An improved generalized estimators for finite population variance of a study variable based on auxiliary information

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Abstract
The information on auxiliary variable has been shown to be relevant in selection and estimation of parameters to gain more precision in estimates of study variable. The transformation of this auxiliary information also aids increase efficiency of estimators. In this article, an improved mixed ratio-product-type exponential estimator is proposed and evaluated using information on auxiliary variable for population variance under simple random sampling. The mathematical expressions for bias and mean squared error (MSE) of the proposed estimator were derived up to first order of approximation. Using real data sets and Monte Carlo simulation study, the performance evaluation of the proposed estimator was considered and compared to the existing estimators. The results of the empirical and simulation studies show that the proposed estimator outperformed the existing estimators in term of MSE and PRE.

Keywords: Auxiliary information, Bias, mean squared error, percentage relative efficiency, variance estimation

Introduction
In a survey process, the use of auxiliary information at the estimation stage usually leads to gain in precision of the estimator of unknown population parameters. Among many ratio, product and regression approaches of estimators are suitable examples, provided that suitable relationship is existing between auxiliary variable and study variable. Many authors have introduced several forms of ratio-type exponential estimators based on different transformation of the original auxiliary variable including Bahl and Tetuja (1991) [6], Grover and Kaur (2011) [10], Singh and Espejo (2003), Kadilar and Cingi (2004) [17], Haq and Shabbir (2014) [12] and others.


Recently, Muneer et al. (2018) and Mohammed et al. (2022) suggested an improved ratio-product type exponential estimator to estimate the finite population variance $S_y^2$. Motivated by the above studies, we consider the importance of estimating variation and propose an improved mixed ratio-product-type exponential estimator of finite population variance $S_y^2$ when the information on parameters of auxiliary variable is available.
Consider a finite population $\Omega = \{\Omega_1, \Omega_2, \Omega_3, \ldots, \Omega_N\}$ of size $N$ units. Let the value of study variable $Y$ and auxiliary variable $X$ be defined as $y_i$ and $x_i$, respectively on $i^{th}$ unit $\Omega_i, i = 1, \ldots, N$. Assuming the population parameters of the auxiliary variable is known. The properties of study variable $Y$ and auxiliary variable $X$, respectively are given as follows.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Study variable</th>
<th>Auxiliary variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sample mean</td>
<td>$\hat{y} = n^{-1} \sum_{i=1}^{n} y_i$</td>
<td>$\hat{x} = n^{-1} \sum_{i=1}^{n} x_i$</td>
</tr>
<tr>
<td>Population mean</td>
<td>$\bar{Y} = N^{-1} \sum_{i=1}^{N} y_i$</td>
<td>$\bar{X} = N^{-1} \sum_{i=1}^{N} x_i$</td>
</tr>
<tr>
<td>Population variance</td>
<td>$S_Y^2 = (N - 1)^{-1} \sum(y_i - \bar{Y})^2$</td>
<td>$S_X^2 = (N - 1)^{-1} \sum(x_i - \bar{X})^2$</td>
</tr>
<tr>
<td>Sample variance</td>
<td>$S_y^2 = (n - 1)^{-1} \sum(y_i - \bar{Y})^2$</td>
<td>$S_x^2 = (n - 1)^{-1} \sum(x_i - \bar{x})^2$</td>
</tr>
<tr>
<td>Coefficient of variation</td>
<td>$C_y = S_y/\bar{Y}$</td>
<td>$C_x = S_x/\bar{X}$</td>
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</tbody>
</table>

To estimate the bias and mean squared error (MSE) of the estimator, the error terms are defined as follows: $e_0 = \frac{s_y^2 - S_y}{S_y^2}$, $e_1 = \frac{s_x^2 - S_x}{S_x^2}$, such that $E(e_0) = E(e_1) = 0$. $E(e_0^2) = \lambda(\delta_0 - 1) = \varphi_0$, $E(e_1^2) = \lambda(\delta_0 - 1) = \varphi_0$, (1 + $\delta_2$) where $\delta_2$ and $\delta_0$ are the population coefficients of kurtosis for variable $Y$ and $X$, respectively.

