International Journal of Statistics and Applied Mathematics

ISSN: 2456-1452
Maths 2022; 7(4): 215-225

www.mathsjournal.com
Received: 16-05-2022
Accepted: 22-06-2022

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Structure of decomposable semigroups of nonnegative r-Potent matrices in $M_n(R)$

Alka Marwaha and Rashmi Sehgal Thukral

Abstract
An r-potent matrix [9] in $M_n(\mathbb{R})$ is an $n \times n$ matrix satisfying $E^r = E$. An idempotent matrix is an r-potent with $r = 2$. A multiplicative semigroup S in $M_n(\mathbb{R})$ is to be said decomposable (see [2]) if there exists a special kind of common invariant subspace called standard invariant subspace for each $A \in S$. A semi-group S of non-negative r-potent matrices in $M_n(\mathbb{R})$ is known (see [5]) to be decomposable if $rank\ (S) > r - 1$ for all S in S. Our contributions in this paper are as follows: We study the structure of decomposable semi-groups of non-negative r-potent matrices in $M_n(\mathbb{R})$. We reduce these decomposable semi-groups into standard block triangular form wherein the diagonal blocks form constant rank indecomposable semi-groups of non-negative r-potents. Under the special condition of fullness, we obtain a block diagonalization of the decomposable semigroup of non-negative r-potent matrices. Lastly, we shall illustrate the complete structure of the maximal indecomposable semigroup of 2-potents (idempotent) with constant rank one.

Keywords: Decomposable semigroups, r-potent matrix, nonnegative

1. Introduction
Let $M_n(\mathbb{R})$ denote the space of all $n \times n$ matrices with entries from the field of real numbers. A semigroup in $M_n(\mathbb{R})$ is said to be reducible if the members of the semigroup have a common nontrivial invariant subspace. The conditions under which specific types of semigroups in $M_n(\mathbb{R})$ are reducible have been studied by mathematicians for decades (see [10]). Further, these conditions are sometimes strong enough to lead to simultaneous triangularizability [7] of the semigroup.

In this paper, we focus on studying conditions for decomposability, that is, reducibility due to the existence of standard invariant subspaces of semigroups of non-negative r-potents (including idempotents). Please note that a standard invariant subspace of a matrix A in $M_n(\mathbb{R})$ is the linear span of the standard basis vectors $\{e_{i_1}, e_{i_2}, \dots e_{i_k}\}$ such that $V\{Ae_{i_1}, Ae_{i_2}, \dots, Ae_{i_k}\} \subseteq V\{e_{i_1}, e_{i_2}, \dots, e_{i_k}\}$,
where $\{e_1, e_2, \dots, e_n\}$ is the standard ordered basis for $\mathbb{R}^n$ and $V\{v_1, v_2, \dots, v_n\}$ denotes the linear span of the vectors $\{v_1, v_2, \dots, v_n\}$.

H. Radjavi proved in [1] that a band in $M_n(\mathbb{R})$ is reducible as well as simultaneously triangularizable. It was further shown in [1] that sub multiplicativity of spectral radius on the members of a semigroup of compact operators represented by matrices with nonnegative entries results in reducibility of the semigroup, even though it may not yield decomposability.

A. Marwaha proved in [2] that under special conditions, a nonzero band in $M_n(\mathbb{R})$ is decomposable and simultaneously triangularizable. The structure of such bands was studied extensively in [2], [3] and [4].

R.S. Thukral determined the conditions for decomposability of nonnegative r-potent matrices in [5]. A general structure for a single decomposable r-potent matrix was also established in [5]. It is pertinent to note that r-potent matrices needed to be studied separately (and beyond idempotent) because an r-potent matrix may or may not be idempotent. For example, the matrix $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ in 3-potent (commonly known as tripotent), but not idempotent.

Consequently, the proof techniques used in [5] were materially different from the techniques used in [2], [3] and [4]. In this paper, we take the results of [5] and study forward the conditions under which a simplified structure of decomposable semigroups of nonnegative r-potent matrices is obtained.

The rest of the paper is structured as follows:

In Section 2, we give an overview of some salient results and definitions for decomposability of semigroups of nonnegative r-potent matrices in $M_n(\mathbb{R})$.

Further, it will be observed that the study of decomposability of a semigroup of nonnegative r-potent matrices reduce to the study of such semigroups of constant rank. This follows from the result that a nonzero ideal of an indecomposable semigroup of $n \times n$ nonnegative matrices is indecomposable.

In Section 3, we first show that the structure of semigroups of r-potent matrices having constant rank can be broken down to a standard block triangular form with nonzero diagonal blocks, with each block constituting a constant rank indecomposable semigroup of r-potent matrices. We then state, in Theorem 3.5, the conditions under which a nonnegative semigroup of r-potent matrices with constant rank can be expressed as a block diagonal representation of indecomposable semigroup of r-potent matrices.We illustrate this theorem with an example.

