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An improved generalized estimators for finite population variance of a study variable based on auxiliary information

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Abstract
The information on auxiliary variable has been shown to be relevant in selection and estimation of parameters to gain more precision in estimates of study variable. The transformation of this auxiliary information also aids increase efficiency of estimators. In this article, an improved mixed ratio-product-type exponential estimator is proposed and evaluated using information on auxiliary variable for population variance under simple random sampling. The mathematical expressions for bias and mean squared error (MSE) of the proposed estimator were derived up to first order of approximation. Using real data sets and Monte Carlo simulation study, the performance evaluation of the proposed estimator was considered and compared to the existing estimators. The results of the empirical and simulation studies show that the proposed estimator outperformed the existing estimators in term of MSE and PRE.

Keywords: Auxiliary information, Bias, mean squared error, percentage relative efficiency, variance estimation

Introduction

In a survey process, the use of auxiliary information at the estimation stage usually leads to gain in precision of the estimator of unknown population parameters. Among many ratio, product and regression approaches of estimators are suitable examples, provided that suitable relationship is existing between auxiliary variable and study variable. Many authors have introduced several forms of ratio-type exponential estimators based on different transformation of the original auxiliary variable including Bahl and Tetuja (1991) [6], Grover and Kaur (2011) [10] Singh and Espejo (2003), Kadilar and Cingi (2004) [17], Haq and Shabbir (2014) [12] and others.

In several cases, estimation of population variance S_y^2 of study variable y is very important, particularly when the population has some extreme values. Based on availability of prior information on parameters of auxiliary variable, many authors have suggested various estimators of population variance S_y^2 including Liu (1974) [18], Das & Tripathi (1978) [8], Isaki (1983) [14], Upadhaya & Singh (1991) [28], Bahl & Tuteja (1991) [6], Ahmed *et al.* (2000) [3], Al-Jararha & Ahmed (2002) [5], Ahmed *et al.* (2003) [4], Upadhyaya *et al.* (2004) [29], Kadilar & Cingi (2006) [15], Shabbir & Gupta (2007) [22], Grover (2010) [9], Singh & Singh (2011), Singh & Malik (2014) [23], Yadav & Kadilar (2014), Haq and Shabbir (2014) [12], Ahmed and Singh (2015) [2], Swain (2015) [27], Subramani & Kumarapandiyan (2015), Yadav *et al.* (2015) [30], Ahmed *et al.* (2016) [1] and among others.

Recently, Muneer *et al.* (2018) and Mohammed *et al.* (2022) suggested an improved ratio-product type exponential estimator to estimate the finite population variance S_y^2. Motivated by the above studies, we consider the importance of estimating variation and propose an improved mixed ratio-product-type exponential estimator of finite population variance S_y^2 when the information on parameters of auxiliary variable is available.

Consider a finite population $\Omega = (\Omega_1, \Omega_2, \Omega_3, \dots, \Omega_N)$ of size N units. Let the value of study variable Y and auxiliary variable X be defined as y_i and x_i, respectively on i^{th} unit $\Omega_i, i = 1, \dots, N$. Assuming the population parameters of the auxiliary variable is known. The properties of study variable Y and auxiliary variable X, respectively are given as follows.

Parameters	**Study variable**	**Auxiliary variable**
Sample mean	$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i$	$\bar{x} = n^{-1}\sum_{i=1}^{n} x_i$
Population mean	$\bar{Y} = N^{-1}\sum_{i=1}^{n} y_i$	$\bar{X} = N^{-1}\sum_{i=1}^{n} x_i$
Population variance	$S_Y^2 = (N-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{Y})^2$	$S_X^2 = (N-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{X})^2$
Sample variance	$S_Y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i - \bar{y})^2$	$S_X^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$
Coefficient of variation	$C_y = S_y/\bar{Y}$	$C_x = S_y/\bar{X}$

To estimate the bias and mean squared error (MSE) of the estimator, the error terms are defined as follows: $e_0 = \frac{s_y^2 - S_y^2}{S_y^2} \Longrightarrow s_y^2 = S_y^2(1+e_0)$ and $e_1 = \frac{s_x^2 - S_x^2}{S_x^2} \Longrightarrow s_x^2 = S_x^2(1+e_0)$, such that $E(e_0) = E(e_1) = 0$, $E(e_0^2) = \lambda(\delta_{40} - 1) = \varphi_{40}, E(e_1^2) = \lambda(\delta_{04} - 1) = \varphi_{04}$ $E(e_0 e_1) = \lambda(\delta_{22} - 1) = \varphi_{22}$ were $\lambda = n^{-1} - N^{-1}$. Also $\delta_{rs} = \frac{\mu_{rs}}{\sqrt{\mu_{20}^r \mu_{20}^r}}$ where $\mu_{rs} = \frac{\sum_{i=1}^{N}(Y_i - \bar{Y})^r (Y_i - \bar{Y})^r}{N-1}$

$\delta_{40} = \beta_{2(y)}$ *and* $\delta_{04} = \beta_{2(x)}$ are the population coefficients of kurtosis for variable Y *and* X, respectively.

