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Credit risk measurement and management using a multivariate Markov chain model

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Abstract
The application on logistic regression has been done on classification and determination of influences that affects the behavior of consumer score. One type of logistic - regression is the cumulative logistic - regression that has the latent - variable linking the functions that determines the consumers' behavior score that are dynamic in nature. The multivariate logistic - regression that describes the dependent nature of credit - risk of an asset is a given asset or portfolio. This paper takes on the credibility theory combining the transitional application that is credit bureau alongside behavioral transitional matrix that is obtained from the performance of consumer and its experience.

Keywords: Logistic regression, asset, portfolio, credibility theory, transition matrix

1. Introduction
The non-regulated finance companies in Kenya do thrive irrespective of the presence and the requirements involved from the Micro-finance Act 2006 where anyone operating micro - finance business is to be licensed. Therefore this depends on the inviolability in contracting the law that preserve them in these businesses. These businesses do operates in an ambiguous way with contracts that are mostly misinterpreted by the borrowers' thereby being non-realistic in payments of interests. Using part 2 of Section 9 (1) (c) from the Micro - finance Act, which describes situations where license can be revoked bringing a possibility of shutting down of business, this occurs when it is being run against the interest of the customer.
Since those who provide credit has a high rate of growth from start-ups in businesses, they develop, tests and use models that can be used to assess the risks over a very short - time period. The main objectives for borrowers usually is to build their businesses not only with very high level from automation but also with a higher or greater cautiousness alongside risk management. The borrowers are therefore required to build a registered analytics that they can use to apply to their data so that this will enable them maintain fair completion within their targeted markets. Therefore these models being developed should be less costly and effective in respect to loan applications. All these considerations in credit - risk modeling make them have a good competitive benefit.

2. Preliminaries
2.1 Markov Chain
It is a Markov process which shows that the pricing process depends only on the current price and not the previous value. It can be discrete with discrete time space.

2.2 Chapman Kolmogorov
It is a formation of equations that can allow calculations of general transitional probabilities in terms of one step probabilities $P_{i,j}^{(n,n+1)}$.
Let $P_{i,j}^{(m,n)}$, the transitional probability in a state j at time n being in a state i at time m

$$\Pr {}^{X_{m+1=j}}\big/_{X_{m=\,i}} = P_{i,j}^{(m,m+1)}$$

The transitional probability from a discrete Markov - chain do obey Chapman-Kolmogorov equation. With one step transitional probability $P_{i,j}^{(n,n+1)}$ the initial probability distribution given as $q_{k= P(X_0=k)}$ used in deducing the probability of any path.

2.3 Homogeneous - Time Markov Chain

Homogeneous - time Markov chain is one where one step transitional probabilities are independent of time and is given as

$$P_{i,j}^{j,n+1} = P_{i,j}$$

Therefore Chapman - Kolmogorov equations usually is given as

$$P_{i,j}^{(n-m)} = \sum_{k\epsilon S} P_{1k}^{(i-m)} P_{kj}^{(n-1)}$$

With normalization condition given as $\sum_{j\epsilon S} P_{ij} = 1$, for all i

3. Main results

3.1 Logistic Regression

The logistic regression is a regression describing the relation between binary response with one or more explanatory variables alongside the application to the logistic – transformation of the dependent variables. Below we represent a simple – logistic model known as the logit – Y with natural – logarithm of odds in the ratio;

$$Y = \ln\left[\frac{\pi(x)}{1-\pi(x\tau)}\right] = \alpha + \beta \text{X}.,$$

Applying antilog to the above equation gives the following equation.

$$\pi = \frac{e^{\alpha+\beta \text{X}}}{1+ e^{\alpha+\beta \text{X}}},$$

where π is the probability derived by the outcome of interest, α is its intercept and β is the coefficient of regression.

From the above equation, the logit of $Y\ on\ X$ is linear and the probability of $Y\ given\ X$ is non-linear making the relationship of $Y\ and\ X$ linear when the relationship is transformed using natural logarithm.

The coefficient of regression β describes the relation that occurs between $logit\ of\ Y\ with\ X$.