In section 4, we discuss the structure of maximal semigroups of nonnegative r-potent matrices in $M_n(\mathbb{R})$ and conclude the paper with complete derivation of the form of an indecomposable band of rank one with nonnegative entries in $M_2(\mathbb{R})$.

2. Conditions for Decomposability of Nonnegative Semigroups of r-potent matrices in $M_n(\mathbb{R})$

We start with some preliminary definitions and an overview of known results.

Definition 2.1 ([2]). A matrix $A \in M_n(\mathbb{R})$ is said to be decomposable if there exists a proper standard invariant subspace of A, as defined in the introduction. Clearly, the definition is equivalent to saying that an $n \times n$ matrix A is decomposable if and only if there exists a permutation matrix P such that $P^{-1}AP = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$ where B and D are square matrices

This equivalent form of decomposability leads to the next definition.

Definition 2.2 ([7]). A semigroup S in $M_n(\mathbb{R})$ is said to be decomposable if there exists a permutation matrix P such that $P^{-1}SP = \begin{pmatrix} S_{11} & S_{12} \\ 0 & S_{22} \end{pmatrix}$ for all $S \in S$, where S_{11} and S_{22} are square matrices of fixed sizes k and $n-k$, respectively.

The following lemma is the key result that yields decomposability of nonnegative semigroups in $M_n(\mathbb{R})$. The proof is due to Proposition III, 8.3 in [6].

Lemma 2.3 ([7]). If a nonnegative semigroup S in $M_n(\mathbb{R})$ has a common zero entry, that is, if for some fixed i and j, the (i,j) entry of every member of S is zero, then S is decomposable.

Definition 2.4 ([7]). A subset $\mathcal{J}$ of a semigroup S is called an ideal if JS and SJ belong to $\mathcal{J}$ for all $J \in \mathcal{J}$ and for all $S \in S$.

It is a well-known result (see [1]) that a nonzero ideal of an irreducible semigroup is irreducible. The counterpart of this result holds for indecomposable semigroups of $n \times n$ matrices with nonnegative entries. While this counterpart was proved in [2], we restate it here for completeness.

Lemma 2.5 ([2]). If S is an indecomposable semigroup of $n \times n$ nonnegative matrices, then so is every nonzero ideal of S.

Proof. Let $\mathcal{J}$ be a nonzero ideal of S. If M is a nontrivial invariant subspace of $\mathcal{J}$, then the standard subspaces generated by the sets $\{JM: J \in \mathcal{J}\}$ and $\{x \in \mathbb{R}^n: J|x| = 0, J \in \mathcal{J}\}$ are both invariant under S. Since $\mathcal{J}$ is nonzero ideal, at least one of them is nontrivial.

Remark 2.6. The above lemma establishes the fact that if a semigroup of nonnegative matrices has an ideal which is decomposable, then the semigroup itself must be decomposable. Also, it is obvious that if the semigroup is decomposable, that is, it has a common nontrivial invariant standard subspace, then every nonzero ideal of it will trivially be decomposable.

Combining Lemma 2.5 and Remark 2.6, we can state the following lemma.

Lemma 2.7. Let $\boldsymbol{S}$ be a semigroup of nonnegative matrices in $M_n(\mathbb{R})$ and $\mathcal{J}$ be the collection of all rank m elements in S. Then S is decomposable if and only if $\mathcal{J}$ is decomposable.

Proof. We only need observe that $\mathcal{J}$ form a nonzero ideal of S and the proof follows from Lemma 2.5 and Remark 2.6.

The next lemma included in the paper is a simple but elegant application of Perron Frobenius Theorem.

Lemma 2.8 ([5]). If A is a nonnegative r-potent matrix in $M_n(\mathbb{R})$ such that $rank(A) > r-1$, then A is decomposable.

Theorem 2.9 ([5]). Let S be a semigroup of nonnegative r-potent matrices of rank$> r-1$. Then S is decomposable.

Proof. Given in ([5, Theorem 14]).

Remark 2.10 ([7]). Suppose $\mathcal{J}$ is the collection of all rank m elements in the nonnegative band S. Then, by Lemma 2.7, S is decomposable if and only if $\mathcal{J}$ is decomposable. Hence, with no loss of generality, we can replace S by $\mathcal{J}$ and thus S can be considered a nonnegative band of constant rank m.

We can summarize this in the following theorem:

Theorem 2.11 ([5]). Suppose S is a semigroup of nonnegative r-potent matrices in $M_n(\mathbb{R})$ such that $rank(S) = m > r - 1$ for all $S \in S$. Then S is decomposable.

Lastly, we state a key result which gives conditions for conditions for decomposability of nonnegative semigroups in $M_n(\mathbb{R})$. The detailed proof is given in [2].