2. Review of the existing estimators

Here, we present existing estimators of the finite population mean that are available in the sampling literature. For brief discussion, the variance and MSEs of all estimators, considered in this section, are obtained up to first order of approximation.

The conventional variance estimator of $\hat{S}_Y^2 = s_y^2$ for population variance is given by

$$\hat{S}_Y^2 \equiv S_Y^2 \varphi_{z_0}$$  \hspace{1cm} (2.1)

Isaki (1983) [14] proposed a ratio-type estimator $\hat{S}_Y^2$ for the population variance of the study variable. The estimator $\hat{S}_Y^2$ and its $MSE(\hat{S}_Y^2)$ are, respectively, given by

$$\hat{S}_Y^2 = s_y^2 \left( \frac{S_y^2}{S_x^2} \right)$$  \hspace{1cm} (2.2)

and

$$MSE(\hat{S}_Y^2) = S_Y^4 \left[ \varphi_{z_0} + \varphi_{z_2} - 2 \varphi_{z_0} \right]$$  \hspace{1cm} (2.3)

the usual regression estimator $\hat{S}_{tr}^2$ and its $MSE(\hat{S}_{tr}^2)$ up to first order of approximation are, respectively, given by

$$\hat{S}_{tr}^2 = s_y^2 + b(s_y^2 s_x^2)(S_y^2 - S_x^2)$$  \hspace{1cm} (2.4)

and

$$MSE_{min}(\hat{S}_{tr}^2) \equiv S_Y^4 \varphi_{z_0} \left( 1 - \rho_{s_y^2 s_x^2}^2 \right)$$  \hspace{1cm} (2.5)

where $b(s_y^2 s_x^2) = \frac{s_y s_x^2}{\varphi_{z_0} \varphi_{z_2}}$ is the sample regression coefficient, and $\rho_{s_y^2 s_x^2}^2 = \frac{\varphi_{z_0}}{\varphi_{z_0} \varphi_{z_2}}$

Singh et al. (1988) proposed a different-type estimator $\hat{S}_d^2$ for population variance, this estimator and its $MSE(\hat{S}_d^2)$ up to first order of approximation are, respectively given by

$$\hat{S}_d^2 = k_1 s_y^2 + k_2 (S_y^2 - s_y^2)$$  \hspace{1cm} (2.6)

and

$$MSE_{min}(\hat{S}_d^2) \equiv S_Y^4 \left[ \frac{\varphi_{z_0} \left( 1 - \rho_{s_y^2 s_x^2}^2 \right)}{1 + \varphi_{z_0} \left( 1 - \rho_{s_y^2 s_x^2}^2 \right)} \right]$$  \hspace{1cm} (2.7)
where \( k_1(\text{opt}) = \frac{\varphi_04}{\varphi_04 + \varphi_{40} \varphi_04 - \varphi_{22}} \) and \( k_1(\text{opt}) = \frac{s_x^2 \varphi_{22}}{s_x^2 (\varphi_{04} + \varphi_{40} \varphi_04 - \varphi_{22})} \) are the optimum values for MSE of the estimator \( \hat{S}_2^2 \).

Bahl and Tuteja (1991) \cite{6} proposed an exponential ratio-type estimator \( \hat{S}_{BT}^2 \) for the population variance of the study variable \( Y \). The \( \hat{S}_{BT}^2 \) and its MSE \((\hat{S}_{BT}^2)\) up to first order of approximation are, respectively given by
\[
\hat{S}_{BT}^2 = s^2y \exp\left(\frac{s_x^2 - s_y^2}{s_x^2 + s_y^2}\right)
\] (2.8)

and
\[
\text{MSE}(\hat{S}_{BT}^2) \equiv S_2^4 \left[ \varphi_{40} + \frac{1}{4} \varphi_{04} - 2 \varphi_{22} \right]
\] (2.9)