Using the above equation of logistic regression we extend the equation to multiple independent variables $X_1, X_2 \dots\dots X_p$, which can be expressed as

$$Y = \ln\left[\frac{\pi(x)}{1-\pi(x\tau)}\right] = \alpha + X_1\beta_1 + X_2\beta_2 + \cdots \dots + X_p\beta_p$$

Then probability of $(Y = outcome\ of\ Interest/X_1 = x_1, X_2 = x_2)$ is given as

$$\pi = \frac{e^{\alpha+X_1\beta_1+X_2\beta_2+\cdots\dots+X_p\beta_p}}{1+ e^{\alpha+X_1\beta_1+X_2\beta_2+\cdots\dots+X_p\beta_p}}$$

We estimate the value of $\alpha\ and\ \beta$ using the maximum likelihood technique.

3.2 Cumulative Logistic Regression

It is analyzing of univariate categorical ordered data where the explanatory variables of log odds are selected from lower response to the higher response categories given by;

If $p(Y \leq j/x) = \pi_1(x) + \pi_2(x) + \cdots \dots \dots . + \pi_j(x)$

We therefore define the cumulative logistic regression as

$$logit\ p(Y \leq j/x) = \log\left(\frac{\left(p(Y \leq j/x)\right)}{\left(1 - p(Y \leq j/x)\right)}\right)$$

$$= \log\left(\frac{\pi_1(x)+\pi_2(x)+\cdots\dots\dots+\pi_j(x)}{\pi_{j+1}(x)+\pi_{j+2}(x)+\cdots\dots\dots+\pi_J(x)}\right)$$

The latent variable can be used to expressed the cumulative model as follows

$$y_i^* = \sum_{k=1}^{K} \beta_k X_{ik} + \varepsilon_i$$

And $y_i = \begin{cases} 1 \; if \; y_i^* \leq \alpha_1 \\ 2 \; if \; \alpha_2 \leq y_i^* \leq \alpha_1 \\ 3 \; if \; \alpha_3 \leq y_i^* \leq \alpha_2 \\ . \\ . \\ . \\ J-1 \; if \; \alpha_{j-2} \leq y_i^* \leq \alpha_{j-1} \\ J \; if \; \alpha_{j-1} \leq y_i^* \leq \alpha_j \end{cases}$

The model will be of the following form of the distribution with error term ε_i giving the logistic distribution as

$$\ln\left[\frac{\Pr(Y_i \leq b)}{\Pr(Y_i > b)}\right] = \alpha_b - \sum_{k=1}^{K} \beta_k X_{ik} \; b = j \ldots \ldots . . j-1,$$

where Y_i gives the response in the i^{th} individual and y_i is the observed variable, y_i^* is latent variable, X_{ik} is the independent variable for the i^{th} individual,

If $j \; is \; less \; than \; k$ then curve in $p(Y \leq k)$ also gives a curve in $p(Y \leq j)$ mapped to $\frac{(\alpha_k - \alpha_j)}{\beta}$ units in the direction of X resulting to;

$$p(Y \leq k / X = x) = p(Y \leq j / X = x + \frac{(\alpha_k - \alpha_j)}{\beta})$$

3.3 Parameter estimations

The common matrix of transition used in credit risks portfolio is based on the information initially derived from the score behavior used.

This results to estimates of $N_e^{(jk)}$ of $N^{(jk)}$is given by $N_e^{(jk)}$=$W_{jk}M^{(jk)} + (1 - W_{jk})N^{(jk)}$for $j, k = 1,2 \ldots \ldots n$ where $0 \leq j, k \leq 1, for \; j, k = 1 \leq ,1,2 \ldots \ldots . . n$ and $Q^{(jk)}$ is the prior matrix for estimation of $N^{(jk)}$.

This proposition given above is what is stated in the multivariate Markov – chain that has stationary distribution of X . Using vector X the occurrence of each state is proportional to the categorical order of ratings

This estimate is denoted by $\dot{X} = (\dot{X}^{(1)}\dot{X}^{(2)\ldots\ldots\ldots}\dot{X}^{(n)})^T$

The proposition given above results to matrix Q given as,

$$\begin{pmatrix} \lambda_{11}P_e^{(11)}\lambda_{12}P_e^{(12)} \ldots \ldots \ldots . . \lambda_{1n}P_e^{(1n)} \\ \lambda_{21}P_e^{(21)}\lambda_{22}P_e^{(22)} \ldots \ldots \ldots \lambda_{2n}P_e^{(2n)} \\ . \\ . \\ . \\ \lambda_{n1}P_e^{(n1)}\lambda_{n2}P_e^{n2} \ldots \ldots \ldots \lambda_{nn}P_e^{(nn)} \end{pmatrix} \hat{X} \approx \hat{X}$$