Theorem 2.12. A semigroup $\boldsymbol{S}$ in $M_n(\mathbb{R})$ with nonnegative matrices is decomposable if and only if $\boldsymbol{S}$ has a common zero entry.

3. Structure of Constant Rank Semi-groups of Nonnegative r-Potent Matrices

In the previous section (refer Theorem 2.11), we have seen that the question of decom-posability for a semigroup of nonnegative idempotent and r-potent matrices reduce to the case of a constant rank ideal in it. This fact exemplifies the importance of studying the constant rank nonnegative semigroup and is a motivation to study their structure. In [2], A. Marwaha has given a detailed structure of constant rank nonnegative semigroups of idempotents called bands. In this paper, we emulate the results for bands and obtain a similar structure for nonnegative semigroups of r-potent matrices.

We state and prove our key result as the next lemma.

By choosing an arrangement of the standard basis $\mathcal{B}$ as $\mathcal{B} = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$, where $\mathcal{B}_1$ and $\mathcal{B}_3$ consist of elements in the null space of $\boldsymbol{S}$ and S^* respectively and $\mathcal{B}_2$ is the complement of $\mathcal{B}_1 \cup \mathcal{B}_3$, we can define a permutation.

Lemma 3.1. Let S be a semigroup nonnegative r-potent matrices in $M_n(\mathbb{R})$ of constant rank one. Then there exists a permutation matrix P such that for each $S \in S, P^{-1}SP$ has the block triangular form $\begin{pmatrix} 0 & XE^{r-1} & XE^{2(r-1)-1}Y \\ 0 & E & E^{r-1}Y \\ 0 & 0 & 0 \end{pmatrix}$ where the diagonal block $S_0 = (E: S \in S)$ constitutes a rank-one indecomposable semigroup of r-potent matrices and $X, Y$are non-negative matrices of suitable size.

By choosing an arrangement of the standard basis $\mathcal{B}$ as $\mathcal{B} = \mathcal{B}_1 \dot\cup \mathcal{B}_2 \dot\cup \mathcal{B}_3$, where $\mathcal{B}_1$ and $\mathcal{B}_3$ consist of elements in the null space of S and S^* respectively and $\mathcal{B}_2$ is the complement of $\mathcal{B}_1 \cup \mathcal{B}_3$, we can define a permutation matrix P such that for each S in $S, P^{-1}SP$ has the matrix form $\begin{pmatrix} 0 & X & Z \\ 0 & E & \gamma \\ 0 & 0 & 0 \end{pmatrix}$ where X, Y, Z are matrices of suitable size.

Since each matrix in S is an r-potent matrix, we have that

$$\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}^r = \begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$$

This gives us the relation

$$\begin{pmatrix} 0 & XE^{r-1} & XE^{r-2}Y \\ 0 & E^r & E^{r-1}Y \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$$

and the equation $E^r = E, X = XE^{r-1}, Y = E^{r-1}Y$ and $Z = XE^{r-2}Y = XE^{r-2}E^{r-1}Y = XE^{2(r-1)-1}Y$.

Clearly, diagonal block $S_0 = \{E: S \in S\}$ form a rank one semigroup because S is a rank one semigroup. Further, S_0 cannot have a zero entry as all the zero rows and zero columns have already been taken out in $\mathcal{B}_1$ and $\mathcal{B}_3$ respectively. Thus, by Theorem 2.12, S_0 is indecomposable.

Lemma 3.2. If $\boldsymbol{S}$ is a semigroup of nonnegative r-potent matrices in $M_n(\mathbb{R})$ with constant rank m, where $m > r - 1$, then $\boldsymbol{S}$ has a standard block triangular form where the diagonal blocks form a constant rank indecomposable semigroup of nonnegative r-potents. Furthermore, this can be done so that no two diagonal blocks are consecutively zero. Therefore, if k be the total number of diagonal blocks, them $k \leq 2m + 1$.

Proof. We shall prove the lemma by induction on m. The case $m = 1$ is proved in Lemma 3.1. Therefore, let $m > 1$. We consider the following two cases:

Case (i): If $r = 2$, then S reduces to nonnegative band with rank $m > 1(\text{semi} - \text{group of idempotents})$. The decomposability of such bands is proved by A. Marwaha in [2]. (refer Lemma 4.2)

Case (ii): If $r > 2$, then since $m > r - 1$ (by hypothesis), S is decomposable by Theorem 2.11 stated earlier in this paper.

Thus, in either case, S is decomposable. Therefore, by Definition 2.1, after a permutation of basis, every $S \in S$ is of the form $\begin{pmatrix} S_{11} & X \\ 0 & S_{22} \end{pmatrix}$, where S_{11} and S_{22} are square matrices.