2. Review of the existing estimators

Here, we present existing estimators of the finite population mean that are available in the sampling literature. For brief discussion, the variance and MSEs of all estimators, considered in this section, are obtained up to first order of approximation.
The conventional variance estimator of $S_y^2 = s_y^2$ for population variance is given by

$$\hat{S}_y^2 \cong S_y^4 \varphi_{40} \tag{2.1}$$

Isaki (1983) [14] proposed a ratio-type estimator $\hat{S}_R^2$for the population variance of the study variable. The estimator $\hat{S}_y^2$ and its $MSE(\hat{S}_R^2)$ are, respectively, given by

$$\hat{S}_R^2 = s_y^2\left(\frac{S_x^2}{s_x^2}\right) \tag{2.2}$$

and

$$MSE(\hat{S}_R^2) = S_y^4[\varphi_{40} + \varphi_{04} - 2\varphi_{22}] \tag{2.3}$$

the usual regression estimator $\hat{S}_{lr}^2$ and its $MSE(\hat{S}_{lr}^2)$ up to first order of approximation are, respectively, given by

$$\hat{S}_{lr}^2 = s_y^2 + b_{(s_y^2 s_x^2)}(S_x^2 - s_x^2) \tag{2.4}$$

and $MSE_{min}(\hat{S}_{lr}^2) \cong S_y^4 \varphi_{40}\left(1 - \rho^2_{(s_y^2 s_x^2)}\right)$ (2.5)

where $b_{(s_y^2 s_x^2)} = \frac{S_y^2 \varphi_{22}}{S_y^2 \varphi_{04}}$ is the sample regression coefficient, and $\rho^2_{(s_y^2 s_x^2)} = \frac{\varphi_{22}}{\sqrt{\varphi_{40}\varphi_{04}}}$

Singh *et al*. (1988) proposed a different-type estimator $\hat{S}_d^2$ for population variance, this estimator and its $MSE(\hat{S}_d^2)$ up to first order of approximation are, respectively given by

$$\hat{S}_d^2 = k_1 s_y^2 + k_2(S_x^2 - s_x^2) \tag{2.6}$$

and

$$MSE_{min}(\hat{S}_d^2) \cong S_y^4\left[\frac{\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)}{1+\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)}\right] \tag{2.7}$$

where $k_{1(opt)} = \frac{\varphi_{04}}{[\varphi_{04}+\varphi_{40}\varphi_{04}-\varphi_{22}^2]}$ and $k_{1(opt)} = \frac{S_x^2\varphi_{22}}{S_y^2[\varphi_{04}+\varphi_{40}\varphi_{04}-\varphi_{22}^2]}$ are the optimum values for MSE of the estimator $\hat{S}_d^2$

Bahl and Tuteja (1991) [6] proposed an exponential ratio-type estimator $\hat{S}_{BT}^2$ for the population variance of the study variable Y. The $\hat{S}_{BT}^2$w and its $MSE(\hat{S}_{BT}^2)$ up to first order of approximation are, respectively given by

$$\hat{S}_{BT}^2 = s_y^2 exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right) \quad (2.8)$$

and

$$MSE(\hat{S}_{BT}^2) \cong S_x^4\left[\varphi_{40}+\frac{1}{4}\varphi_{04}-2\varphi_{22}\right] \quad (2.9)$$

Shabbir and Gupta (2007) [22] proposed a different-ratio type exponential estimator $\hat{S}_{SG}^2$. The estimator and its MSE are given by

$$\hat{S}_{SG}^2 = w_1 s_y^2 + w_2(S_x^2 - s_x^2) \quad (2.10)$$

and

$$MSE_{min}(\hat{S}_{GR}^2) \cong \frac{S_y^4}{64}\left[\frac{-4\varphi_{04}^{\,2}-16\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)(\varphi_{40}-4)}{1+\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)}\right] \quad (2.11)$$

where $d_{1(opt)} = \frac{\varphi_{04}}{8}\left[\frac{8-\varphi_{04}}{\varphi_{04}+\varphi_{40}\varphi_{04}-\varphi_{22}^2}\right]$ and $d_{2(opt)} = \frac{S_y^2}{8S_x^2}\left[\frac{-4\varphi_{40}+\varphi_{04}+8\varphi_{22}-\varphi_{22}\varphi_{04}+4\varphi_{40}\varphi_{04}-4\varphi_{22}^2}{\varphi_{40}+\varphi_{40}\varphi_{04}-\varphi_{22}^2}\right]$ are the optimum values for MSE of the estimator $\hat{S}_{SG}^2$

Swain (2015) [27] proposed a generalized estimator for population variance. The estimator and its MSE up to first order of approximation are, respectively, given by

$$\hat{S}_{SW}^2 = s_y^2\left(k\left(\frac{S_x^2}{s_x^2}\right)^g(1-k)\left(\frac{s_x^2}{S_x^2}\right)^q\right)^\delta \quad (2.12)$$

and

$$MSE_{min}(\hat{S}_{SW}^2) \cong S_y^4\varphi_{40}\left[1-\rho^2_{(s_y^2 s_x^2)}\right] \quad (2.13)$$

where $g, h\ and\ \delta = (1,-1)$ are real and free parameters to be chosen suitably and $k = \frac{\delta h + \frac{\varphi_{22}}{\varphi_{04}}}{\delta(g+h)}$ is the optimum value for the estimator $\hat{S}_{SW}^2$

Yadav *et al.* (2015) [30] proposed a generalized variance estimator $\hat{S}_{YG}^2$. the estimator and its MSE up to first order of approximation are, respectively, given by

$$\hat{S}_{YG}^2 = \left(u_1 s_y^2 + u_2(S_x^2 - s_x^2)\right)\left\{\gamma\left(\frac{aS_x^2+b}{as_x^2+b}\right)+(1-\gamma)\left(\frac{a(S_x^2-s_x^2)}{a(S_x^2+s_x^2)+b}\right)\right\} \quad (2.14)$$

and

$$MSE_{min}(\hat{S}_{YG}^2) = S_y^4\left[1-\frac{1}{4}g^2(1+\gamma)^2\varphi_{04}-\frac{1-\frac{1}{8}g^2(1+3\gamma+4\gamma^2)\varphi_{04}}{1-\frac{1}{4}g^2(1+3\gamma)\varphi_{04}^2+\varphi_{04}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)}\right] \quad (2.15)$$

where

$g = \frac{aS_x^2}{s_x^2+b}$, $u_{1(opt)} = \left(\frac{1-\frac{1}{8}g^2(1+3\gamma+4\gamma^2)\varphi_{04}}{1-\frac{1}{4}g^2(1+3\gamma)\varphi_{04}^2+\varphi_{04}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)}\right)$ and $u_{2(opt)} = \frac{S_y^2}{S_x^2}\left(\frac{1}{2}g(1+\gamma)+u_{1(opt)}\left(\frac{\varphi_{22}}{\varphi_{04}}-g(1+\gamma)\right)\right)$ are the optimum values that minimized the MSE of the estimator.

