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Finite single integral representation for the generalized polynomial set $D_n\{(x_k),y\}$

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Abstract
In the present paper, an attempt has been made to express a Finite Single Integral Representation for the Generalized Polynomial set $D_n\{(x_k),y\}$ has been defined by means generating relations which contains the product of generalized hypergeometric functions and lauricella function in the notation of Burchnall and Chaundy [1]. Many interesting new results may be obtained as particular cases on separating the parameters.
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1. Introduction

We define the certain hypergeometric polynomial set of n-variables by means of generating functions.

$$(\xi+vt^e)^{-\sigma} F\left[\begin{matrix}(a_r) & \\ & \mu_1 y^{-e_1}t^{e_1}\\ (b_s) & \end{matrix}\right]$$

$$F\left[\begin{matrix}(A_p);(\alpha_g);(\gamma_{u_k})\\ \mu x_1^{t_1}t, \mu_2 x_2^{e_2}y^{-e_2}t^{e_2}\ldots\ldots\mu_k x_k^{e_k}t^{e_k}\\ (B_q);(\beta_h);(\delta_{v_k})\end{matrix}\right]$$

$$\sum_{n=0}^{\infty} D_{n,e;e_1,e_2;\ldots\ldots e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v_k})}^{v;\xi,\sigma;\mu;\mu_1;\mu_2;\ldots\ldots\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{u_k});}\{(x_k),y\}t^n. \tag{1.1}$$

Where $v,\xi,\sigma,\mu,\mu_1\mu_2,\ldots\ldots\mu_k$ are real and $e,e_1,e_2,\ldots\ldots e_k,r_1$ are non-negative integers.

The left hand sides of (1.1) contains the product of generalized hypergeometric functions involving Lauricella functions in the notation of Burchnall and Chaundy [1]. The generalized polynomial set contains a number of parameters, for simplicity we shall denote it

$$D_{n,e;e_1,e_2;\ldots\ldots e_k,r_1;(b_s);(B_q);(\beta_h);(\delta_{v_k})}^{v;\xi,\sigma;\mu;\mu_1;\mu_2;\ldots\ldots\mu_k;(\alpha_r)(A_p);(\alpha_g);(\gamma_{u_k});}\{(x_k),y\} \text{ by } D_n\{(x_k),y\}.$$

Where *n* denotes the order of the generalized polynomial set $D_n\{(x_k),y\}$.

After little simplification (1.1) gives.

$$D_n\{(x_k),y\}=\xi^{\sigma}\sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\cdots\cdots\cdots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1\cdots\cdots-e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times\frac{\left[\left(A_p\right)\right]_{n-em-e_1m_1-(e_2-1)m_2\cdots\cdots(e_k-1)m_k}\left[\left(\alpha_g\right)\right]_{n-em-e_1m_1-e_2m_2\cdots\cdots e_km_k}}{\left[\left(B_q\right)\right]_{n-em-e_1m_1-(e_2-1)m_2\cdots\cdots(e_k-1)m_k}\left[\left(\beta_h\right)\right]_{n-em-e_1m_1-e_2m_2\cdots\cdots e_km_k}}$$

$$\times\frac{\left[\left(\gamma_{u_k}\right)\right]_{mk}\left[\left(a_r\right)\right]_{m_1}\left(\sigma_m\right)_m(-v)^m\,\mu_1^{m_1}\left(\mu_2x_2^{e_2}\right)^{m_2}\cdots\cdots\cdots\left(\mu_kx_x^{e_k}\right)^{m_k}}{\left[\left(\delta_{v_k}\right)\right]_{m_k}\left[\left(b_s\right)\right]_{m_1}\xi^m\quad m!\quad m_1!\quad y^{e_1m_1+e_2m_2}\quad m_2!\quad m_k!}$$

$$\times\frac{\left(\mu x_1^{r_1}\right)^{n-em-e_1m_1-e_2m_2-e_3m_3\cdots\cdots e_xm_k}}{\left(n-em-e_1m_1-e_2m_2-\cdots\cdots-e_km_k\right)!}\qquad(1.2)$$

2. Notations

(i)

$$(n)=1,2\ldots.n-1,.$$

$$\left(a_p\right)=a_1,a_2,\ldots.a_p$$

(ii)

$$\left[\left(a_p\right)\right]=a_1,a_2,\ldots\ldots a_p.$$

$$\left[\left(a_p\right)\right]_n=\left(a_1\right)_n,\left(a_2\right)_n\cdots\cdots\left(a_p\right)_n$$