Consider the two diagonal blocks, $S_1 = \{S_{11}: S \in S\}$ and $S_2 = \{S_{22}: S \in S\}$. Then S_1 and S_2 form nonzero, nonnegative semigroups of r-potent matrices.

Further, it can be easily proved that S_1 and S_2 are constant rank semigroups. The techniques proving this are given in [2] (Lemma 4.2).

Evidently, the ranks of S_1 and S_2 are less than $rank(S) = m$. Consequently, induction applies to both S_1 and S_2 and we obtain the desired result.

Finally, if any two consecutive diagonal blocks are zero except the block (1, 2), then the fact that all matrices are r-potent will result in the matrices reducing to zero, which is absurd. This justifies the assertion that no two successive diagonal blocks are zero.

We next state a definition from [2].

Definition 3.3. A semigroup $\mathcal{S}$ in $M_n(\mathbb{R})$ of nonnegative matrices is called a full semigroup if $\mathcal{S}$ has no common zero row and no common zero column.

Lemma 3.4. Let $\mathcal{S}$ be a full semigroup of nonnegative r-potent matrices in $M_n(\mathbb{R})$ with constant rank one. Then $\mathcal{S}$ is indecomposable.

Proof. According to Lemma 3.1, if $\mathcal{S}$ is a full semigroup of nonnegative r-potent matrices of constant rank one, it must be indecomposable, for otherwise it will have zero columns and zero rows and its structure will be of the form $\begin{pmatrix} 0 & XE^{r-1} & XE^{2(r-1)-1}Y \\ 0 & E & E^{r-1}Y \\ 0 & 0 & 0 \end{pmatrix}$ with the diagonal block $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ constituting a rank one indecomposable band.

The next theorem holds true for nonnegative semigroups that are full. A similar result was proved for idempotent in [2]. In this theorem, we prove the result for semigroups of nonnegative r-potent matrices.

Theorem 3.5. Let $\mathcal{S}$ be a semigroup of nonnegative r-potent matrices in $M_n(\mathbb{R})$ with constant rank $m > r-1$.

(i) If $\mathcal{S}$ is full, then there exists a permutation matrix P such that for any $S \in \mathcal{S}, P^{-1}SP$ has the block diagonal form $\begin{bmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_m \end{bmatrix}$ where each $\mathcal{S}_i = \{S_i : S \in \mathcal{S}\}$ is an indecomposable semigroup of constant rank nonnegative r-potent matrices.

(ii) In general, there is a permutation matrix Q such that for each $S \in \mathcal{S}, Q^{-1}SQ$ has the upper block triangular form $\begin{pmatrix} 0 & XE^{r-1} & XE^{2(r-1)-1}Y \\ 0 & E & E^{r-1}Y \\ 0 & 0 & 0 \end{pmatrix}$ where matrices X, Y are of appropriate size and $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ is the same as in case (i).

Proof.

(i) For $m = 1$, the result holds by Lemma 3.4. We prove the theorem by induction on m. Now, let $m > 1$, then by Lemma 3.2, each S in $\mathcal{S}$ can be assumed to have the form $\begin{pmatrix} S_{11} & X_1 \\ 0 & S_{22} \end{pmatrix}$ where the diagonal block $\mathcal{S}_1 = \{S_{11} : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_{22} : S \in \mathcal{S}\}$ form nonzero semigroups of constant rank less than m.

Further, fullness of $\mathcal{S}$ implies that $\mathcal{S}_1$ has no common zero column and $\mathcal{S}_2$ has no common zero row. If we take $E = \begin{pmatrix} E_1 & X \\ 0 & E_2 \end{pmatrix}$ to be any arbitrary but fixed element in $\mathcal{S}$, it can be proved that $\mathcal{S}_1 \times \mathcal{S}_2 = 0$ ……… (1)

For details of the proof, we refer the reader to Lemma 4.5 in [2].

Since $\mathcal{S}_1$ has no common zero column, equation (1) reduces to $X\mathcal{S}_2 = 0$ and combined further with the fact that $\mathcal{S}_2$ has no common zero, it implies that $X = 0$. Therefore, $E = \begin{pmatrix} E_{11} & 0 \\ 0 & E_{22} \end{pmatrix}$.

This shows that any general element S in $\mathcal{S}$ is of the form $\begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} \end{pmatrix}$ where $\mathcal{S}_1 = \{S_{11} : S \in \mathcal{S}\}$ and $\mathcal{S}_2 = \{S_{22} : S \in \mathcal{S}\}$ are nonnegative full semigroups with constant rank less than m. Consequently, induction applies and $\mathcal{S}$ is of the desired form.