the minimum MSE of the estimator $(\hat{S}_{YG}^2)$ at $(\gamma, a, b) = (1,1,0)$ also given by

$$MSE_{min}(\hat{S}^2_{YG}) \cong S_y^4 \left[\frac{\frac{1}{S_y^4}MSE(S^2_{lr})(1-\varphi_{04})}{1-\varphi_{04}+\frac{1}{S_y^4}MSE(S^2_{lr})}\right] \quad (2.16)$$

Yadav and Kadilar (2014) proposed ratio-product-ratio type estimator for $\hat{S}^2_{YK}$, is given by

$$\hat{S}^2_{YK} = \hat{S}^2_y = \alpha_1\left(\frac{(1-\beta_1)s_x^2+\beta_1 S_x^2}{\beta_1 s_x^2+(1-\beta_1)S_x^2}\right) + (1-\alpha_1)\left(\frac{\beta_1 s_x^2+(1-\beta_1)S_x^2}{a(1-\beta_1)s_x^2+\beta_1 S_x^2}\right) \quad (2.17)$$

The minimum MSE of the estimator $\hat{S}^2_{YK}$ is similar to regression estimator $\hat{S}^2_{lr}$ (2.5) at optimum values $(\alpha_1,\beta_1) = \frac{\varphi_{04}-\varphi_{22}}{2\varphi_{04}}, 0$.

Singh and Malik (2014) [23] proposed an improve estimator $\hat{S}^2_{SM}$ for population variance, given by

$$\hat{S}^2_{SM} = \hat{S}^2_y\left(v_1 s_y^2 + v_2(S_x^2 - s_x^2)\right) exp\left\{\psi\frac{a(S_x^2-s_x^2)}{a(S_x^2+s_x^2)+2b}\right\} \quad (2.18)$$

Where v_1 and v_2 are suitably chosen constants, ψ takes values $+1$ and -1 for ratio and product type estimators and a, b be the known population parameters of the auxiliary variables. The MSE of $\hat{S}^2_{SM}$ at optimum values of

$$v_{1(opt)} = \frac{1}{4}\left\{\frac{-12\varphi_{04}\varphi_{22}+\varphi_{04}^2+16\varphi_{22}^2-8\varphi_{04}+2\varphi_{40}^2}{\varphi_{04}^2-4\varphi_{04}\varphi_{22}+8\varphi_{22}^2-2\varphi_{40}\varphi_{04}-2\varphi_{04}-\varphi_{40}^2}\right\}$$

and

$$v_{1(opt)} = \frac{1}{4S_y^2}\left\{\frac{-6\varphi_{04}\varphi_{22}+\varphi_{04}^2+8\varphi_{22}^2-4\varphi_{04}+8\varphi_{22}-8\varphi_{04}\varphi_{22}+4\varphi_{40}\varphi_{04}}{\varphi_{04}^2-4\varphi_{04}\varphi_{22}+8\varphi_{22}^2-2\varphi_{40}\varphi_{04}-2\varphi_{04}-\varphi_{40}^2}\right\} \text{ is given by}$$

$$MSE_{min}(\hat{S}^2_{SM}) \cong \frac{S_y^4}{64}\left[\frac{\varphi_{04}\{\varphi_{04}(\varphi_{04}+8\varphi_{22})+16MSE(\hat{S}^2_{lr})(\varphi_{04}-4)+16\varphi_{22}(\varphi_{22}-\varphi_{04})\}}{-\varphi_{04}(1+\varphi_{40}+2\varphi_{22})+4\varphi_{22}^2}\right] \quad (2.19)$$

Yakub and Shabbir (2016) [2] proposed a class of estimator for population variance S_y^2 and given by

$$\hat{S}^2_{YS} = \hat{S}^2_y\left(\theta_1 s_y^2 + \theta_2(S_x^2 - s_x^2)\right)\left(\frac{aS_x^2+b}{as_x^2+b}\right)\left\{\frac{1}{2}exp\left(\frac{a(S_x^2-s_x^2)}{a(S_x^2+s_x^2)+b}\right) + \frac{1}{2}exp\left(\frac{a(s_x^2-S_x^2)}{a(s_x^2+S_x^2)+2b}\right)\right\} \quad (2.20)$$

Where θ_1 and θ_2 are suitably chosen constants and a and b be the known parameters of the auxiliary variable. The minimum MSE at optimum values of $\theta_1 = \frac{\varphi_{04}}{2}\left\{\frac{1+7(1-\varphi_{04})}{\varphi_{04}^2+4(\varphi_{04}(1-\varphi_{04}+\varphi_{40})-\varphi_{22}^2)}\right\}$ and $\theta_2 = \frac{S_y^2}{2S_x^2}\left\{\frac{\varphi_{22}+(7\varphi_{22}-8\varphi_{04})(1-\varphi_{04})+8\varphi_{40}\varphi_{04}-8\varphi_{22}^2}{\varphi_{04}^2+4(\varphi_{04}(1-\varphi_{04}+\varphi_{40})-\varphi_{22}^2)}\right\}$ given by

$$MSE_{min}(\hat{S}^2_{YS}) \cong \frac{S_y^4}{16}\left[\frac{\frac{64}{S_y^4}(1-\varphi_{04})MSE(\hat{S}^2_{lr})-\varphi_{04}^2}{\varphi_{04}+4(1-\varphi_{40})+2\varphi_{22})+\frac{4}{S_y^4}MSE(\hat{S}^2_{lr})}\right] \quad (2.21)$$