Shabbir and Gupta (2007) \cite{22} proposed a different-ratio type exponential estimator \( \hat{S}_{SB}^2 \). The estimator and its MSE are given by
\[
\hat{S}_{SB}^2 = w_1s_y^2 + w_2\left( s_x^2 - s_y^2 \right)
\] (2.10)

and
\[
\text{MSE}_{\text{min}}(\hat{S}_{SB}^2) \equiv S_2^4 \left[ \varphi_{40} + \frac{1}{4} \varphi_{04} - \frac{1}{2} \varphi_{22} \right]
\] (2.11)

where \( d_1(\text{opt}) = \frac{\varphi_{04}}{\varphi_{04} + \varphi_{40} \varphi_04 - \varphi_{22}} \) and \( d_2(\text{opt}) = \frac{s_y^2}{S_2^2} \left[ \frac{-4 \varphi_{40} + \varphi_{04} + 8 \varphi_{22} - \varphi_{22} \varphi_{04} + 4 \varphi_{40} \varphi_04 - 4 \varphi_{22}^2}{\varphi_{04} + \varphi_{40} \varphi_04 - \varphi_{22}} \right] \) are the optimum values for MSE of the estimator \( \hat{S}_{SB}^2 \).

Swain (2015) \cite{27} proposed a generalized estimator for population variance. The estimator and its MSE up to first order of approximation are, respectively, given by
\[
\hat{S}_{SW}^2 = s^2y \left( k \left( \frac{s_x^2}{s_y^2} \right)^{g} \left( 1 - k \right) \left( \frac{s_x^2}{s_y^2} \right)^{h} \right)^{\delta}
\] (2.12)

and
\[
\text{MSE}_{\text{min}}(\hat{S}_{SW}^2) \equiv S_2^4 \varphi_{40} \left[ 1 - \rho^2(\hat{S}_{SW}^2) \right]
\] (2.13)

where \( g, h \text{ and } \delta = (1, -1) \) are real and free parameters to be chosen suitably and \( k = \frac{\delta h + \varphi_{22}}{\delta (g+h)} \) is the optimum value for the estimator \( \hat{S}_{SW}^2 \).

Yadav et al. (2015) \cite{30} proposed a generalized variance estimator \( \hat{S}_{YG}^2 \). The estimator and its MSE up to first order of approximation are, respectively, given by
\[
\hat{S}_{YG}^2 = \left( u_1s_y^2 + u_2\left( S_x^2 - s_y^2 \right) \right) \left[ \gamma \left( \frac{a_s^2 + b}{as_s^2 + b} \right) + (1 - \gamma) \left( \frac{a(s_x^2 - s_y^2)}{a(s_x^2 + s_y^2) + b} \right) \right]
\] (2.14)

and
\[
\text{MSE}_{\text{min}}(\hat{S}_{YG}^2) = S_2^4 \left[ 1 - \frac{1}{4} g^2 \left( 1 + \gamma \right)^2 \varphi_{04} - \frac{1 - \frac{1}{4} g^2 (1 + 3\gamma + 4 \gamma^2) \varphi_{04}}{1 - \frac{1}{4} g^2 (1 + 3 \gamma) \varphi_{04} + \varphi_{04} \left( 1 - \rho^2(\hat{S}_{YG}^2) \right)} \right]
\] (2.15)

where
\[
g = \frac{a^2}{s_x^2 + b} u_1(\text{opt}) = \left( \frac{1 - \frac{1}{4} g^2 (1 + 3 \gamma + 4 \gamma^2) \varphi_{04}}{1 - \frac{1}{4} g^2 (1 + 3 \gamma) \varphi_{04} + \varphi_{04} \left( 1 - \rho^2(\hat{S}_{YG}^2) \right)} \right) \text{ and } u_2(\text{opt}) = \frac{S_2^2}{S_2^2} \left( \frac{1}{4} g (1 + \gamma) + u_1(\text{opt}) \left( \frac{\varphi_{22}}{\varphi_{04}} - g (1 + \gamma) \right) \right)
\]

are the optimum values that minimized the MSE of the estimator.

the minimum MSE of the estimator \((\hat{S}_{YG}^2)\) at \((\gamma, a, b) = (1, 1, 0)\) also given by
\( M_{\text{min}}(\hat{S}_{xy}) \cong S_{xy}^2 \left[ \frac{\text{MSE}(\hat{S}_{xy})}{\text{MSE}(\hat{S}_{xy})} \right] \) \hspace{1cm} (2.16)

