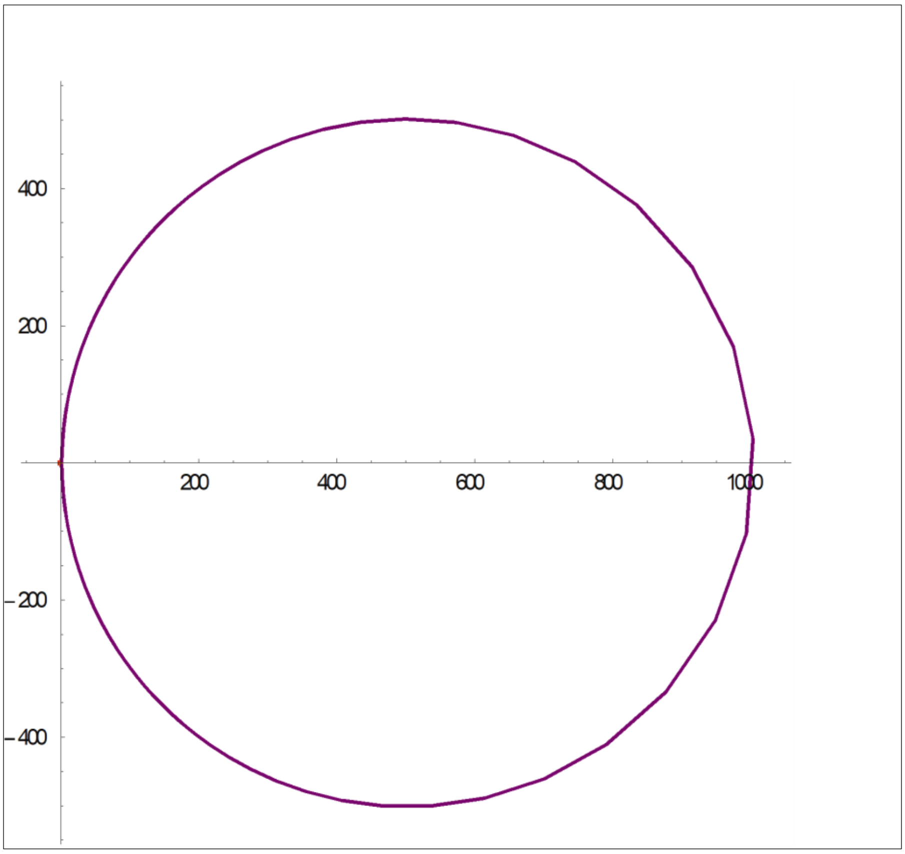

Fig 2: A-stable method for r =3 at d = -1 and s = -6.5

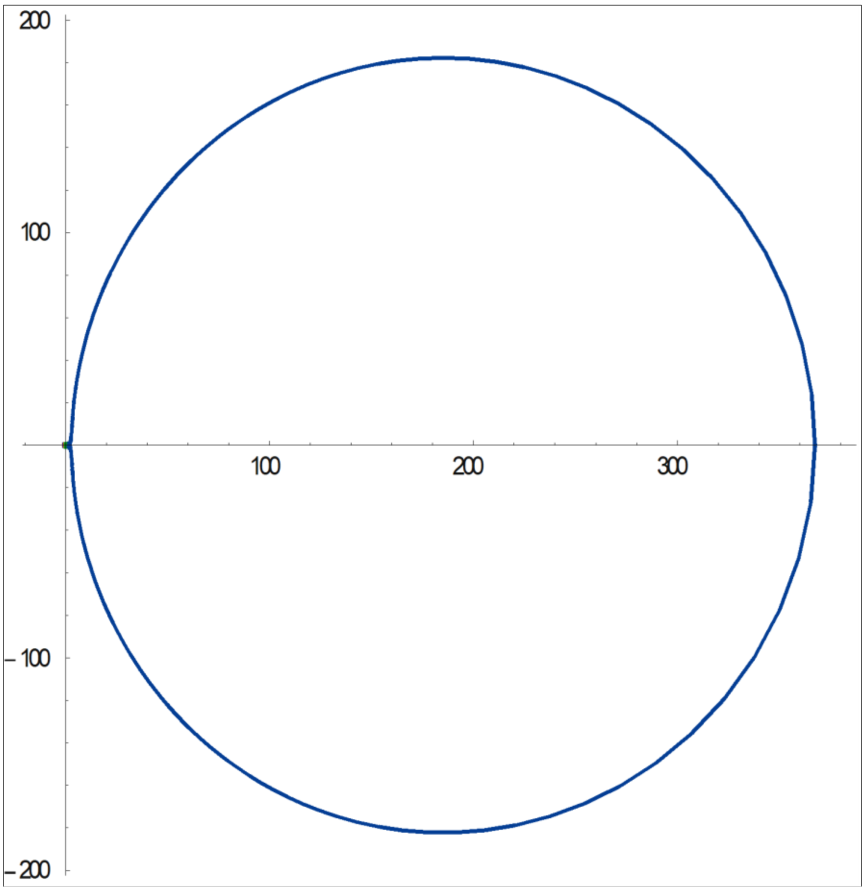

Fig 3: A-stable method for r = 4, d = -0.35 and s = -6.8

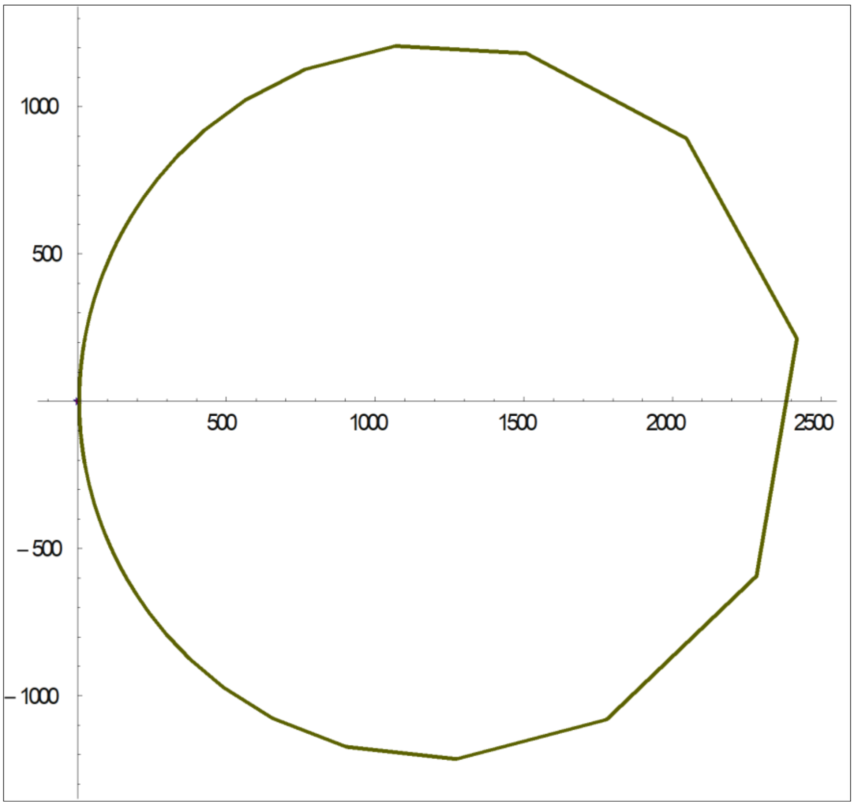

Fig 4: A-stable method for r =5, d = -0.16 and s = -10.2

Implementation of the Method

This section considers numerical experiments of the proposed block methods for solution of system of first order IVPs in ODEs.

Problem 1:

$$y' = \frac{y(1-y)}{2(y-1)}, y(0) = \frac{5}{6}, 0 \le x \le 1$$

Exact Solution

$$y(x) = \frac{1}{2} + \sqrt{\frac{1}{4} - \frac{5}{36}e^{-x}} \tag{16}$$

Problem 2:

$$y' = -\frac{y^3}{2}, y(0) = 1, 0 \le x \le 4$$

Exact Solution

$$y(x) = \frac{1}{\sqrt{1+x}}$$

Problem 3:

$$y' = y, y(0) = 1, h = 0.1$$

Exact Solution

$y(x) = e^x$ [1].

Problem 4: Consider the system of equations

$$X'(t) = 998X(t) + 1998Y(t), X(0) = 1,$$

$$Y'(t) = -999X(t) - 1999Y(t), Y(0) = 0,$$

The exact solutions are

$$X(t) = 2e^{-t} - e^{-1000t}$$

$Y(t) = e^{-1000t} - e^{-t}$. [2].