(ii) In the general case, we first consider the same arrangement of the basis $\mathcal{B}$ as in Lemma 3.1. Then, with respect to this permutation of basis, every element S of $\mathcal{S}$ is of the form $\begin{pmatrix} 0 & X & Z \\ 0 & E & Y \\ 0 & 0 & 0 \end{pmatrix}$

Since $S^r = S$, we have

$E^r = E,$

$X = E^{r-1},$

$Y = E^{r-1}Y,$

$Z = XE^{2(r-1)-1}Y$

These equations imply that $\mathcal{S}_0 = \{E : S \in \mathcal{S}\}$ cannot have a common zero row or column. Therefore, $\mathcal{S}_0$ is a full nonnegative semigroup of constant rank m, and hence is of the form given in (i) above.

Example 3.6. Let us illustrate the above theorem with the help of the following example:

$$\begin{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 10 & 0 & 0 & 0 & \dots & 0 \\ 0 \end{bmatrix}_{n-1\times n-1} & 0 \\ & [1]_{1\times 1} \end{bmatrix}_{n\times n}$$

$$A^{n-1} = I$$

Therefore

$$A^n = A$$

$$rank(A) = n > n-1$$

Therefore A is decomposable.

Consider the semigroup generated by A

$$S = \{1, A, A^2, \dots, A^{n-2}\}$$

where

$$A^2 = \begin{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 \end{bmatrix}_{n-1\times n-1} & 0 \\ 0 & [1]_{1\times 1} \end{bmatrix}_{n\times n}$$

$$\vdots$$

$$A^{n-1} = \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{n-1\times n-1} & 0 \\ 0 & [1]_{1\times 1} \end{bmatrix}_{n\times n}$$

S is a constant rank semigroup where rank of each member is n.
Further, each member has a common zero entry.
Therefore, S is decomposable. Also, S is a full semigroup. Therefore S has a block diagonal representation where each diagonal block is a constant rank indecomposable semigroup of n-potents.

4. Structure of Maximal Semigroups of Nonnegative r-Potent Matrices $in\ M_n(\mathbb{R})$

We have seen in Theorem 3.5 that under the special condition of fullness, a semigroup of nonnegative r-potent matrices has a block-diagonal representation where each block is indecomposable semigroup of constant rank nonnegative r-potent matrices. In particular, if S is taken to be a maximal semigroup, then it readily follows that the semigroups S_i are also maximal. In part (ii) of Theorem 3.5, S_0 and the collection of all X, Y are maximal too.
In fact, the converse of part (i) of Theorem 3.5 is also true in case the semigroups S_i are taken to be maximal. In other words, a direct sum of maximal, indecomposable, nonnegative semigroups of r-potents in $M_n(\mathbb{R})$ is a maximal semigroup. The tools required for proving this are given in [2] in lemmas 4.7, 4.8 and Theorem 4.9. Using these techniques and the results proved in Theorem 3.5, the following theorem can be proved.

Theorem 4.1 Let $\boldsymbol{S}$ be a nonnegative semigroup in $M_n(\mathbb{R})$ of r-potent matrices of constant rank m. Then,

(i) If S is full, then S is maximal if and only if

$$S = \left\{ \begin{pmatrix} S_1 & & & \\ & S_2 & & \\ & & \ddots & \\ & & & S_k \end{pmatrix} = S_i \in S_i, k \le m \right\}$$

where each S_i is a maximal constant rank indecomposable semigroup of nonnegative r-potent matrices.

(ii) In general, if S is maximal, then

$$S = \left\{ \begin{pmatrix} 0 & XE^{r-1} & XE^{2(r-1)-1}Y \\ 0 & E & E^{r-1}Y \\ 0 & 0 & 0 \end{pmatrix} : E \in S, X \in X, Y \in \mathcal{Y} \right\},$$

where S, is a direct sum as in part (i) and $X, \mathcal{Y}$ are the entire sets of nonnegative matrices of suitable size.
We consider the paper by illustrating the process of obtaining the complete structure of maximal indecomposable semigroups of rank one 2-potent matrices or idempotents. We shall refer to a semigroup of idempotents as a band.

5. A Complete Structure of Maximal Indecomposable Bands (2-Potents or Idempotent) of Rank One

Example 5.1. Let S be an indecomposable band of rank one with nonnegative entries in $M_2(\mathbb{R})$. Any elements in S is of the form $A = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \in S, 0 \leq x \leq 1, y, z \geq 0$ with $yz = x(1-x)$

Case 1: $x = 0 \Rightarrow yz = 0$

Thus $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 0 \\ z & 1 \end{pmatrix}$

Subcase (i)

Suppose $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Let $B = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}, bc = a(1-a)$ be any arbitrary element in $\boldsymbol{S}$. Then

$BA = \begin{pmatrix} 0 & b \\ 0 & 1-a \end{pmatrix}$ and $AB = \begin{pmatrix} 0 & 0 \\ c & 1-a \end{pmatrix}$

Also

$$BAB = \begin{pmatrix} bc & b(1-a) \\ c(1-a) & (1-a)^2 \end{pmatrix}$$

and

$$ABA = \begin{pmatrix} 0 & 0 \\ 0 & 1-a \end{pmatrix}$$

All these must be idempotent simultaneously.
Now ABA is an idempotent if and only if $a = 0$ or $a = 1$.
If $a = 0, BA = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ and $AB = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$ which are both idempotents.