Yadav and Kadilar (2014) proposed ratio-product-ratio type estimator for \( \hat{S}_{xy}^2 \), is given by
\[
\hat{S}_{xy}^2 = \alpha_1 \left( \frac{(1-\beta_1)s_y^2 + \beta_1 s_x^2}{(1-\beta_1)s_y^2 + \beta_1 s_x^2} \right) + (1 - \alpha_1) \left( \frac{\beta_2 s_y^2 + (1-\beta_2)s_x^2}{(1-\beta_2)s_y^2 + \beta_2 s_x^2} \right)
\] \hspace{1cm} (2.17)

The minimum MSE of the estimator \( \hat{S}_{xy}^2 \) is similar to regression estimator \( \hat{S}_{xy}^2 \) (2.5) at optimum values \( (\alpha_1, \beta_1) = \frac{\theta_1}{2\theta_1}, 0 \).

Singh and Malik (2014) proposed an improve estimator \( \hat{S}_{SM}^2 \) for population variance, given by
\[
\hat{S}_{SM}^2 = S_{xy}^2 \left( v_1 s_y^2 + v_2 (S_x^2 - s_y^2) \right) \exp \left\{ \psi \frac{a(s_y-s_x)}{a(s_z^2+s_y^2)+b} \right\}
\] \hspace{1cm} (2.18)

Where \( v_1 \) and \( v_2 \) are suitably chosen constants, \( \psi \) takes values +1 and −1 for ratio and product type estimators and \( a, b \) be the known population parameters of the auxiliary variables. The MSE of \( \hat{S}_{SM}^2 \) at optimum values of
\[
v_1(\text{opt}) = \frac{1}{4} \left\{ -\frac{12\theta_2 \psi_2 + 4\theta_2}{4\theta_2 + 4\theta_1 + 16\theta_1 + 8\theta_1} \right\}
\]
and
\[
v_2(\text{opt}) = \frac{1}{4} \left\{ -\frac{\theta_2 \psi_2 + 4\theta_2}{4\theta_2 + 4\theta_1 + 16\theta_1 + 8\theta_1} \right\}
\]

\[ M_{\text{min}}(\hat{S}_{xy}) \cong S_{xy}^2 \left[ \frac{\text{MSE}(\hat{S}_{xy})}{\text{MSE}(\hat{S}_{xy})} \right] \] \hspace{1cm} (2.19)

Yakub and Shabbir (2016) \([21]\) proposed a class of estimator for population variance \( S_y^2 \) and given by
\[
\hat{S}_{YS}^2 = \hat{S}_y^2 \left( \theta_1 s_y^2 + \theta_2 (S_x^2 - s_y^2) \right) \left( \frac{a(s_y-s_x)}{a(s_z^2+s_y^2)+b} \right) + \frac{1}{2} \exp \left( \frac{a(s_y-s_z)^2}{a(s_z^2+s_y^2)+b} \right)
\] \hspace{1cm} (2.20)

Where \( \theta_1 \) and \( \theta_2 \) are suitably chosen constants and \( a \) and \( b \) be the known parameters of the auxiliary variable. The minimum MSE at optimum values of \( \theta_1 = \frac{\theta_2}{2\theta_2} \left\{ 1+\frac{\theta_2}{\theta_1} \right\} \) and \( \theta_2 = \frac{\theta_2}{2\theta_2} \left\{ \frac{\theta_2}{\theta_1} + 4\theta_1 - 4\theta_2 \right\} \) given by
\[
M_{\text{min}}(\hat{S}_{YS}) \cong S_{xy}^2 \left[ \frac{\text{MSE}(\hat{S}_{xy})}{\text{MSE}(\hat{S}_{xy})} \right] \] \hspace{1cm} (2.21)