Muneer *et al*. (2018) proposed an improved class of estimator for variance and given by

$$\hat{S}^2_M = s_y^2\left(\omega_1\left(\frac{S_x^2}{s_x^2}\right) + \omega_2\left(\frac{s_x^2}{S_x^2}\right)\right) exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right) \quad (2.22)$$

where ω_1 and ω_2 are suitably chosen constants. The minimum MSE of $\hat{S}^2_M$ up to first order of approximation, at optimum values

$$\omega_{1opt)} = \frac{1}{8}\left[\frac{16\varphi_{22}^2+6\varphi_{22}\varphi_{40}-24\varphi_{22}\varphi_{04}-16\varphi_{40}\varphi_{04}-\varphi_{04}^2-16\varphi_{22}-8\varphi_{04}}{16\varphi_{22}^2-16\varphi_{22}\varphi_{04}-4\varphi_{40}\varphi_{04}+\varphi_{04}^2-4\varphi_{04}}\right]$$

and

$$\omega_{2(opt)} = \frac{1}{8}\left[\frac{48\varphi_{22}^2-16\varphi_{22}\varphi_{40}-72\varphi_{22}\varphi_{04}+16\varphi_{40}\varphi_{04}+21\varphi_{04}^2+16\varphi_{22}-24\varphi_{04}}{16\varphi_{22}^2-16\varphi_{22}\varphi_{04}-4\varphi_{40}\varphi_{04}+\varphi_{04}^2-4\varphi_{04}}\right] \text{ is given by}$$

$$MSE_{min}(\hat{S}^2_M) \cong \frac{S_y^4}{16}\left[\frac{\begin{array}{c}64\varphi_{22}^2\varphi_{40}-48\varphi_{22}^2\varphi_{04}-128\varphi_{22}\varphi_{40}\varphi_{04}+48\varphi_{22}\varphi_{04}^2\\+64\varphi_{40}\varphi_{04}^2+9\varphi_{04}^3+64\varphi_{22}^2-64\varphi_{40}\varphi_{04}\end{array}}{16\varphi_{22}^2-16\varphi_{22}\varphi_{04}-4\varphi_{40}\varphi_{04}+\varphi_{04}^2-4\varphi_{04}}\right] \quad (2.23)$$

Mohammed *et al*. (2022) proposed a generalized estimator for variance and is given by

$$\hat{S}^2_{MZA} = \left(d_1 s_y^2\left\{\frac{1}{2}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2}\right)\right\}^{\alpha} + d_2(S_x^2 - s_x^2)\right) exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right) \quad (2.24)$$

Where $d_i(i = 1,2)$ are unknown constants whose values to be determined later, and α is suitably chosen constant. The minimum MSE of $\hat{S}^2_{MZA}$ up to first order of approximation, at optimum values

$$d_{1(opt)} = \frac{1+\left(A-\frac{1}{2}\right)\varphi_{04}}{1+\left(2A-\frac{3}{4}\right)\varphi_{04}+\left(\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}\right)} \text{ and } d_{1(opt)} = \frac{S_y^2}{S_x^2}\left\{\frac{1}{2} - d_{1(opt)}\left(1-\frac{\varphi_{22}}{\varphi_{04}}\right)\right\} \text{ given by}$$

$$MSE_{min}(\hat{S}^2_{MZA}) \cong s_y^4\left[\left(1-\frac{\varphi_{04}}{4}\right) - \frac{\left\{1+\left(A-\frac{1}{2}\right)\varphi_{04}\right\}^2}{1+\left(2A-\frac{3}{4}\right)\varphi_{04}+\left(\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}\right)}\right] \quad (2.25)$$

where $A = \frac{1}{2}\alpha + \frac{3}{8}$

3. Proposed Estimator

In the line with the direction of study carried out by Mohammed *et al*. (2022), we proposed an improved class of combined estimators for estimating the finite population variance s_y^2 under method of simple random sampling (SRS). The proposed estimator is given by

$$\hat{S}^2_{New} = \left[\omega_1 s_y^2\left\{\frac{1}{4}\left(\frac{S_x^2}{s_x^2}+\frac{s_x^2}{S_x^2}\right)\left(exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right)+exp\left(\frac{s_x^2-S_x^2}{s_x^2+S_x^2}\right)\right)\right\}^{\gamma} + \omega_2(S_x^2 - s_x^2)\right] exp\left(\frac{S_x^2-s_x^2}{S_x^2+s_x^2}\right) \quad (3.1)$$

where ω_1 and ω_2 are unknown constants whose values are to be determined such that the MSE's are minimum, γ is a suitably chosen constant.

Rewrite (3.1) in term of errors, we obtain

$$\hat{S}^2_{New} = \left[\omega_1 S_y^2(1+e_0)\left\{\frac{1}{4}((1+e_1)^{-1}+1+e_1)\left(exp\left(-\frac{1}{2}e_1\left(1+\frac{1}{2}e_1\right)^{-1}\right)+ exp\left(\frac{1}{2}e_1(1+\frac{1}{2}e_1)^{-1}\right)\right)\right\}^{\gamma} - \omega_2 S_x^2 e_1\right] exp\left(-\frac{1}{2}e_1(1+\frac{1}{2}e_1)^{-1}\right) \quad (3.2)$$