Let $\lambda_{jk}^{\sim 1} = \lambda_{jk}W_{jk}$ and $\lambda_{jk}^{\sim 2}$=$\lambda_{jk}(1 - W_{jk})$.it can be shown that $\lambda_{jk}^{\sim 1} + \lambda_{jk}^{\sim 2}$=$\lambda_{jk}$, in each j, k=1, 2…n

$$\min_{\lambda^{\sim 1}\lambda^{\sim 2}} O_j$$

$$\begin{cases} subject \; to \begin{pmatrix} O_j \\ O_j \\ O_j \\ O_j \\ . \\ . \\ O_j \\ O_j \end{pmatrix} \geq \hat{X}^{(j)} - B_j \begin{pmatrix} \lambda_{j1}^{\sim 1} \\ \lambda_{j1}^{\sim 2} \\ \lambda_{j2}^{\sim 1} \\ \lambda_{j2}^{\sim 2} \\ . \\ . \\ \lambda_{jn}^{\sim 1} \\ \lambda_{jn}^{\sim 2} \end{pmatrix}, \begin{pmatrix} O_j \\ O_j \\ O_j \\ O_j \\ . \\ . \\ . \\ O_j \\ O_j \end{pmatrix} \geq \widehat{-X}^{(j)} + B_j \begin{pmatrix} \lambda_{j1}^{\sim 1} \\ \lambda_{j1}^{\sim 2} \\ \lambda_{j2}^{\sim 1} \\ \lambda_{j2}^{\sim 2} \\ . \\ . \\ . \\ \lambda_{jn}^{\sim 1} \\ \lambda_{jn}^{\sim 2} \end{pmatrix} \end{cases}$$

$O_j \geq 0$

$\sum_{k=1}^{n}(\lambda_{jk}^{\sim 1} + \lambda_{jk}^{\sim 2}) = 1, \lambda_{jk}^{\sim 1} \geq 0 \; and \; \lambda_{jk}^{\sim 2} \geq 0, \forall j, k$ where

$$B_j = Q^{(j1)}\hat{X}^{(1)}P^{(j1)}/Q^{(j2)}\hat{X}^{(2)}/P^{(j2)}\hat{X}^{(2)}/ \ldots/Q^{(jn)}\hat{X}^{(n)}/P^{(jn)}\hat{X}^{(n)}$$

3.4 Credit Risk Measures

Define

$$P^{(j)}_{t+1/t} := P^{(j1)}_{t+1/t}, P^{(j2)}_{t+1/t}, \ldots, P^{(jn)}_{t+1/t}$$

$$P^{ji}_{t+1/t} := P\left(\left\{Y^{(j)}_{t+1} = e_i\right\}/\mathcal{F}_t\right) = E_\rho(\langle Y^{(j)}_{t+1}, e_i\rangle / \mathcal{F}_t$$

$$= E_p\left(\langle Y^{(j)}_{t+1}, e_i\rangle\right) \mathsf{I}_{Y_t=(e_{i_1}e_{i_2},\ldots,e_{i_n})}$$

From equation (1) below,

$$X^{(j)}_{t+1} = \sum_{j=1}^{n} \lambda_{jk}\, P^{(jk)} X^{(k)}_t, for\ j = 1,2, \ldots . n.$$

The above equation has unknown parameters which can be estimated:

$$X^{(j)}_{t+1} = \sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X^{(k)} \approx \sum_{k=1}^{n} (\lambda^{\sim 1}_{jk} N^{(jk)} + \lambda^{\sim 2}_{jk} M^{(jk)}) X^{(k)}_t, for\ j = 1,2 \ldots, n.$$