Muneer et al. (2018) proposed an improved class of estimator for variance and given by
\[
\hat{S}_{M}^2 = S_y^2 \left( \omega_1 \left( \frac{s_y^2}{s_z^2} \right) + \omega_2 \left( \frac{s_z^2}{s_y^2} \right) \right) \exp \left( \frac{s_y^2-s_z^2}{s_z^2+s_y^2} \right)
\] \hspace{1cm} (2.22)

where \( \omega_1 \) and \( \omega_2 \) are suitably chosen constants. The minimum MSE of \( \hat{S}_{M}^2 \) up to first order of approximation, at optimum values
\[
\omega_1(\text{opt}) = \frac{1}{8} \left\{ \frac{16\theta_2^2 + 8\theta_2 \psi_1^2 + 24\theta_2 \psi_1^2}{16\theta_2^2 + 8\theta_2 \psi_1^2} \right\}
\]
and
\[
\omega_2(\text{opt}) = \frac{1}{8} \left\{ \frac{16\theta_2^2 + 8\theta_2 \psi_1^2 + 24\theta_2 \psi_1^2}{16\theta_2^2 + 8\theta_2 \psi_1^2} \right\}
\]
is given by
\[
M_{\text{min}}(\hat{S}_{M}) \cong S_{xy}^2 \left[ \frac{\text{MSE}(\hat{S}_{xy})}{\text{MSE}(\hat{S}_{xy})} \right] \] \hspace{1cm} (2.23)

Mohammed et al. (2022) proposed a generalized estimator for variance and is given by
\[
\hat{S}_{MZA} = d_1 s_y^2 \left\{ \frac{s_y^2}{s_z^2} \right\}^\alpha + d_2 (S_x^2 - s_y^2) \exp \left( \frac{s_y^2-s_z^2}{s_z^2+s_y^2} \right)
\] \hspace{1cm} (2.24)
Where $d_i (i = 1, 2)$ are unknown constants whose values to be determined later, and $\alpha$ is suitably chosen constant. The minimum MSE of $\hat{S}_{MZA}^2$ up to first order of approximation, at optimum values

$$d_1(\text{opt}) = \frac{1+(\frac{1}{2})\phi_{04}}{1+(2\alpha-\frac{1}{2})\phi_{04} + \phi_{40} - \phi_{22}}$$

and $d_1(\text{opt}) = \frac{s^2}{s^2} \left( \frac{1}{2} - d_1(\text{opt}) \left( 1 - \frac{\phi_{22}}{\phi_{04}} \right) \right)$ given by

$$\text{MSE}_{\text{min}}(\hat{S}_{MZA}^2) \cong \mathcal{S}_y \left( 1 - \frac{\phi_{04}}{4} - \frac{\left( 1+(\frac{1}{2})\phi_{04} \right)}{1+(2\alpha-\frac{1}{2})\phi_{04} + (\phi_{40} - \phi_{22})} \right)^2$$

(2.25)

where $A = \frac{1}{2} \alpha + \frac{3}{6}$

### 3. Proposed Estimator

In the line with the direction of study carried out by Mohammed et al. (2022), we proposed an improved class of combined estimators for estimating the finite population variance $s_y^2$ under method of simple random sampling (SRS). The proposed estimator is given by

$$\hat{S}_{\text{New}}^2 = \left[ \omega_1 s^2 \left( \frac{1}{4} \left( \frac{s_x^2}{s_z^2} + \frac{s_y^2}{s_z^2} \right) \exp \left( \frac{s_x^2 - s_y^2}{s_z^2 + s_z^2} \right) + \exp \left( \frac{s_x^2 - s_y^2}{s_z^2 + s_z^2} \right) \right) \right]^\gamma + \omega_2 (S_x^2 - s_y^2) \exp \left( \frac{s_x^2 - s_y^2}{s_z^2 + s_z^2} \right)$$

(3.1)

where $\omega_1$ and $\omega_2$ are unknown constants whose values are to be determined such that the MSE's are minimum, $\gamma$ is a suitably chosen constant.

