International Journal of Statistics and Applied Mathematics 2022; 7(4): 136-141

International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2022; 7(4): 136-141

www.mathsjournal.com
Received: 19-04-2022
Accepted: 23-05-2022

Mfon Okon Udo
Department of Mathematics, Akwa Ibom State University, Ikot Akpaden, Mkpat Enin, Nigeria

Michael Precious Ineh
Department of Mathematics and Computer Science, Ritman University, Ikot Ekpene, Nigeria

Elisha John Inyang
Department of Statistics, University of Uyo, Uyo, Nigeria

Peter Benneth
Department of Mathematics and Statistics, Federal University Otueke, Nigeria

Corresponding Author:
Elisha John Inyang
Department of Statistics, University of Uyo, Uyo, Nigeria

Solving nonlinear Volterra integral equations by an efficient method

Mfon Okon Udo, Michael Precious Ineh, Elisha John Inyang and Peter Benneth

Abstract
The purpose of this article is to use M-iteration method to approximate the solution a nonlinear Volterra integral equations in Banach spaces. Our results are achieved through the concept of fixed point theory. The results in this paper are new and interesting.

Keywords: Fixed point, banach space, strong convergence, nonlinear integral equation

Introduction
Fixed point theory has received massive attention for some decades now. This is as a result of its application to certain areas in applied science and engineering such as: Optimization theory, Game theory, Approximation theory, Dynamic theory, Fractals and many other subjects.
One of the first fixed point theorems is the Banach fixed point theorem. This theorem is also known as the Banach contraction principle. Banach contraction principle is important as a source of existence and uniqueness theorem in diverse branches of sciences. This theorem gives a demonstration of the unifying power of functional analytic methods and usefulness of fixed point theory.
The Banach contraction principle uses the Picard iterative method which is defined as follows:

$$\psi_{s+1} = G\psi_s, \forall s \in \mathbb{N}, \tag{1.1}$$

for contraction mappings in a complete metric space. It is well known that this principle does not hold for nonexpansive mappings since Picard iteration method fails to converge to the fixed point of nonexpansive mappings even when the existence of fixed point is guaranteed in a complete metric space.
Some many authors have constructed several iterative methods for approximating the fixed points of nonexpansive mappings and other more general classes of mappings. An efficient iterative method is one which; converges to the fixed point of an operator, has a better rate of convergence, gives data dependent result and guarantees stability with respect to G.
Some notable iterative schemes in the existing literature includes: Mann iteration [19], Ishikawa iteration [16], Noor iteration [22], Argawal *et al*. iteration [2], Abbas and Nazir iteration [1], SP iteration [23], S* iteration [15], CR iteration [8], Normal-S iteration [25], Picard-S iteration [12], Thakur iteration [31], Thakur iteration [32], M iteration [34], M* iteration [33], Garodia and Uddin iteration [10], Two-Step Mann iteration [30] and many others.
Very recently, Ullah and Arshad [34] defined M iteration scheme as follows:

$$\begin{cases} k_s = (1-r_s)\psi_s + r_s G\psi_s, \\ \quad \eta_s = Gk_s, \forall s \geq 1, \\ \qquad \psi_{s+1} = G\eta_s, \end{cases} \tag{1.2}$$

where $\{r_s\}$ is a sequence in [0,1]. The authors showed that (1.2) converges faster than several existing iterative methods.

On the other hand, several problems which arise in mathematical physics, engineering,biology, economics and etc., lead to mathematical models described by nonlinear integral equations (see [20] and the references therein). In particular, Volterra-Fredholm integral equations arise from parabolic boundary value problems, from the mathematical modeling of the spatio-temporal development of an epidemic, and from various physical and biological models (see) [21, 36].