$$\Delta(a;b)=\frac{b}{a},\frac{b+1}{a},\ldots..\frac{b+a-1}{q}$$

$$\Delta[a;b]=1-\frac{b}{a},1-\frac{b+1}{a},\ldots\ldots1\frac{b+a-2}{a}$$

(iii) $$\Delta_k\left[(a;b)\right]=\prod_{r=1}^{a}\left(\frac{a+r-1}{a}\right)_k=\left(\frac{b}{a}\right)_k\left(\frac{b+1}{a}\right)_k\cdots\cdots\left(\frac{b+a-1}{a}\right)_k$$

$$\Delta_k\left[m;\left(a_p\right)\right]=\prod_{k=1}^{p}\prod_{r=1}^{m}\left(\frac{a_i+r-1}{m}\right)_k$$

$$\Delta_{k+1}(b;a)=\Delta_k(b;a)\Delta(b;a+1).$$

$$\Gamma\left[a+\frac{(m)}{m}\right]=\prod_{r=1}^{m}\Gamma\left(a+\frac{r}{m}\right).$$

(iv)
$$\Gamma\left[a+\frac{(m)+(b_q)}{m}\right]=\prod_{r=1}^{m}\prod_{i=1}^{q}\Gamma\left(a+\frac{r+b_i}{m}\right).$$

$$\Gamma\left[\left(m:(a_p)\right)\right]=\prod_{i=1}^{p}\prod_{r=1}^{m}\Gamma\left(\frac{a_j+r-1}{m}\right).$$

$$\Gamma\left[(a;b)\right]=\prod_{r=1}^{a}\Gamma\left(\frac{b+r-1}{a}\right).$$

(v)
$$\Gamma(a\pm b)=\Gamma(a+b)\Gamma(a-b.)$$
$$\Gamma_{**}(a+b)=\Gamma(a+b)\Gamma(a+b).$$

3. Theorem: For $e_2>1,\ldots\ldots,e_k>1$

$$D_n\{(x_k),y\}=\frac{n!\Gamma(\alpha+a)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(\alpha+a+n)\Gamma(\beta+\alpha+1)}$$

$$\times P\int_{-1}^{1}(1-x)^{a}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$$

$$\times F^{1+p+g+r:\,u_1,u_2\ldots\ldots\ldots u_k}_{q+h+s:\,V_1,V_2,\ldots\ldots\ldots\ldots V_k}\Big[[(-n);e,e_1,e_2\ldots\ldots..e_k)],\underline{\qquad}$$

$$[(1-(B_q)-n):e,e_1,e_2-1,e_k-1],[(1-(\beta_h)-n):e,e_1,e_2\ldots e_k],$$

$$[(1-(A_p)-n):e,e_1,e_2-1,e_k-1],[(1-(\alpha_g)-n):e,e_1,e_2\ldots e_k],$$

$$[(\alpha_r):1],[(\gamma_{u_1}):1][(\gamma_{u_2}):1]\ldots[(\gamma_{u_k}):1],[\sigma:1],[(\mu+v+1):2]$$

$$[(b_s):1],[(\delta_{V_1}):1],[(\delta_{V_2}):1]\ldots[(\delta_{V_k}):1],[(\mu):1],[(v):1]$$

$$\times\frac{-v(-1)^{e(r+s+p+q+g+h+1)}}{\xi\left(\mu x_1^{r_1}\right)^{e}},\frac{\mu_1(-1)^{e_1(r+s+p+q+g+h+1)}}{\left(\mu x_1^{r_1}y\right)^{e_1}},$$

$$\times\frac{\mu_2x_2^{e_2}(-1)^{e_2(r+s+p+q+g+h+1)+p+q}}{\left(\mu x_1^{r_1}y\right)^{e_2}},$$

$$\ldots\ldots\ldots\ldots..,\times\frac{\mu_kx_k^{e_k}(-1)^{e_k(r+s+g+h+p+q+1)+p+q}}{\left(\mu x_1^{r_1}\right)^{e_k}}\Big]\,dx\ \ldots. \quad (3.1)$$

Proof: We have

$$I_1=\int_{-1}^{1}(1-x)^{\alpha}(1+x)^{\beta}P_n^{(\alpha,\beta)}(x)$$

$$\times\sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\ldots\ldots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2\ldots\ldots e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times \frac{[(A_p)]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}[(\alpha_g)]_{n-em-e_1m_1-e_2m_2......e_km_k}}{[(B_q)]_{n-em-e_1m_1-(e_2-1)m_2......(e_k-1)m_k}[(\beta_h)]_{n-em-e_1m_1-e_2m_2......e_km_k}}$$

$$\times \frac{[(\alpha_{u_k})]_{m_k}[(a_r)]_{m_1}(\sigma)_m(-v)^m\mu_1^{m_1}(\mu_2x_2^{e_2})^{m_2}\ldots\ldots(\mu_kx_k^{e_k})^{m_k}}{[(\beta_{v_k})]_{m_k}[(b_s)]_{m_1}\xi^m\, m!\; m_1!\; y^{e_1m_1+e_2m_2}m_2!\; m_k!}$$

$$\times \frac{(\mu x_1^{r_1})^{n-em-e_1m_1-e_2m_2......e_km_k}(\alpha-a)_{m_1}(\alpha+\beta+n+1)_m dx}{(n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k)(\alpha-a+n)_{m_1}}$$