Rewrite (3.1) in term of errors, we obtain

$$S_{\text{New}}^2 = \left[ \omega_1 s^2 \left( 1 + e_0 \right) \left( \frac{1}{4} \left( 1 + e_i \right)^{-1} + 1 + e_i \right) \exp \left( -\frac{1}{2} e_1 \left( 1 + \frac{1}{2} e_i \right) \right) + \exp \left( \frac{1}{2} e_1 \left( 1 + \frac{1}{2} e_i \right) \right) \right]^\gamma - \omega_2 S_x^2 = \exp \left( -\frac{1}{2} e_1 \left( 1 + \frac{1}{2} e_i \right) \right)$$

(3.2)

Expand (3.2) and retain the errors to second degree order using Taylor series, we obtain

$$S_{\text{New}}^2 = \left[ \omega_1 s^2 \left( 1 + e_0 \right) \left( \frac{1}{4} \left( 2 + e_i^2 \right) \left( 2 + \frac{1}{4} e_i^2 \right) \right)^\gamma - \omega_2 S_x^2 \right] \left( 1 - \frac{1}{2} e_1 + \frac{3}{8} e_i^2 \right)$$

(3.3)

$$\hat{S}_{\text{New}}^2 - S_y^2 = -S_y^2 + \omega_1 S_{\text{New}}^2 \left[ 1 + e_0 - \frac{1}{2} e_1 - \omega_2 R e_1 - \frac{1}{2} e_0 e_1 + \theta e_i^2 + \omega_2 R e_i^2 \right]$$

(3.4)

where $\theta = \frac{s}{6} \gamma + \frac{3}{8} \alpha$ and $R = \frac{s^2}{s_z^2}$

Taking the expectation of (3.4), the bias of the proposed estimator, up to the first order of approximation, given by

$$\text{Bias}(\hat{S}_{\text{New}}^2) = -S_y^2 + \omega_1 S_{\text{New}}^2 \left[ 1 - \frac{1}{2} \phi_{22} + \theta \phi_{04} + \omega_2 R \phi_{04} \right]$$

(3.5)

Squaring and taking the expectation of (3.4), we obtain the MSE of the proposed estimator by using first order of approximation and is given by

$$\text{MSE}(\hat{S}_{\text{New}}^2) = S_y^2 \left[ 1 + \omega_1 \left[ 1 + \phi_{40} + \frac{1}{4} \phi_{04} - 2 \phi_{22} + 2 \theta \phi_{04} \right] - \omega_1 \left[ 1 + 2 \theta \phi_{04} - \phi_{22} \right] - 2R \omega_1 \omega_2 \left( \phi_{22} - \phi_{04} \right) + \omega_2 R^2 \phi_{04} - \omega_2 R \phi_{04} \right]$$

(3.6)

Differentiate partially (3.6) w.r.to $\omega_1$ and $\omega_2$ and equate to zero, we have

$$\left[ 2(1 + \phi_{40} + \frac{1}{4} \phi_{04} - 2 \phi_{22} + 2 \theta \phi_{04}) \right] \left[ \frac{\omega_1}{\omega_2} \right] = \left[ -2 - 2 \theta \phi_{04} + \phi_{22} \right]$$

After simplifying, we obtain the optimum value of $\omega_1$ and $\omega_2$, that is,

$$\omega_1(\text{opt}) = \frac{1 + \phi_{04} - \frac{1}{2} \phi_{04}}{1 + \left( 2 \theta - \frac{3}{2} \right) \phi_{04} + \phi_{40} - \phi_{22}}$$

and $\omega_2(\text{opt}) = R \left[ \frac{1}{2} - \frac{(1 + \theta \phi_{04} - \frac{1}{2} \phi_{04}) \left( 1 - \frac{\phi_{22}}{\phi_{04}} \right)}{1 + \left( 2 \theta - \frac{3}{2} \right) \phi_{04} + \phi_{40} - \phi_{22}} \right].$

Substituting $\omega_1(\text{opt})$ and $\omega_2(\text{opt})$ into (3.6), we obtain

$$\text{MSE}(\hat{S}_{\text{New}}^2) \cong S_y^4 \left[ 1 - \frac{\phi_{04}}{4} - \frac{(1 + \theta \phi_{04} - \frac{1}{2} \phi_{04})^2}{1 + \left( 2 \theta - \frac{3}{2} \right) \phi_{04} + \phi_{40} - \phi_{22}} \right]$$