In this article, we will use M-iterative method (1.2) to solve the following Volterra-Fredholm integral equation which have been considered by Lungu and Rus [18]:

$$u(\psi,\eta) = g(\psi,\eta,h(u(\psi,\eta))) + \int_0^{\psi}\int_0^{\eta} k(\psi,\eta,m,n,u(m,n))dmdn, \tag{1.3}$$

For all $\psi,\eta \in \Re_+$. Let $(\Omega,|.|)$ be a Banach space. Let $\tau > 0$ and

$$X_\tau = \left\{u \in C(\Re_+{}^2\Omega) \middle| \exists M(u) > 0 : |u(\psi,\eta)| e^{-\tau(\psi+\eta)} \le M(u)\right\}$$

We now consider Bielecki's norm on X_τ as follows:

$$\|u\|_\tau = \sup_{\psi,\eta\in\Re_+} \left(|u(\psi,\eta)| e^{-\tau(\psi+\eta)}\right)$$

Obviously, $(X_\tau, \|.\|_\tau)$ is a Banach space (see) [5].

The following result which was given by Lungu and Rus [18] will be useful in proving our main result.

Theorem 1.1. [18] Suppose the following conditions are fulfilled:

(V_1) $g \in C(\Re_+{}^2 \times \Omega, \Omega), K \in C(\Re_+{}^4 \times \Omega, \Omega)$

(V_2) $h : X_\tau \to X_\tau$ is such that

$\exists l_h > 0 : |h(u(\psi,\eta)) - h(v(\psi,\eta))| \le l_h \|u - v\| . e^{\tau(\psi+\eta)}$, for all $\psi,\eta \in \Re_+$ and $u, v \in X_\tau$;

(V_3) $\exists l_g > 0 : |g(\psi,\eta,e_1) - g(\psi,\eta,e_2)| \le l_g |e_1 - e_2|$ for all $\psi,\eta \in \Re_+$ and $e_1, e_2 \in \Omega$;

(V_4) $\exists l_K(\psi,\eta,m,n_1) : |K(\psi,\eta,m,n,e_1) - K(\psi,\eta,m,n,e_2)| \le l_K(\psi,\eta,m,n_1)|e_1 - e_2|$,

for all $\psi,\eta,m,n \in \Re_+$ and $e_1, e_2 \in \Omega$;

(V_5) $l_K \in C(\Re_+{}^4, \Re_+)$ and $\int_0^{\psi}\int_0^{\eta} l_K(\psi,\eta,m,n)e^{\tau(m+n)}dmdn \le le^{\tau(\psi+\eta)}$, for all $\psi,\eta \in \Re_+$;

(V_6) $l_g l_h + l < 1$.

Then, the equation (1.3) has a unique solution $z \in X_\tau$ and the sequence of successive approximations

$$u_{s+1}(\psi,\eta) = g(\psi,\eta,h(u_s(\psi,\eta))) + \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,u_s(m,n))dmdn, \tag{1.4}$$

for all $s \in \mathbb{N}$ converges uniformly to z.

We now give our main result in this section.

Theorem 1.2. Let $\{\psi_s\}$ be M-iterative method defined by (1.2) with sequences $\{r_s\}$ and $\{p_s\}$ in [0,1] such that $\sum_{s=0}^{\infty} r_s = \infty$. If all the conditions (V_1) - (V_6) in theorem 8.1 are satisfied, then the equation (1.3) has a unique solution z in X_τ and the A^* iterative sequence (1.2) converges strongly to z.

Proof. Let $\{\psi_s\}$ be an iterative sequence generated by A^* iterative method (1.2) for the operator $A: X_\tau \to X_\tau$ defined by

$$A(u(\psi,\eta)) = g(\psi,\eta,h(u(\psi,\eta))) + \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,u(m,n))dmdn \tag{1.5}$$

We will prove that $\psi_s \to 0$ as $s \to \infty$. Using (1.2), we obtain

$$\|\psi_{s+1} - z\|_\tau = \sup_{\psi,\eta\in\Re_+} (|A(\eta_s(\psi,\eta)) - A(z(\psi,\eta))| e^{-\tau(\psi+\eta)}).$$

Now,

$$\begin{aligned}
&|A(\eta_s(\psi,\eta)) - A(z(\psi,\eta))| \\
&\le |g(\psi,\eta,h(\eta_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))| \\
&+ \left|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,\eta_s(m,n))dmdn - \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\right| \\
&\le l_g |h(\eta_s(\psi,\eta)) - h(z(\psi,\eta))| \\
&+ \int_0^{\psi}\int_0^{\eta} |K(\psi,\eta,m,n,\eta_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn| \\
&\le l_g l_h \|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} + \int_0^{\psi}\int_0^{\eta} l_K(\psi,\eta,m,n)|\eta_s(m,n)) - z(m,n)|dmdn \\
&\le l_g l_h \|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} + l\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)} \\
&= (l_g l_h + l)\|\eta_s - z\|_\tau e^{\tau(\psi+\eta)}
\end{aligned}$$