Also

$$BAB = \begin{pmatrix} bc & b \\ c & 1 \end{pmatrix}$$

Now $(BAB)^2 = BAB$ if and only if $bc = 0$.

If $b = 0, B = \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix}$ and then $S = \left\{ \begin{pmatrix} 0 & 0 \\ c & 1 \end{pmatrix} : c \geq 0 \right\}$ which is decomposable.

Similarly, if $c = 0, B = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ and then $\boldsymbol{S} = \left\{ \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} : b \geq 0 \right\}$ which is clearly decomposable.

Now, if $a = 1, BA = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, AB = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ and $BAB = \begin{pmatrix} bc & 0 \\ 0 & 0 \end{pmatrix}$

All three are idempotents if and only $b = c = 0$ which gives $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Subcase (ii)

Next suppose that $A = \begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}, y \neq 0$.

If $y \neq 1$, by a similarity transformation of the form $\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}$, we can take $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

(Observe that $\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$.

Then

$BA = \begin{pmatrix} 0 & a+b \\ 0 & c+1-a \end{pmatrix}$,

$AB = \begin{pmatrix} c & 1-a \\ c & 1-a \end{pmatrix}$.

$ABA = \begin{pmatrix} 0 & c+1-a \\ 0 & c+1-a \end{pmatrix}$ and $BAB = \begin{pmatrix} c(a+b) & (1-a)(a+b) \\ c(c+1-a) & (1-a)(c+1-a) \end{pmatrix}$

Now BA is an idempotent if and only if either $a+b=0$ or $c+1-a=0$

Clearly, $c+1-a \neq 0$. Therefore, we must we have $c=a$.

Further, if $a=0$, then $c=0$ and so $B = \begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix}$ which give $S = \left\{\begin{pmatrix} 0 & b \\ 0 & 1 \end{pmatrix} : b \geq 0\right\}$, a decomposable semigroup.

And if $a \neq 0$, then $ba = a(1-a) \Rightarrow b = 1-a$ and so $B = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}$.

Then $\boldsymbol{S} = \left\{\begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix} : \leq a < 1\right\}$ which is an indecomposable semigroup.

Subcase (iii)

Let $A = \begin{pmatrix} 0 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$.

Since A is a transpose of $\begin{pmatrix} 0 & z \\ 0 & 1 \end{pmatrix}$, same as Subcase (ii), we shall obtain

$S = \left\{\begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix} : 0 \leq x \leq 1\right\}$.

This completes the case when $x = 0$.

Case II $x = 1 \Rightarrow yz = 0$

Again, there are three possibilities $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Subcase (i) $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Let $B = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix} \in S$ be arbitrary, where $bc = a(1-c)$.

Then

$$BA = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$$
$$AB = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$
$$BAB = \begin{pmatrix} a^2 & ac \\ ca & c^2 \end{pmatrix} \text{ and } ABA = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

If $a = 0, BA = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}$ which is an idempotent $\Leftrightarrow c = 0$;

$AB = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ which is an idempotent $\Leftrightarrow b = 0$;

and $ABA = BAB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Thus $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and so $S = \left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$.

If $a = 1, BA = \begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix}, AB = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$ which are idempotents; and $BAB = \begin{pmatrix} 1 & c \\ c & c^2 \end{pmatrix}$ which is an idempotent if and only if $c = 0$. Thus $S = \left\{\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} : b \geq 0\right\}$

Subcase (ii) $A = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}, y \neq 0$. With no loss of generality, we can take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ if $y \neq 1$. Then

$$AB = \begin{pmatrix} a + c & b + 1 - a \\ 0 & 0 \end{pmatrix}$$

$$BA = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$$

$$ABA = \begin{pmatrix} a + c & a + c \\ 0 & 0 \end{pmatrix}, \text{ and}$$

$$BAB = \begin{pmatrix} a^2 + ac & ab + a(1 - a) \\ 2ca & bc + c(1 - a) \end{pmatrix}$$

All should be idempotents simultaneously.
Now AB is an idempotent if and only if $a + c = 0, b + 1 - a = 0$ which is not possible or if $a + c = 1$.
Therefore $a + c = 1$. If $a = 0$, then $c = 1$ and so

$AB = \begin{pmatrix} 1 & b + 1 \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, ABA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, BAB = \begin{pmatrix} 0 & 0 \\ 0 & b + 1 \end{pmatrix}$ which are idempotents if and only if $b = 0$. Thus $B = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

If $a \neq 0$, then $c = 0$ and so $AB = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}, BA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, ABA = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, BAB = \begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix}$

Thus $S = \left\{\begin{pmatrix} 1 & b \\ 0 & 0 \end{pmatrix} : b \geq 0\right\}$.