Expand (3.2) and retain the errors to second degree order using Taylor series, we obtain

$$\hat{S}^2_{New} = \left[\omega_1 S_y^2(1+e_0)\left\{\frac{1}{4}(2+e_1^2)\left(2+\frac{1}{4}e_1^2\right)\right\}^{\gamma} - \omega_2 S_x^2 e_1\right]\left(1-\frac{1}{2}e_1+\frac{3}{8}e_1^2\right) \quad (3.3)$$

$$\hat{S}^2_{New} - S_y^2 = -S_y^2 + \omega_1 S_y^2\left[1+e_0-\frac{1}{2}e_1 - \omega_2 R e_1 - \frac{1}{2}e_0 e_1 + \theta e_1^2 + \omega_2 R e_1^2\right] \quad (3.4)$$

where $\theta = \frac{5}{8}\gamma + \frac{3}{8}$ *and* $R = \frac{S_y^2}{S_x^2}$

Taking the expectation of (3.4), the bias of the proposed estimator, up to the first order of approximation, given by

$$Bias(\hat{S}^2_{New}) = -S_y^2 + \omega_1 S_y^2\left[1-\frac{1}{2}\varphi_{22}+\theta\varphi_{04}+\omega_2 R\varphi_{04}\right] \quad (3.5)$$

Squaring and taking the expectation of (3.4), we obtain the MSE of the proposed estimator by using first order of approximation and is given by

$$MSE(\hat{S}^2_{New}) = S_y^4\left[1+\omega_1^2\left\{1+\varphi_{40}+\frac{1}{4}\varphi_{04}-2\varphi_{22}+2\theta\varphi_{04}\right\} - \omega_1\{2+2\theta\varphi_{04}-\varphi_{22}\} - 2R\omega_1\omega_2\{\varphi_{22}-\varphi_{04}\} + \omega_2^2 R^2\varphi_{04} - \omega_2 R\varphi_{04}\right] \quad (3.6)$$

Differentiate partially (3.6) w.r.to ω_1 and ω_2 and equate to zero, we have

$$\begin{bmatrix} 2(1+\varphi_{40}+\frac{1}{4}\varphi_{04}-2\varphi_{22}+2\theta\varphi_{04}), & -2(\varphi_{22}-\varphi_{04}) \\ -2(\varphi_{22}-\varphi_{04}), & 2R^2\varphi_{04}\end{bmatrix}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix} = \begin{bmatrix}-2-2\theta\varphi_{04}+\varphi_{22}\\ -\omega_2 R\varphi_{04}\end{bmatrix}$$

After simplifying, we obtain the optimum value of ω_1 and ω_2, that is,

$$\omega_{1(opt)} = \frac{1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}}{1+\left(2\theta-\frac{3}{4}\right)\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}} \text{ and } \omega_{2(opt)} = R\left[\frac{1}{2} - \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)\left(1-\frac{\varphi_{22}}{\varphi_{04}}\right)}{1+\left(2\theta-\frac{3}{4}\right)\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right].$$

Substituting $\omega_{1(opt)}$ and $\omega_{2(opt)}$ into (3.6), we obtain

$$MSE(\hat{S}^2_{New}) \cong S_y^4\left[1-\frac{\varphi_{04}}{4} - \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\left(2\theta-\frac{3}{4}\right)\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] \quad (3.7)$$

Special Cases of Estimator in (3.1)

On putting $\alpha = 1$ in (3.1), the estimator is defined as:

$$\hat{S}^2_{New1} = \left[\omega_1 s_y^2 \left\{\frac{1}{4}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2}\right)\left(exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) + exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right)\right)\right\} + \omega_2(S_x^2 - s_x^2)\right] exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) \quad (3.8)$$

The optimum values of ω_1 and ω_2 for estimator in (3.8) are $\omega_{1(opt)} = \frac{1+\frac{1}{2}\varphi_{04}}{1+\frac{5}{4}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}$ and $\omega_{2(opt)} = R\left[\frac{1}{2} - \frac{\left(1+\frac{1}{2}\varphi_{04}\right)\left(1-\frac{\varphi_{22}}{\varphi_{04}}\right)}{1+\frac{5}{4}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right]$ and the minimum MSE, up to the first order of approximation, given by

$$MSE_{min}\left(\hat{S}^2_{New1}\right) \cong S_y^4\left[1 - \frac{\varphi_{04}}{4} - \frac{\left(1+\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{4}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] \quad (3.9)$$

On putting $\alpha = 2$ in (3.1), the estimator is defined as:

$$\hat{S}^2_{New2} = \left[\omega_1 s_y^2 \left\{\frac{1}{4}\left(\frac{S_x^2}{s_x^2} + \frac{s_x^2}{S_x^2}\right)\left(exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) + exp\left(\frac{s_x^2 - S_x^2}{s_x^2 + S_x^2}\right)\right)\right\}^2 + \omega_2(S_x^2 - s_x^2)\right] exp\left(\frac{S_x^2 - s_x^2}{S_x^2 + s_x^2}\right) \quad (3.10)$$

The optimum values of ω_1 and ω_2 for estimator in (3.10) are $\omega_{1(opt)} = \frac{1+\frac{9}{8}\varphi_{04}}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}$ and $\omega_{2(opt)} = R\left[\frac{1}{2} - \frac{\left(1+\frac{9}{8}\varphi_{04}\right)\left(1-\frac{\varphi_{22}}{\varphi_{04}}\right)}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right]$ and the minimum MSE, up to the first order of approximation, given by

$$MSE_{min}\left(\hat{S}^2_{New2}\right) \cong S_y^4\left[1 - \frac{\varphi_{04}}{4} - \frac{\left(1+\frac{9}{8}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] \quad (3.11)$$

4. Relative Efficiency comparisons

Here, the theoretical efficiency of the proposed class of estimators were carried out and the criteria is discussed under which the estimator performs better than other existing estimators based on the following conditions.