Let $[V]^i$ in the i[th] element be denoted by the column vector V, for each $j = 1,2, \ldots n\ and\ i = 1,2, \ldots n$. Then we have,

$$P^{(ji)}_{t+1/t} = E_\rho(\langle Y^{(j)}_{t+1}, e_i\rangle) \mathsf{I}_{Y_t=(e_{1_i}e_{i_2},\ldots e_{i_n})}$$

$$=\rho\left(\left\{Y^{(j)}_{t+1} = e_i\right\}\right) \mathsf{I}_{Y_t=e_{i_1}e_{i_2},\ldots,e_{i_n}}$$

$$= \left[X^{(j)}_{t+1}\right]^i \mathsf{I}_{Y_t=(e_{i_1}e_{i_2}\ldots e_{i_n})} = \left[X^{(j)}_{t+1}\right]^i \mathsf{I}_{X_t=(e_{i_1}e_{i_2},\ldots,e_{i_n})}$$

$$= \left[\sum_{k=1}^{n} \lambda_{jk} P^{(jk)} X^{(k)}\right]^i \mathsf{I}_{X_t=e_{i_1},e_{i_2},\ldots e_{i_n}} \approx \left[\sum_{k=1}^{n} (\lambda^{\sim 1}_{jk} Q^{(jk)} + \lambda^{\sim 2}_{jk} \hat{P}^{(jk)}) X^{(k)}_t\right]^i \mathsf{I}_{X_t=(e_{i_1},e_{i_2},\ldots e_{i_n})}$$

This implies that

$$E_\rho(\mathcal{L}_{t+1}(Y_{t+1})/\mathcal{F}_t) = \sum_{j=1}^{n}\sum_{i=1}^{m} \langle L^j_{t+1}, e_i\rangle \rho(\left\{Y^{(j)}_{t+1} = e_i\right\}/\mathcal{F}_t)$$

$$= \sum_{j=1}^{n}\sum_{i=1}^{m} \langle L^j_{t+1}, e_i\rangle P^{(ji)}_{t+1/t} \approx \sum_{j=1}^{n}\sum_{i=1}^{m} \langle L^j_{t+1}, e_i\rangle \left[\sum_{k=1}^{n} (\lambda^{\sim 1}_{jk} N^{(jk)} + \lambda^{\sim 2}_{jk} \widehat{M}^{(jk)}) X^{(k)}_t\right]^i \mathsf{I}_{X_t=(e_{i_1},e_{i_2},\ldots e_{i_n})}$$

This conditional joint predictively distribution, Y_{t+1} is derived from the information set $\mathcal{F}_t$ which is important in evaluating credit - value at risk (VaR).

$$P^{(j)}_{t+1/t} := \left(P^{j1}_{t+1/t}, P^{j2}_{t+1/t}, \ldots \ldots P^{jn}_{t+1/t}\right)^T \approx \sum_{k=1}^{n} (\lambda^{\sim 1}_{jk} Q^{(jk)} + \lambda^{\sim 2}_{jk} \hat{P}^{(jk)}) X^{(k)}_t \mathsf{I}_{X_t=(e_{i_1},e_{i_2},\ldots,e_{i_n})}$$

$Y^{(1)}_{t+1}, Y^{(2)}_{t+1}, \ldots \ldots \ldots Y^{(n)}_{t+1}$ are resulting conditionally independent variables given by $\mathcal{F}_t\ or\ Y_t$ therefore the conditional joint predictively distribution $P_{t+1/t}\ of\ Y_{t+1}$,is derived from the information $\mathcal{F}_t$ that is completely given as $P_{t+1/t} := \left(P^{j1}_{t+1/t}, P^{j2}_{t+1/t}, \ldots \ldots P^{jn}_{t+1/t}\right)^T$, where $P_{t+1/t}$ is given as (n x m) in the dimensional - probability matrix.

This conditional joint predictively distribution has the cumulative loss $\mathcal{L}_{t+1}$ equated to $\mathcal{L}_{t+1}(\widetilde{k})$ that is derived as:

$$\rho(\mathcal{L}_{t+1} = \mathcal{L}_{t+1}(\widetilde{k})/\mathcal{F}_t) = \sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}} \prod_{j=1}^{n} P^{(ji_j)}_{t+1/t} \approx \sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}} \left\{\prod_{j=1}^{n}\left[\sum_{k=1}^{n} (\lambda^{\sim 1}_{jk} Q^{(jk)} + \lambda^{\sim 2}_{jk} \hat{P}^{(jk)}) X^{(k)}_t \mathsf{I}_{X_t=(e_{i_1},e_{i_2},\ldots,e_{i_n})}\right]^{i_j}\right\}$$

The VaR of the asset has a level of probability $\alpha \epsilon (0,1) at\ time\ t+1$ given by the information in the market denoted as $\mathcal{F}_t$ is given by $VaR_{\alpha,}\rho(\mathcal{L}_{t+1}/\mathcal{F}_t) := inf\{L \epsilon \mathcal{R} / \rho(\mathcal{L}_{t+1} \geq L / \mathcal{F}_t) \leq \alpha \}$