Table 1: Numerical solution for problem 1

X	Theoretical Solution	3PBEBDF ORDERP=6 [17]	Proposed Method Order P=6	Error Between Exact Solution and 3PBEBDF [17]	Error Between Exact Solution and Proposed Method of order p=6
0.0	1.8333333	1.8333333	1.8333333	0	0
0.1	0.8526020	0.8527450	0.8526350	0.0001430	0.0000420
0.2	0.8691712	0.8690573	0.8691573	0.0001139	0.0000128
0.3	0.8835474	0.8829767	0.8829767	0.0005707	0.0002505
0.4	0.8961060	0.8960859	0.8960959	0.0000201	0.0000101
0.5	0.9071359	0.9065019	0.9068239	0.0006340	0.0003120
0.6	0.9168647	0.9155497	0.9167617	0.0013150	0.0001030
0.7	0.9254760	0.9253058	0.9254158	0.0001702	0.0000602
0.8	0.9331203	0.9321562	0.9328783	0.0009641	0.0002420
0.9	0.9399227	0.9381680	0.9391680	0.0017547	0.0007547
1.0	0.9459884	0.9439650	0.9449762	0.0020234	0.0010122

Table 2: Numerical solution for problem 2

X	Theoretical Solution	3PBEBDF ORDER P=6 [17]	Proposed Method Order P=6	Absolute Error Between Exact Solution and 3PBEBDF [17]	Error Between Exact Solution and Proposed Method of order p =6
0.0	1.0000000	1.0000000	1.0000000	0	0
0.1	0.9534626	0.9531365	0.9533494	0.0003261	0.0001132
0.2	0.9128709	0.9131317	0.9128106	0.0002608	0.0000603
0.3	0.8770580	0.8783667	0.8767493	0.0013087	0.0003087
0.4	0.8451543	0.8452111	0.8451295	0.0000568	0.0000248
0.5	0.8164966	0.8180010	0.8161924	0.0015044	0.0003042

Table 3: Numerical solution for problem 3

X	Theoretical Solution	Ajileye *et al.* [1]	Proposed Methods at Order p=4	Error Between Exact solution and Ajileye *et al.* [1]	Error Between Exact solution and The proposed methods at p=4
0.1	1.1501709187	1.1501709178	1.1501709186	0.0000000009	0.0000000001
0.2	1.2214027581	1.2214027576	1.2214027580	0.0000000005	0.0000000001
0.3	1.3498588075	1.3498588067	1.3498588073	0.0000000008	0.0000000002
0.4	1.4918246976	1.4918246964	1.4918246977	-0.0000000012	-0.0000000001
0.5	1.6487212707	1.6487212269	1.6487212704	0.0000000438	0.0000000003
0.6	1.8221188003	1.8221187982	1.8221188012	0.0000000021	-0.0000000009
0.7	2.0137527074	2.0137527047	2.0137527072	0.0000000027	0.0000000002
0.8	2.2255409284	2.2255409250	2.2255409281	0.0000000034	0.0000000003
0.9	2.4596031111	2.4596031068	2.4596031101	0.0000000043	0.0000000001
1.0	2.7182818284	2.7182818241	2.7182818283	0.0000000042	0.0000000001

Table 4: for problem 4

h		Exact Solution X(t)	Proposed Method of Order P=4	Error Between Exact and The Proposed Method of Order p=4
0.0	x(t)	1.000000000	1.000000000	0.0000000000
	y(t)	0.000000000	0.000000000	0.0000000000
0.2	x(t)	1.637461506	1.637461506	0.0000000000
	y(t)	-0.8187307531	-0.8187307507	0.0000000002
0.4	x(t)	1.340640092	1.340640092	0.0000000000
	y(t)	-0.6703200460	-0.6703200464	0.0000000004
0.6	x(t)	1.097623272	1.097623272	0.0000000000
	y(t)	-0.5488116361	-0.5488116386	0.0000000025
0.8	x(t)	0.899353050	0.899399929	0.000046879
	y(t)	-0.4493289641	-0.4493289662	0.0000000021
1.0	x(t)	0.736470389	0.736458088	0.000012301
	y(t)	-0.3678794412	-0.3678794425	0.0000000013

Conclusion

A new family of block methods have been developed using a polynomial interpolant as a basis function. The family of the block methods has small error constant at high order which is a good property of every efficient numerical integrator. The boundary Locus shows that the method is A-stable for all values of r = 2, 3, 4, and 5with varying stability parameters s and d.

Numerical Integrator for stiff IVPs in ODEs must have a wider region of absolute stability and which also our proposed block method has and therefore, it is suitable for stiff IVPs in ODEs it also satisfies other stability requirements like consistency and zero stability. The method was demonstrated in three stiff problems and results are displayed in table 1, 2, 3,.. This result shows high competitiveness with the exact solutions and some existing numerical methods. Therefore proposed block method is suitable and also recommended for system of first order IVPs in ODEs.

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