If $c \neq 0$, then $bc = ca$ which implies $b = c$.

Therefore $B = \begin{pmatrix} a & a \\ 1 - a & 1 - a \end{pmatrix}$ which gives that

$$S = \{\begin{pmatrix} a & a \\ 1 - a & 1 - a \end{pmatrix} : 0 \leq a \leq 1\}.$$

Subcase (III) $A = \begin{pmatrix} 1 & 0 \\ z & 0 \end{pmatrix}$. A is simply the transpose of the above case and therefore, we shall obtain $S = \left\{\begin{pmatrix} a & 1 - a \\ a & 1 - a \end{pmatrix} : 0 \leq a \leq 1\right\}$.

Case (III) Let $0 < x < 1$.

Claim. By similarity transformation with a positive diagonal matrix, we can assume with no loss of generality that $\begin{pmatrix} x & x \\ 1 - x & 1 - x \end{pmatrix} \in S$ for some $0 < x < 1$. Choose a,b>0 such that $ay = bx$ where $\begin{pmatrix} x & y \\ z & 1 - x \end{pmatrix} \in S, yz = x(1 - x)$. Then

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} x & y \\ z & 1 - x \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} ax & ay \\ bz & b(1 - x) \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} x & ab^{-1}y \\ a^{-1}bz & 1 - x \end{pmatrix} = \begin{pmatrix} x & x \\ 1 - x & 1 - x \end{pmatrix}$$

$$\left(\because a^{-1}bz = {}^{y}/_{x} \cdot z = \frac{x(1 - x)}{x} = 1 - x\right)$$

Observe that taking similarity transformation with $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ does not alter the structure of S and this proves the claim. Therefore, let $A = \begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix} \in S$ for some $a: 0 < a < 1$.

Let $B = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix}, yz = x(1-x), 0 < x < 1, y, z > 0$ be any arbitrary element of S.

Then

$$AB = \begin{pmatrix} a(x+z) & a(y+1-x) \\ (1-a)(x+z) & (1-x)(y+1-x) \end{pmatrix},$$

$$BA = \begin{pmatrix} ax + y(1-a) & ax + y(1-a) \\ az + (1-x)(1-a) & az + (1-a)(1-x) \end{pmatrix}$$

$$ABA = \begin{pmatrix} a^2(x+z) + a(1-a)(y+1-x) & a^2(x+z) + a(1-a)(y+1-x) \\ a(1-a)(x+z)(1-a)^2(y+1-x) & a(1-a)(x+z) + (1-a)^2(y+1-x) \end{pmatrix}$$

$$BAB = \begin{pmatrix} ax^2 + (1-a)xy + axz + (1-a)yz & axy + (1-a)y^2 + ax(1-x) + (1-a)y(1-x) \\ axz + (1-a)x(1-x) + az^2 + (1-a)(1-x)z & ayz + (1-a)(1-x)y + az(1-x) + (1-a)(1-x) \end{pmatrix}$$

It can be easily checked that AB, BA, ABA and BAB are idempotents if and only if $(2a-x)x + (1-a)y + az = a$ (1)

Subcase 1 $a = \frac{1}{2}$. Then equation (1) becomes $y + z = 1$. Also
$yz = x(1-x)$

$\Leftrightarrow \; y(1-y) = x(1-x)$
$\Leftrightarrow \; (y-x)(y+x-1) = 0$
$\Leftrightarrow \; y = x$ or $y = 1-x$

Now $y = x \Rightarrow B = \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}$ and so $S = \left\{\begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix} : 0 < x < 1\right\}$

and $y = 1-x \Longrightarrow B = \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}$ and so $\mathcal{S} = \left\{\begin{pmatrix} x & 1-x \\ x & 1-x \end{pmatrix} : 0 < x < 1\right\}$

Subcase II $\quad a \neq \frac{1}{2}$. Equation (1) can be written as

$$x - y = a(2x - y + z - 1)$$
$$= a\left(2x - y + \frac{x(1-x)}{y} - 1\right) \text{ as } z = \frac{(1-x)}{y}$$

$\Leftrightarrow \; y(x-y) = a(2xy - y^2 + x - x^2 - y)$
$= a(x-y)(y-x+1)$

$\Leftrightarrow \; (x-y)(y - ay + ax - a) = 0$

If $y = x$, then

$z = 1-x$ and $\mathcal{S} = \left\{\begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix} : 0 < x < 1\right\}$