(3.7)
Special Cases of Estimator in (3.1)

On putting \( \alpha = 1 \) in (3.1), the estimator is defined as:

\[
S_{\hat{Y}_{new1}}^2 = \left[ \omega_1 s_y^2 \left\{ \frac{1}{4} \left( \frac{s_x^2}{s_y^2} + \frac{s_y^2}{s_x^2} \right) \left( \exp \left( \frac{s_x^2 - s_y^2}{s_x^2 + s_y^2} \right) + \exp \left( \frac{s_y^2 - s_x^2}{s_x^2 + s_y^2} \right) \right) + \omega_2 (S_x^2 - S_y^2) \right\} \exp \left( \frac{s_x^2 - s_y^2}{s_x^2 + s_y^2} \right) \right] (3.8)
\]

The optimum values of \( \omega_1 \) and \( \omega_2 \) for estimator in (3.8) are \( \omega_{1(\text{opt})} = \frac{1 + \frac{5}{2} \varphi_{04}}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \) and \( \omega_{2(\text{opt})} = R \left[ 1 + \frac{\varphi_{04} - \frac{\varphi_{22}}{\varphi_{04}}}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] \) and the minimum MSE, up to the first order of approximation, given by

\[
MSE_{\min}(S_{\hat{Y}_{new1}}^2) \cong S_y^2 \left[ 1 - \frac{\varphi_{04}}{4} - \frac{(1 + \frac{5}{2} \varphi_{04})^2}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] \] (3.9)

On putting \( \alpha = 2 \) in (3.1), the estimator is defined as:

\[
S_{\hat{Y}_{new2}}^2 = \left[ \omega_1 s_y^2 \left\{ \frac{1}{4} \left( \frac{s_x^2}{s_y^2} + \frac{s_y^2}{s_x^2} \right) \left( \exp \left( \frac{s_x^2 - s_y^2}{s_x^2 + s_y^2} \right) + \exp \left( \frac{s_y^2 - s_x^2}{s_x^2 + s_y^2} \right) \right) + \omega_2 (S_x^2 - S_y^2) \right\} \exp \left( \frac{s_x^2 - s_y^2}{s_x^2 + s_y^2} \right) \right] (3.10)
\]

The optimum values of \( \omega_1 \) and \( \omega_2 \) for estimator in (3.10) are \( \omega_{1(\text{opt})} = \frac{1 + \frac{5}{2} \varphi_{04}}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \) and \( \omega_{2(\text{opt})} = R \left[ 1 + \frac{\varphi_{04} - \frac{\varphi_{22}}{\varphi_{04}}}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] \) and the minimum MSE, up to the first order of approximation, given by

\[
MSE_{\min}(S_{\hat{Y}_{new2}}^2) \cong S_y^2 \left[ 1 - \frac{\varphi_{04}}{4} - \frac{(1 + \frac{5}{2} \varphi_{04})^2}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] \] (3.11)

4. Relative Efficiency comparisons

Here, the theoretical efficiency of the proposed class of estimators were carried out and the criteria is discussed under which the estimator performs better than other existing estimators based on the following conditions.

i. From (2.1) and (3.11): \( Var(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if

\[
S_y^2 \left[ \varphi_{40} - 1 + \frac{\varphi_{04}}{4} + \frac{(1 + \theta \varphi_{04} - \frac{1}{2} \varphi_{04})^2}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] > 0
\]

ii. From (2.3) and (3.11): \( MSE(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if

\[
S_y^2 \left[ \varphi_{40} - \varphi_{22} - 1 + \frac{5 \varphi_{04}}{4} + \frac{(1 + \theta \varphi_{04} - \frac{1}{2} \varphi_{04})^2}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] > 0
\]

iii. From (2.5, 2.13, 2.17) and (3.11): \( MSE(S_{\hat{Y}_{new}}^2), MSE(S_{\hat{Y}_{new}}^2), MSE(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if

\[
S_y^2 \left[ \varphi_{40} \left( 1 - \rho_{(s_x s_y)}^2 \right) - 1 + \frac{\varphi_{04}}{4} + \frac{(1 + \theta \varphi_{04} - \frac{1}{2} \varphi_{04})^2}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right] > 0
\]