i. From (2.1) and (3.11): $Var\left(\hat{S}_y^2\right) - MSE_{min}\left(\hat{S}^2_{New}\right) > 0, if$

$$S_y^4\left[\varphi_{40} - 1 + \frac{\varphi_{04}}{4} + \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] > 0$$

ii. From (2.3) and (3.11): $MSE\left(\hat{S}_R^2\right) - MSE_{min}\left(\hat{S}^2_{New}\right) > 0, if$

$$S_y^4\left[\varphi_{40} - \varphi_{22} - 1 + \frac{5\varphi_{04}}{4} + \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] > 0$$

iii. From (2.5, 2.13, 2.17) and (3.11): $MSE\left(\hat{S}_{lr}^2\right), MSE\left(\hat{S}_{SW}^2\right), MSE\left(\hat{S}_{YK}^2\right) - MSE_{min}\left(\hat{S}^2_{New2}\right) > 0, if$

$$S_y^4\left[\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right) - 1 + \frac{\varphi_{04}}{4} + \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] > 0$$

iv. From (2.7) and (3.11): $MSE\left(\hat{S}_{GR}^2\right) - MSE_{min}\left(\hat{S}^2_{New}\right) > 0, if$

v.

$$S_y^4\left[\frac{-\varphi_{04}^2 - 4\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)(\varphi_{40}-4)}{16\left(1+\varphi_{40}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)\right)} - 1 + \frac{\varphi_{04}}{4} + \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] > 0$$

vi. From (2.15) and (3.11):

$$MSE\left(\hat{S}_{YG}^2\right) - MSE_{min}\left(\hat{S}^2_{New}\right) > 0, if\ S_y^4\left[\frac{\varphi_{04}}{4}\left(1-g^2(1+\gamma)^2\right) - \frac{1-\frac{1}{8}g^2(1+3\gamma+4\gamma^2)\varphi_{04}}{1-\frac{1}{4}g^2(1+3\gamma)\varphi_{04}^2+\varphi_{04}\left(1-\rho^2_{(s_y^2 s_x^2)}\right)} + \frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right] > 0$$

vii. From (2.21) and (3.11): $MSE\left(\hat{S}_{YS}^2\right) - MSE_{min}\left(\hat{S}^2_{New}\right) > 0, if$

$$S_y^4\left[\frac{\frac{64}{S_y^4}(1-\varphi_{04})MSE(\hat{S}_{lr}^2)-\varphi_{04}^2}{16\left(\varphi_{04}+4(1-\varphi_{40})+2\varphi_{22})+\frac{4}{S_y^4}MSE(\hat{S}_{lr}^2)\right)}-1+\frac{\varphi_{04}}{4}+\frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right]>0$$

viii. From (2.23) and (3.11): $MSE(\hat{S}_M^2)-MSE_{min}(\hat{S}_{New}^2)>0, if$

$$S_y^4\left[\frac{\begin{matrix}64\varphi_{22}^2\varphi_{40}-48\varphi_{22}^2\varphi_{04}-128\varphi_{22}\varphi_{40}\varphi_{04}+48\varphi_{22}\varphi_{04}^2\\+64\varphi_{40}\varphi_{04}^2-9\varphi_{04}^3+64\varphi_{22}^2-64\varphi_{40}\varphi_{04}\end{matrix}}{16(16\varphi_{22}^2-16\varphi_{22}\varphi_{04}-4\varphi_{40}\varphi_{04}+\varphi_{04}^2-4\varphi_{04})}-1+\frac{\varphi_{04}}{4}+\frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\frac{5}{2}\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}\right]>0$$

ix. From (2.25) and (3.11): $MSE(\hat{S}_{MZA}^2)-MSE_{min}(\hat{S}_{New}^2)=$

$$S_y^4\left[\frac{\left(1+\theta\varphi_{04}-\frac{1}{2}\varphi_{04}\right)^2}{1+\left(2\theta-\frac{3}{4}\right)\varphi_{04}+\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}}-\frac{\left\{1+\left(A-\frac{1}{2}\right)\varphi_{04}\right\}^2}{1+\left(2A-\frac{3}{4}\right)\varphi_{04}+\left(\varphi_{40}-\frac{\varphi_{22}^2}{\varphi_{04}}\right)}+\right]>0\text{ , If }\theta>A$$

Note: The above-mentioned conditions (i)–(viii) are always satisfied. Henceforth, the proposed estimator $\hat{S}_{New}^2$ will always outperformed all other estimators considered here.

5.0 Performance Evaluation of the Proposed Estimator $\hat{S}_{New}^2$

To determine the performance of the proposed estimator over the existing estimators, we have considered the real and simulated data sets.

5.1. Numerical Study

For demonstration of the performance of the proposed exponential-type estimator, we considered some set of populations which earlier used by many authors. The description of the populations is presented in Table 1

Table 1: Parameters of the data sets and corresponding source (Muneer *et al*. (2018)

Parameters	Data Set 1 Gujarati (2004) [11]	Data Set 2 Mukherjee *et al.* (1998) [19]	Data Set 3 Singh and Mangat (1996) [24]	Data Set 4 http://www.osservatorionazionalerifiuti.it (2004)	Data 5 Murthy (1967) [20]
N	23	64	39	103	80
n	10	19	18	40	20
$\bar{Y}$	39.699	5.549	26.8433	62.6212	51.8264
$\bar{X}$	1035.065	141.5	106.2	556.5541	11.2646
S_y^2	54.360	2.277	38.99	8345.718	336.9757
S_x^2	381735	5772.67	124.1286	372300.5	70.6634
$\rho_{S_y^2S_x^2}$	0.8277	0.2971	0.8899	0.657	0.7941
$\beta_{2(y)}$	2.03	2.773	2.4032	37.1279	2.2667
$\beta_{2(x)}$	2.696	2.341	2.993	17.8738	2.8664
δ_{22}	2.094	1.458	2.4882	17.222	2.2209
λ	0.1565	0.037	0.0458	0.0153	0.0375

Data set 1: y is the food cost of family employment and x is the weekly income of families.

Data Set 2: y is the number of literate persons in the village and x is the number of workers in the village.

Data Set 3: y is the leaf area for the newly developed strain of wheat and x is the weight of leaves.

Data Set 4: y is the total amount (tons) of recyclable-waste collection in Italy in 2003 and x is the number of inhabitants living in Italy in 2003.