Hence,

$$\|\psi_{s+1} - z\|_\tau \le (l_g l_h + l)\|\eta_s - z\|_\tau \tag{1.6}$$

Again from (1.2), we have

$$\|\eta_s - z\|_\tau = \sup_{\psi,\eta\in\Re_+} (|A(g_s(\psi,\eta)) - A(z(\psi,\eta))| e^{-\tau(\psi+\eta)}).$$

Now,

$$\begin{aligned}
&|A(g_s(\psi,\eta)) - A(z(\psi,\eta))| \\
&\le |g(\psi,\eta,h(g_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))| \\
&+ \left|\int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,g_s(m,n))dmdn - \int_0^{\psi}\int_0^{\eta} K(\psi,\eta,m,n,z(m,n))dmdn\right| \\
&\le l_g |h(g_s(\psi,\eta)) - h(z(\psi,\eta))| \\
&+ \int_0^{\psi}\int_0^{\eta} |K(\psi,\eta,m,n,g_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn|
\end{aligned}$$

$$\leq l_g l_h \|g_s - z\|_\tau e^{\tau(\psi+\eta)} + \int_0^\psi \int_0^\eta l_K(\psi,\eta,m,n)|g_s(m,n)) - z(m,n)|dmdn$$
$$\leq l_g l_h \|g_s - z\|_\tau e^{\tau(\psi+\eta)} + l\|g_s - z\|_\tau e^{\tau(\psi+\eta)}$$
$$= (l_g l_h + l)\|g_s - z\|_\tau e^{\tau(\psi+\eta)}$$

Hence,

$$\|\eta_s - z\|_\tau \leq (l_g l_h + l)\|g_s - z\|_\tau \tag{1.7}$$

Finally,

$$\|k_s - z\|_\tau = \|((1-r_s)\psi_s + r_s A\psi_s) - z\|_\tau$$
$$= \|((1-r_s)(\psi_s - z) + r_s(A\psi_s - z)\|_\tau \tag{1.8}$$
$$\leq (1-r_s)\|\psi_s - z\|_\tau + r_s\|A\psi_s - z\|_\tau$$

Now,

$$\|A\psi_s - Az\|_\tau = \sup_{\psi,\eta\in\Re_+} (|A(\psi_s(\psi,\eta)) - A(z(\psi,\eta))|e^{-\tau(\psi+\eta)})$$

And

$$|A(\psi_s(\psi,\eta)) - A(z(\psi,\eta))|$$
$$\leq |g(\psi,\eta,h(\psi_s(\psi,\eta))) - g(\psi,\eta,h(z(\psi,\eta)))|$$
$$+\left|\int_0^\psi \int_0^\eta K(\psi,\eta,m,n,\psi_s(m,n))dmdn - \int_0^\psi \int_0^\eta K(\psi,\eta,m,n,z(m,n))dmdn\right|$$
$$\leq l_g |h(\psi_s(\psi,\eta)) - h(z(\psi,\eta))|$$
$$+\int_0^\psi \int_0^\eta |K(\psi,\eta,m,n,\psi_s(m,n)) - K(\psi,\eta,m,n,z(m,n))dmdn|$$
$$\leq l_g l_h \|\psi_s - z\|_\tau e^{\tau(\psi+\eta)} + \int_0^\psi \int_0^\eta l_K(\psi,\eta,m,n)|\psi_s(m,n)) - z(m,n)|dmdn$$
$$\leq l_g l_h \|\psi_s - z\|_\tau e^{\tau(\psi+\eta)} + l\|\psi_s - z\|_\tau e^{\tau(\psi+\eta)}$$
$$= (l_g l_h + l)\|\psi_s - z\|_\tau e^{\tau(\psi+\eta)}$$