If $y \neq x$, then

$y - a(y + 1 - x) = 0$

$$\Rightarrow y = \frac{a(1-x)}{1-a}$$

Write $b = \frac{a}{1-a}$. Then

$y = b(1-x)$

and $z = \frac{x(1-x)}{b(1-x)} = \frac{1}{b}x$

Therefore $\mathcal{S} = \left\{ \begin{pmatrix} x & b(1-x) \\ \frac{1}{b}x & 1-x \end{pmatrix} : 0 < x < 1 \right\}$

Consider the diagonal $\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x & b(1-x) \\ \frac{1}{b}x & 1-x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} x & b(1-x) \\ x & b(1-x) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} x & 1-x \\ x & 1-x \end{pmatrix}$$

Also

$$\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} a & a \\ 1-a & 1-a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} a & a \\ b(1-a) & b(1-a) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & b^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} a & b^{-1}a \\ b(1-a) & 1-a \end{pmatrix}$$
$$= \begin{pmatrix} a & \dfrac{1-a}{a} \cdot a \\ a & 1-a \end{pmatrix} = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}$$

Thus, after the similarity transformation and possible transposition, an indecomposable band of rank one matrices with nonnegative entries is

$\boldsymbol{S}$=$\left\{ \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix} : 0 \leq x \leq 1 \right\}$

It only remains to be proved that $\mathcal{S} = \left\{ 0, \begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}, I : 0 \leq x \leq 1 \right\}$ is maximal.

Suppose $A = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}, bc = a(1-a)$.

Now $B = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \in \mathcal{S}$

Then

$$AB = \begin{pmatrix} 1/2\,(a+b) & 1/2\,(a+b) \\ 1/2\,(c+1-a) & 1/2\,(c+1-a) \end{pmatrix}$$

and $BA = \begin{pmatrix} 1/2(a+c) & 1/2\,(b+1-a) \\ 1/2(a+c) & 1/2\,(b+1-a) \end{pmatrix}$

AB, BA are idempotents if and only if $b + c = 1 \Leftrightarrow c = 1 - b$.

If $b, c > 0$, then $A \in \mathcal{S}$.

Suppose $b = 0$. Then $c = 1$ which implies $a(1-a) = 0$, that is, $a = 0$ or $a = 1$.

Now $a = 0 \Rightarrow A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$

If $C = \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} \in \mathcal{S}, 0 < x < 1, y, z > 0, yz = x(1-x)$, then

$$AC = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ x+z & y+1-x \end{pmatrix}$$

AC is an idempotent $\Leftrightarrow y + 1 - x = 0 \Leftrightarrow x = y \Rightarrow z = 1 - x$.

Thus C=$\begin{pmatrix} x & x \\ 1-x & 1-x \end{pmatrix}$. But C is an arbitrary element of S. Therefore, a$\neq 0$.

Let $a = 1$ which gives $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$.

Again $AC = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ z & 1-x \end{pmatrix} = \begin{pmatrix} x & y \\ x & y \end{pmatrix}$.

AC is an idempotent $\Leftrightarrow x + y = 1 \Longrightarrow y = 1 - x \Longrightarrow z = x$.

Thus $C = \begin{pmatrix} x & 1-x \\ x & 1-x \end{pmatrix}$. But again C is arbitrary. Therefore $a \neq 1$.

Similarly, we can check that the possibilities $b = 1, c = 0, a = 0$ and $b = 1, c = 0, a = 1$ lead to similar contradictions.
Thus AB, BA are idempotents if and only if $b + c = 1, b, c > 0, 0 < a < 1$ in which case $A \in \mathcal{S}$.
Hence $\mathcal{S}$ is maximal.

6. Conclusion

It will be interesting and worthwhile to study the geometric characterization of maximal constant rank semigroups of nonnegative r-potent operators in infinite dimensions. For this, we shall consider $\mathcal{X}$ to be a separable, locally compact Hausdorff space and μ a Borel measure on $\mathcal{X}$ such that $\mu(\mathcal{X}) < \infty$ and let $\mathcal{L}^2(\mathcal{X})$ denote the Hilbert space of (equivalent classes) complex-valued measurable and square integrable function on $\mathcal{X}$. Conditions leading to decomposability of a single nonnegative r-potent operator defined on $\mathcal{L}^2(\mathcal{X})$ and semigroups of nonnegative r-potents operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ have been studied in [11].

In particular, A. Marwaha in her paper [12] has independently obtained results of decomposability of semigroups of nonnegative idempotents (known as bands) in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ and also studied their structure in entirety of constant rank bands in view of their decomposability.

Further, A. Marwaha, in her paper [13] has obtained a geometric characterization of maximal bands of constants rank in general, and of constant rank one, in particular.

It will be noteworthy to study the conditions which yield such a geometric characterization for maximal semigroups of nonnegative r-potents of constant rank.

7. References

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