Data Set 5: $y$is the output for 80 factories in a region and x is the fixed capital.

The numerical values that satisfied the condition for efficiency of the proposed estimator based on set of data described earlier, is presented in Table 2 and it is found that the proposed estimator satisfied all efficiency conditions. The MSEs and PREs results based on the data sets considered, are presented in Table 3. The expression given in (5.1) is used to obtain the percentage relative efficiency (PREs) of the proposed and the existing estimators

$$PRE=\frac{Var(S_y^2)}{minMSE(*)}\times 100, *=\ S_y^2, S_R^2, S_{lr}^2, \dots, S_{NEW}^2 \quad (5.1)$$

In Table 3, it is observed that the proposed estimator S_{NEW2}^2 have the smallest MSE as compared to existing estimators for all data sets considered. Thus, the proposed estimator outperformed all the existing estimators.
Note: the minimum $MSE(S_{YG}^2)$, $MSE(S_W^2)$ $MSE(S_{YK}^2)$, *and* given in Table 3 for $(\gamma, a, b), (k, \delta)$ *and* $(\alpha_1, \beta_1) = (1, 1, 0), (1,1)$ *and* $\left(\frac{\varphi_{04}-\varphi_2}{2\varphi_{04}}, 0\right)$.

Table 2: Numerical Values Conditions for Estimator S_{NEW2}^2

	Conditions				
Parameters	**Data 1**	**Data 2**	**Data 3**	**Data 4**	**Data 5**
S_y^2	170.82>0	0.1017>0	61.34>0	32516752>0	4633.71>0
S_R^2	88.64>0	0.1836>0	16.62>0	15932852>0	2183.53>0
$S_{lr}^2, S_W^2, S_{YK}^2$	52.95>0	0.0716>0	10.80>0	15905962>0	1232.89>0
S_d^2	51.98>0	0.0541>0	10.69>0	10679282>0	1198.51>0
S_{BT}^2	58.91>0	0.0783>0	16.32>0	19730572>0	1421.73>0
S_{SG}^2	49.03>0	0.0498>0	10.16>0	9384592>0	1130.07>0
S_{YG}^2	51.87>0	0.0533>0	10.68>0	9401712>0	1195.97>0
S_{Ys}^2	50.08>0	0.0495>0	10.39>0	8452032>0	1151.50>0
S_M^2	37.65>0	0.0339>0	8.26>0	7840882>0	863.56>0
S_{MZA1}^2	36.71>0	0.0332>0	7.79>0	5896072>0	840.59>0
S_{MZA2}^2	14.07>0	0.0120>0	3.06>0	2005090>0	323.32>0
S_{NEW1}^2	31.81>0	0.0283>0	6.80>0	4941162>0	728.73>0
S_{NEW2}^2	-	-	-	-	-

Table 3: MSE and PRE values of the estimators with respect to S_y^2

	Data 1		Data 2		Data 3		Data 4		Data 5	
Parameters	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
S_y^2	172.03	100.00	0.3401	100.00	63.81	100.00	38482180	100.00	5393.89	100.00
S_R^2	89.86	191.45	0.4220	80.59	19.09	334.25	21898280	175.73	2943.71	183.23
$S_{lr}^2, S_W^2, S_{YK}^2$	54.17	317.59	0.3101	109.68	13.28	480.65	21871390	175.95	1993.07	270.63
S_d^2	53.19	323.41	0.2926	116.24	13.16	484.84	16644710	231.20	1958.69	275.38
S_{BT}^2	60.13	286.11	0.3168	107.36	18.79	339.55	25696000	149.76	2181.91	247.21
S_{SG}^2	50.25	342.34	0.2882	118.01	12.63	505.22	15350020	250.70	1890.25	285.35
S_{YG}^2	53.09	324.03	0.2917	116.58	13.15	485.11	15367140	250.42	1956.15	275.74
S_{Ys}^2	51.30	335.35	0.2880	118.09	12.86	496.08	14417460	266.91	1911.68	282.15
S_M^2	38.87	442.64	0.2724	124.88	10.73	594.63	13806310	278.73	1623.74	332.19
S_{MZA1}^2	37.93	453.54	0.2717	125.18	10.27	621.59	11861500	324.43	1600.78	336.95
S_{MZA2}^2	15.29	1125.06	0.2504	135.81	5.53	1153.58	7970518	482.81	1083.51	497.82
S_{NEW1}^2	33.03	520.80	0.2667	127.51	9.27	688.57	10906590	352.83	1488.91	362.27
S_{NEW2}^2	1.22	14132.86	0.2385	142.63	2.47	2581.11	5965428	645.09	760.18	709.55

5.2 Simulation Study

A simulation study is conducted to verify the theoretical results discussed in section 3. We generated three artificial population of size N=1000 from a multivariate normal distribution of the same means $\mu = [\bar{Y}, \bar{X}\ ] = [2, 2]$and different covariance matrices are as follows.

Table 4: Simulation Results for MSEs and PREs of different Estimators with respect to S_y^2

		Population 1		Population 2		Population 3	
n	**Estimator**	**MSE**	**PRE**	**MSE**	**PRE**	**MSE**	**PRE**
50	S_y^2	3.0662	100.00	3.7855	100.00	1.3628	100.00
	S_R^2	5.5261	55.49	4.1712	90.75	0.6824	199.70
	$S_{lr}^2, S_W^2, S_{YK}^2$	3.0305	101.18	3.0121	125.68	0.5950	229.04
	S_d^2	2.9208	104.98	2.9237	129.48	0.5853	232.85
	S_{BT}^2	3.5280	86.91	3.0305	124.91	0.6817	199.91
	S_{SG}^2	2.8861	106.24	2.8873	131.11	0.5765	236.37
	S_{YG}^2	2.9167	105.13	2.9203	129.62	0.5849	232.99
	S_{Ys}^2	2.8875	106.19	2.8905	130.96	0.5785	235.58
	S_M^2	2.8267	108.47	2.7800	136.17	0.5600	243.33
	S_{MZA1}^2	2.7488	111.55	2.7406	138.12	0.5397	252.48
	S_{MZA2}^2	2.5634	119.62	2.5328	149.46	0.4796	284.17
	S_{NEW1}^2	2.7063	113.30	2.6936	140.54	0.5266	258.80
	S_{NEW2}^2	2.4560	124.85	2.4101	157.07	0.4423	308.13
100	S_y^2	1.4524	100.00	1.7931	100.00	0.6455	100.00
	S_R^2	2.6176	55.49	1.9758	90.75	0.3232	199.70
	$S_{lr}^2, S_W^2, S_{YK}^2$	1.4355	101.18	1.4268	125.68	0.2818	229.04