We have from [2]

$$\int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}P_n^{(\alpha,\beta)}\,dx = \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha-a-n)\Gamma(\beta+n+1)}{n!\Gamma(\alpha-a)\Gamma(\alpha+\beta+n+1)}$$

$$= \frac{\sigma}{\xi}\sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\ldots\ldots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2......e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times \frac{[(A_p)]_{n-em-e_1m_1-(e_{2-1})m_2......(e_k-1)m_k}[(\alpha_g)]_{n-em-e_1m_1-e_2m_2......e_km_k}}{[(B_q)]_{n-em-e_1m_1-(e_{2-1})m_2......(e_k-1)m_k}[(\beta_h)]_{n-em-e_1m_1-e_2m_2......e_km_k}}$$

$$\times \frac{[(\gamma_{u_k})]_{m_k}[(a_r)]_{m_1}(\sigma)_m(-v)^m\mu_1^{m_1}(\mu_2x_2^{e_2})^{m_2}\ldots\ldots(\mu_kx_k^{e_k})^{m_k}}{[(\delta_{v_k})]_{m_k}[(b_s)]_{m_1}\xi^m\, m!\; m_1!\; y^{e_1m_1+e_2m_2}m_2!\; m_k!}$$

$$\times \frac{(\mu x_1^{r_1})^{n-em-e_1m_1-e_2m_2.......e_km_k}(\alpha-a)_{m_1}(\alpha+\beta+n+1)_{m_1}}{(n-em-e_1m_1-e_2m_2\ldots\ldots e_km_k)2^{m_1}(\alpha-a+n)_{m_1}\,n!}$$

$$\times \frac{2^{\alpha+m_1+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha+m_1-a+n)\Gamma(\beta+n+1)}{\Gamma(\alpha+m_1+\beta+n+1)\Gamma(\alpha-a)}$$

$$= \frac{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\alpha-a+n)\Gamma(\beta+n+1)\xi^{\sigma}}{n!\;\Gamma(\alpha-a)\Gamma(\alpha+\beta+n+1)}$$

$$\times P\sum_{m=0}^{\left[\frac{n}{e}\right]}\sum_{m_1=0}^{\left[\frac{n-em}{e_1}\right]}\sum_{m_2=0}^{\left[\frac{n-em-e_1m_1}{e_2}\right]}\ldots\ldots\sum_{m_k=0}^{\left[\frac{n-em-e_1m_1-e_2m_2.......e_{k-1}m_{k-1}}{e_k}\right]}$$

$$\times \frac{[1-(B_q)-n]_{em+e_1m_1+(e_2-1)m_2+\,......+\,(e_k-1)m_k}}{[1-(A_P)-n]_{em+e_1m_1+(e_2-1)m_2+\,......+\,(e_k-1)m_k}}$$

$$\times \frac{[1-(\beta_h)-n]_{\,em-e_1m_1-e_2m_2+\,........+\,e_km_k}[(\gamma_{u_1})]_{m_1}[(\gamma_{u_2})]_{m_2}}{[1-(\alpha_g)-n]_{em-e_1m_1-e_2m_2+\,........+\,e_km_k}[(\delta_{v_1})]_{m_1}[(\delta_{v_2})]_{m_2}}$$

$$\times \frac{[(\gamma_{u_k})]_{m_k}[a_r]_{m_1}(\sigma)_m(-n)_{em+e_1m_1+e_2m_2+\,........+\,e_km_k}}{[(\delta_{v_k})]_{m_k}[(b_s)]_{m_1}\,m!}$$

$$\times \frac{(-v)^m(-1)^{(r+s+p+q+g+h+1)em}\mu_1^m(-1)^{(r+s+p+q+g+h+1)e_1m_1}}{(\mu x_1^{r_1})^{em}(\mu x_1^{r_1})^{e_1m_1}m_1!}$$

$$\times \frac{(\mu_2x_2^{e^2})^{m_2}(-1)^{\{e_2(1+r+s+p+q+g+h)+p+q\}+m_2}\ldots\ldots\mu_k^{m_k}x_k^{e_k}(-1)^{\{e_k(r+s+p+q+g+h+1)+p+q\}+m_k}}{(\mu x_1^{r_1}\,y)^{e_2m_2}m_2!\,(\mu x_1^{r_1}\,y)^{e_km_k}m_k!} \tag{3.2}$$

The single terminating factor $(-n)_{em+e_1m_1+e_2m_2+\ldots\ldots+e_km_k}$ makes all summation in (3.2) runs up to ∞ and finally achieve.