Suppose $K^\star$ gives a positive integer in $\{1,2, \ldots \ldots M\}$ that is to say

$$\rho(\mathcal{L}_{t+1} \geq / \mathcal{L}_{t+1}(K^\star)/\mathcal{F}_t) = \sum_{k=K^\star}^{M} \rho(\mathcal{L}_{t+1} = \mathcal{L}_{t+1}(\widetilde{k}) / \mathcal{F}_t) \leq \alpha$$

$$\rho(\mathcal{L}_{t+1} \geq / \mathcal{L}_{t+1}(K^\star)/\mathcal{F}_t) = \sum_{k=K^\star+1}^{M} \rho(\mathcal{L}_{t+1} = \mathcal{L}_{t+1}(\widetilde{k}) / \mathcal{F}_t) > \alpha$$

Then we have

$$VaR_{\alpha,}\rho(\mathcal{L}_{t+1}/\mathcal{F}_t) = \mathcal{L}_{t+1}(K^\star)$$

$$ES_\alpha(\mathcal{L}_{t+1}/\mathcal{F}_t) = \frac{1}{\alpha} E_p(\mathcal{L}_{t+1} \boldsymbol{I}_{\{\mathcal{L}_{t+1} \geq / \mathcal{L}_{t+1}(K^\star)\}} + A(\alpha)$$

The adjusted term $A(\alpha)$ is given as;

$$A(\alpha) := \mathcal{L}_{t+1}(K^\star)\left[1 - \frac{\rho(\mathcal{L}_{t+1} \geq / \mathcal{L}_{t+1}(K^\star)/\mathcal{F}_t)}{\alpha}\right]$$

$$ES_\alpha(\mathcal{L}_{t+1}/\mathcal{F}_t) = \frac{1}{\alpha}\left[\sum_{k=K^\star}^{M} \mathcal{L}_{t+1}(\widetilde{k}) \rho(\mathcal{L}_{t+1} = \mathcal{L}_{t+1}(\widetilde{k})/\mathcal{F}_t) - \mathcal{L}_{t+1}(K^\star) * \rho(\mathcal{L}_{t+1} \geq / \mathcal{L}_{t+1}(K^\star)/\mathcal{F}_t) - \alpha\right]$$

$$= \frac{1}{\alpha}\left\{\sum_{k=K^\star}^{M} \mathcal{L}_{t+1}(k)\left(\sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}} \prod_{j=1}^{n} P_{t+1/t}^{j i_j}\right) - \mathcal{L}_{t+1}(K^\star) * \left[\sum_{k=K^\star}^{M}\left(\sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}} \prod_{j=1}^{n} P_{t+1/t}^{j i_j}\right) - \alpha\right]\right\}$$

$$\approx \frac{1}{\alpha}\left\{\sum_{k=K^\star}^{M} \mathcal{L}_{t+1}(k)\left\{\sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}}\left\{\left[\sum_{k=1}^{n}\left(\lambda_{jk}^{\sim 1} Q^{(jk)} + \lambda_{jk}^{\sim 2} \hat{P}^{(jk)}\right) X_t^{(k)} \mid_{X_t=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})}\right]^{i_j}\right\}\right\} - \mathcal{L}_{t+1}(K^\star)\left\{\sum_{k=K^\star}^{M}\left\{\sum_{(i_1 i_2,\ldots,i_n)\epsilon I_{t+1,\check{k}}}\left\{\left[\sum_{k=1}^{n}\left(\lambda_{jk}^{\sim 1} Q^{(jk)} + \lambda_{jk}^{\sim 2} \hat{P}^{(jk)}\right) X_t^{(k)} \mid_{X_t=(e_{i_1}, e_{i_2}, \ldots, e_{i_n})}\right]^{i_j}\right\}\right\} - \alpha\right\}\right\}$$

4. Conclusion

In this paper Logistic regression has been applied to demonstrate how classification of consumers based on their behavioral characteristics. Behavioral scoring is an important feature in managing credit risks for the existing clients and their future performance. Our model targeted the unbanked population, therefore use of logistic regression is vital component as it provides the underwriting features for important classification. Other methods of classification for instance the neutral networks, discriminant and classification trees analysis can also be used for comparison purposes. Another area for further research is changing of the objective from the estimation of probability of default to profit maximization.

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