Thus,

$$\|A\psi_s - Az\|_\tau \leq (l_g l_h + l)\|\psi_s - z\|_\tau \tag{1.9}$$

From (1.8) and (1.9), we obtain

$$\|k_s - z\|_\tau \leq (1-r_s)\|g_s - z\| + r_s(l_g l_h + l)\|g_s - z\|_\tau$$
$$= \left[1 - r_s\left\{1 - \left(l_g l_h + l\right)\right\}\right]\|\psi_s - z\|_\tau \tag{1.10}$$

By (1.6), (1.7) and (1.10), we have

$$\|\psi_{s+1} - z\|_\tau \leq (l_g l_h + l)^2\left[1 - r_s\left\{1 - (l_g l_h + l)\right\}\right]\|\psi_s - z\|_\tau$$

Recalling from assumption (C_6) that $l_g l_h + l < 1$. Thus, from (1.10), we obtain

$$\|\psi_{s+1} - z\|_\tau \le \left[1 - r_s\left\{1 - (l_g l_h + l)\right\}\right]\|\psi_s - z\|_\tau \tag{1.11}$$

Inductively, from (1.10), we have

$$\|\psi_{s+1} - z\|_\tau \le \|\psi_0 - z\|_\tau \prod_k^s \left[1 - r_k\left\{1 - (l_g l_h + l)\right\}\right] \tag{1.12}$$

Since $r_k \in [0,1]$ for all $k \in \mathbb{N}$ and assumption (C_6) gives

$$1 - r_k\left\{1 - (l_g l_h + l)\right\} < 1$$

From classical analysis, we know that $1 - \psi \le e^{-\psi}$ for all $\psi \in [0,1]$. Thus, (1.12) becomes

$$\|\psi_{s+1} - z\|_\tau \le \|\psi_0 - z\|_\tau \, e^{-\left[1 - r_k\left\{1 - (l_g l_h + l)\right\}\right]\sum_{k=0}^{s} r_k}$$

which yields $\lim_{s\to\infty}\|\psi_s - z\|_\tau = 0$. This completes the proof.

Conclusion

In this paper, the concept of fixed point has been employed to approximate the solution of a nonlinear Volterra integral equation. The method used in our results is well known to be efficient. Our results complement, improve and generalize several results in this research direction.

Availability of data and material

The data used to support the findings of this study are included within the article.

Conflicts of interest

The authors declare no conflict of interests.

References

1. Abbas M, Nazir T. A new faster iteration process applied to constrained inimization and feasibility problems. Mat Vesn. 2014;66:223-234.
2. Agarwal RP, Regan DO, Sahu DR. Iterative construction of fixed points of nearly asymptotically nonexpansive mappings, J Nonlinear Convex Anal. 2007;8:61-79.
3. Berinde V. Picard iteration converges faster than Mann iteration for a class of quasicontractive operators, Fixed Point Theory Appl. 2004;2:97-105.
4. Berinde V. On the approximation of fixed points of weak contractive mapping, Carpath J Math. 2003;19:7-22.
5. Bielecki A. Uneremarque sur l'application de la methode de Banach--Cocciopoli-Tichonovdans la thorie de l'equation $s = f(x, y, z, p, q)$, Bull. Pol. Acad. Sci. Math. 1956;4:265-357.
6. Bocharov GA, Rihan FA. Numerical modelling inbiosciences using delay differential equations," Journal of Computationaland Applied Mathematics. 2000;125(1-2):183-199.
7. Chatterjea SK. Fixed point theorems, C R AcadBulg Sci. 1972;25:727-730.
8. Chugh R, Kumar V, Kumar S. Strong convergence of a new three step iterative scheme in Banach spaces, AmericanJ Comp Math. 2012;2:345-357.
9. Coman GH, Pavel G, Rus I, *et al*. Introduction in the Theory of Operational Equation, Ed. Dacia, Cluj--Napoca, 1976.
10. Garodia C, Uddin I, A new fixed point algorithm for finding the solution of a delay differentialequation, AIMS Mathematics. 5(4):3182--3200. OI:10.3934/math.2020205.
11. Garodia C, Uddin I. Solution of a nonlinear integral equation via new fixed point iteration process, arXiv:1809.03771v1 [math.FA] 11 Sep 2018.
12. Gursoy F, Karakaya V. A Picard--S hybrid type iteration method for solving a differential equation with retarded argument. 2014;arXiv:1403.2546v2.
13. Hämmerlin G, Hoffmann KH. Numerical Mathematics. Springer, Berlin, 1991.
14. Harder MA. Fixed point theory and stability results for fixed point iteration procedures. PhD thesis, University of Missouri-Rolla, Missouri, 1987.
15. Karahan I, Ozdemir M. A general iterative method for approximation of fixed points and their applications, AdvFixed Point Theory. 2013;3:510-526.