$$= \frac{2^{\alpha+\beta+1}\Gamma(a+1)\Gamma(\alpha-a+n)\Gamma(\beta+n+1)}{n!\,\Gamma(\alpha-a)\Gamma(\alpha+\beta+n+1)} D_n\{(x_k), y\}$$

4. Particular Cases of (3.1)

Separating the term corresponding to $m_2 = 0 = \ldots\ldots = m_k \Rightarrow x_2 = 0 = \ldots. = x_k$ in (3.2) & we obtained a number of results as specializing the remaining parameters:

i) on making the substitution p= 0 = q = g = u_1 = v_1= r_1= e =1= ξ= v = σ= μ = μ_1 = e = y = e_1; x_1=$\frac{1}{x}$ and replace(a_r) by(A_p) and (b_s) by (b_q) in (3.1), we obtained

$${}_1F_1(-n; b; x) = \frac{\Gamma(\alpha-a)\Gamma(\alpha+\beta+n+1)}{2^{\alpha+\beta+1}\Gamma(a+1)\Gamma(\alpha-a+n)\Gamma(\beta+\alpha+1)n!}$$

$$\times F\begin{bmatrix} -n, (a), \Delta(\alpha-a; 1), (\alpha+\beta+n+1:1); \\ x \\ (b_q), (\alpha+a+n; 1); \end{bmatrix} \mathrm{dx}$$

ii) On putting r = 0 = q = g = u_1 = v_1= r_1= u_1 = v_1; 1+λ_2, b_1 = -1,e =1,e = 1 = v = ξ = y = μ = μ_1 = s = e_1; x_1= y in (3.1), we obtained

$$A_n^{(\lambda_2)}(y) = \frac{\Gamma\,(\alpha-a)\Gamma(\alpha+\beta+n+1y^n}{2^{\alpha+\beta+1}n!\Gamma(a+1)\Gamma(\alpha-a+n)\Gamma(\beta+n+1)}$$

$$\times \int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}$$

$$\times F\begin{bmatrix} -n, 1+\lambda_2(\alpha-a; 1), (\alpha+\beta+n+1:1); \\ \frac{1}{y} \\ -1, (\alpha-a+n:1); \end{bmatrix} dx$$

Where $A_n^{(\lambda_2)}(y)$ are the srivastava polynomials [4]

iii) On putting r = 0 = s =p = q = h = u_1 = v_1; e = e_1 = g = 1 = r_1= ξ = σ = v = μ = μ_1 = y; α_1= 1+ λ, x_1=$\frac{-1}{y}$ in (3.1), we obtained

$$A_n^{(\lambda)}(y) = \frac{\Gamma\,(\alpha-a)\Gamma(\alpha+\beta+n+1)(-1)^n(1+\lambda)}{n!\,2^{\alpha+\beta+1}\Gamma(a+1)\Gamma(\alpha-a+n)\Gamma(\beta+n+1)}$$

$$\times \int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta} \times F\begin{bmatrix} -n, (\alpha-a; 1), (\alpha+\beta+n+1:1); \\ -y \\ -\lambda-n; (\alpha-a+n; 1); \end{bmatrix} dx$$

Where $A_n^{(\lambda)}(y)$ are the polynomials defined by Srivastava K.N [3]

iv) On putting r =0 = q = g = u_1 = v_1; r_1= 1 =s = ξ = σ = = v = μ = e = e_1 = μ_1 = y; b_1=1+α and x_1=$\frac{1}{y}$ in (3.1), we obtained

$$L_n^{(\alpha)}(y) = \frac{(1+\alpha)_n\Gamma(\alpha-a)\Gamma(\alpha+\beta+n+1)}{y^n n!\,2^{\alpha+\beta+1}\Gamma(a+1)\Gamma(\alpha-a+n)\Gamma(\beta+n+1)}$$

$$\times \int_{-1}^{1}(1-x)^{\alpha}\,(1+x)^{\beta}$$

$$\times F\begin{bmatrix} -n, (\alpha-a; 1), (\alpha+\beta+n+1:1); \\ y \\ 1+\alpha; (\alpha-a+n; 1); \end{bmatrix} \mathrm{dx}$$

Where $l_n^{(\alpha)}(y)$ are the generalized laguerre polynomials

5. References

1. Burchnall JL, Chaundy TW. Expansions of Appell's Double Hypergeometric Functions, Quart. Journal of Mathematics. Oxford. 1941;12:112-118.
2. Rainville ED. Special Functions. Mac Millan Co. New York, 1960.
3. Srivastava KN. Some Polynomials related to the Laguerre Polynomial. Journal of Indian Mathematics, Soc. 1964;28(2):43-59.
4. Srivastava PN. Some Operational Formulas and Generalized Generating Functions, Math Education. 1974;8:19-23.