	S_d^2	1.4104	102.98	1.4066	127.48	0.2796	230.84
	S_{BT}^2	1.6711	86.91	1.4355	124.91	0.3229	199.91
	S_{SG}^2	1.4025	103.56	1.3983	128.23	0.2777	232.48
	S_{YG}^2	1.4100	103.01	1.4063	127.51	0.2796	230.88
	S_{Ys}^2	1.4033	103.50	1.3995	128.13	0.2782	232.07
	S_M^2	1.3888	104.58	1.3729	130.60	0.2738	235.79
	S_{MZA1}^2	1.3699	106.03	1.3638	131.48	0.2691	239.89
	S_{MZA2}^2	1.3247	109.64	1.3135	136.51	0.2547	253.42
	S_{NEW1}^2	1.3596	106.82	1.3525	132.58	0.2660	242.69
	S_{NEW2}^2	1.2981	111.89	1.2833	139.73	0.2457	262.78
200	S_y^2	0.6455	100.00	0.7969	100.00	0.2869	100.00
	S_R^2	1.1634	55.49	0.8781	90.75	0.1437	199.70
	$S_{lr}^2, S_W^2, S_{YK}^2$	0.6380	101.18	0.6341	125.68	0.1253	229.04
	S_d^2	0.6330	101.98	0.6301	126.48	0.1248	229.84
	S_{BT}^2	0.7427	86.91	0.6380	124.91	0.1435	199.91
	S_{SG}^2	0.6314	102.23	0.6285	126.81	0.1244	230.57
	S_{YG}^2	0.6330	101.98	0.6301	126.48	0.1248	229.85
	S_{Ys}^2	0.6316	102.20	0.6287	126.75	0.1245	230.38
	S_M^2	0.6287	102.68	0.6233	127.86	0.1236	232.04
	S_{MZA1}^2	0.6248	103.32	0.6215	128.24	0.1227	233.81
	S_{MZA2}^2	0.6155	104.88	0.6111	130.40	0.1198	239.53
	S_{NEW1}^2	0.6227	103.67	0.6192	128.71	0.1221	235.02
	S_{NEW2}^2	0.6099	105.83	0.6049	131.75	0.1179	243.32
300	S_y^2	0.3766	100.00	0.4649	100.00	0.1674	100.00
	S_R^2	0.6786	55.49	0.5122	90.75	0.0838	199.70
	$S_{lr}^2, S_W^2, S_{YK}^2$	0.3722	101.18	0.3699	125.68	0.0731	229.04
	S_d^2	0.3705	101.64	0.3685	126.14	0.0729	229.51
	S_{BT}^2	0.4333	86.91	0.3722	124.91	0.0837	199.91
	S_{SG}^2	0.3699	101.79	0.3680	126.34	0.0728	229.93
	S_{YG}^2	0.3705	101.65	0.3685	126.14	0.0729	229.51
	S_{Ys}^2	0.3700	101.77	0.3681	126.30	0.0728	229.82
	S_M^2	0.3690	102.05	0.3662	126.95	0.0725	230.79
	S_{MZA1}^2	0.3676	102.42	0.3656	127.17	0.0722	231.81
	S_{MZA2}^2	0.3644	103.33	0.3620	128.42	0.0712	235.09
	S_{NEW1}^2	0.3669	102.63	0.3648	127.44	0.0720	232.51
	S_{NEW2}^2	0.3625	103.87	0.3598	129.19	0.0705	237.24

Population 1: $\Sigma_{yx} = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}, \rho_{yx} = 0.3209,$

Population 2: $\Sigma_{yx} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}, \rho_{yx} = 0.6746$ and

Population 3: $\Sigma_{yx} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}, \rho_{yx} = 0.3209$

We considered different sample sizes (50, 100, 200 and 300) for each population and the values of MSE's for all estimators are computed as presented in Tale 4, using 5000 Monte Carlo samples of different sizes selected from each population. The PREs for all estimators are computed by using expression in (5.1).

Observing the results in table 4, the PRE of the estimator S_{NEW2}^2 is always having the highest value across the three-population considered and performed better than other existing estimators. Besides, it can be seen that estimator S_{lr}^2, S_{SW}^2 *and* S_{YK}^2 performed equally. he

7. Conclusion

In this study, an improved class of exponential estimator for finite population variance S_y^2 utilizing auxiliary information in simple random sampling has been proposed. Bias and MSE expression of the proposed estimator are derived up to first order of approximation. Using real data sets and Monte Carlo simulation study, the theoretical and numerical performance of the proposed estimator were carried out and compared to the existing estimators. The findings shown that the proposed estimator S_{NEW2}^2 outperformed the unit variance estimator and other existing estimators in term of MSE and PRE. Also, it is observed that increase in sample size and correlation coefficient led to gain in efficiencies of all estimators considered for the three populations (see Table 4). Estimators S_{lr}^2, S_{SW}^2 *and* S_{YK}^2 are found to perform equally in all data sets. Based on these findings, the proposed estimator S_{NEW2}^2 is recommended for efficient estimation of population variance under simple random sampling.

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