16. Ishikawa S. Fixed points by a new iteration method. Proc Am Math Soc. 1974;44:147-150.17.
17. Kannan R. Some results on fixed point. Bull Calcutta Math Soc. 1968;10:71-76.
18. Lungu N, Rus IA. On a functional Volterra-Fredholm integral equation, via Picard operators. Journal of Mathematical Inequalities. 2009;3(4):519-527.
19. Mann WR. Mean value methods in iteration, Proc. Am. Math. Soc. 1953;4:506-510.
20. Maleknejad K, Torabi P. Application of fixed point method for solving Volterra-Hammerstein integral equation, U.P.B. Sci. Bull., Series A. 2012;74(1).
21. Maleknejad K, Hadizadeh M. A New computational method for Volterra-Fredholm integral equations, Comput. Math. Appl. 1999;37:1-8.
22. Noor MA. New approximation schemes for general variational inequalities, J Math Anal Appl. 2000;251:217-229.
23. Phuengrattana W, Suantai S. On the rate of convergence of Mann, Ishikawa, Noor and SP-iterations for continuous functions on an arbitrary interval, J. Comput. Appl. Math. 2011;235:3006-3014.
24. Rihan FA, Abdelrahman DH, Al-Maskari F, Ibrahim F, Abdeen MA. Delay differential model for tumour-immune response with chemoimmunotherapy and optimal control, Computational and Mathematical Methods in Medicine. Article ID982978. 2014;2014:15.
25. Sahu DR, Petrusel A. Strong convergence of iterative methods by strictly pseudocontractive mappings in Banachspaces. Nonlinear Anal Theory Methods Appl. 2011;74:6012-6023.
26. Schu J. Weak and strong convergence to fixed points of asymptotically nonexpansive mappings, B. Aust. Math. Soc. 1991;43:153-159.
27. Senter HF, Dotson WG. Approximating fixed points of nonexpansive mapping, Proc. Amer. Math. Soc. 1974;44:375-380.
28. Soltuz SM, Grosan T. Data dependence for Ishikawa iteration when dealing with contractive like operators. Fixed Point Theory Appl. 2008. doi:10.1155/2008/242916.
29. Suzuki T. Fixed point theorems and convergence theorems for some generalized nonexpansive mappings, J. Math. Anal. Appl. Math. 2008;340:1088-10995.
30. Thianwan S. Common fixed points of new iterations for two asymptotically nonexpansiven on self-mappings in a Banach space. Journal of Computational and Applied Mathematics. 2009;224:688-695.
31. Thakur D, Thakur BS, Postolache M. A new iterative scheme for numerical reckoning fixed points of Suzuki’s generalized nonexpansive mappings, Appl. Math. Comput. 2016;275:147-155.
32. Thakur BS, Thakur D, Postolache M. A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized nonexpansive mappings, Appl. Math. Comput. 2016;275:147-155.
33. Ullah K, Arshad M. New iteration process and numerical reckoning fixed points in Banach spaces, University Politehnica of Bucharest Scientific Bulletin Series A. 2017;79:113-22.
34. Ullah K, Arshad M. Numerical reckoning fixed points for Suzuki’s generalized nonexpansive mappings via new iteration process, Filomat. 2018;32:187-196.
35. Ullah K, Arshad M. New three-step iteration process and fixed point approximation in Banach spaces, Journal of Linear and Topological Algebra. 2018;7(2):87-100.
36. Wazwaz AM. A reliable treatment for mixed Volterra-Fredholm integral equations, Appl. Math. Comput. 2002;127:405-414.
37. Weng X. Fixed point iteration for local strictly pseudocontractive mapping, Proc. Am. Math. Soc. 1991;113:727-731.
38. Zamfirescu T, Fixed point theorems in metric spaces, Arch. Math. (Basel). 1972;23:292-298.