iv. From (2.7) and (3.11): \( MSE(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if

\[
S_y^2 \left[ -\psi_{40} \frac{(1 - \rho_{(s_x s_y)}^2)}{\psi_{40} + 4} \left( 1 + \frac{1 + \theta \varphi_{04} - \frac{1}{2} \varphi_{04}}{1 + \frac{5}{2} \varphi_{04} + \varphi_{40} - \frac{\varphi_{22}}{\varphi_{04}}} \right) \right] > 0
\]

v. From (2.15) and (3.11):

\[
MSE(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if \( S_y^2 \left[ \varphi_{40} \left( 1 - g^2(1 + \gamma)^2 \right) - \frac{1 - \psi_{22} (1 + 3 \gamma + 4 \gamma^2) \psi_{40}}{1 - \psi_{22} (1 + 3 \gamma) \psi_{40} + \psi_{40} \left( 1 - \rho_{(s_x s_y)}^2 \gamma \right)} \right] > 0
\]

vi. From (2.21) and (3.11): \( MSE(S_{\hat{Y}_{new}}^2) - MSE_{\min}(S_{\hat{Y}_{new}}^2) > 0 \), if
\[
S^4_\gamma \left[ \frac{64\varphi_{04} - 128\varphi_{22}\varphi_{40} + 128\varphi_{22}\varphi_{04} + 128\varphi_{04}\varphi_{40} + 128\varphi_{22}\varphi_{40} - 128\varphi_{04}\varphi_{40}}{16(16\varphi_{22} - 16\varphi_{22}\varphi_{04} - 4\varphi_{40}\varphi_{04} + 4\varphi_{04})} \right] - 1 + \frac{\varphi_{04}}{4} + \frac{(1 + \varphi_{04})^2}{1 + 2\varphi_{04} + \frac{\varphi_{22}}{\varphi_{04}}} > 0
\]

viii. From (2.23) and (3.11): \(MSE(S^4_\gamma) - MSE_{\min}(S^2_{NEW}) > 0\), if

\[
S^4_\gamma \left[ \frac{64\varphi_{04} - 128\varphi_{22}\varphi_{40} + 128\varphi_{22}\varphi_{04} + 128\varphi_{04}\varphi_{40} + 128\varphi_{22}\varphi_{40} - 128\varphi_{04}\varphi_{40}}{16(16\varphi_{22} - 16\varphi_{22}\varphi_{04} - 4\varphi_{40}\varphi_{04} + 4\varphi_{04})} \right] - 1 + \frac{\varphi_{04}}{4} + \frac{(1 + \varphi_{04})^2}{1 + 2\varphi_{04} + \frac{\varphi_{22}}{\varphi_{04}}} > 0
\]

ix. From (2.25) and (3.11):

\[
MSE(S^2_{M2A}) - MSE_{\min}(S^2_{NEW}) = 0, \text{If } \theta > A
\]

Note: The above-mentioned conditions (i)–(viii) are always satisfied. Henceforth, the proposed estimator \(S^2_{NEW}\) will always outperform all other estimators considered here.

5.0 Performance Evaluation of the Proposed Estimator \(S^2_{NEW}\)

To determine the performance of the proposed estimator over the existing estimators, we have considered the real and simulated data sets.

5.1. Numerical Study

For demonstration of the performance of the proposed exponential-type estimator, we considered some set of populations which earlier used by many authors. The description of the populations is presented in Table 1.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>23</td>
<td>64</td>
<td>39</td>
<td>103</td>
<td>80</td>
</tr>
<tr>
<td>(n)</td>
<td>10</td>
<td>19</td>
<td>18</td>
<td>40</td>
<td>20</td>
</tr>
<tr>
<td>(y)</td>
<td>39.699</td>
<td>5.549</td>
<td>26.843</td>
<td>62.621</td>
<td>51.8264</td>
</tr>
<tr>
<td>(x)</td>
<td>1035.065</td>
<td>141.5</td>
<td>106.2</td>
<td>556.5541</td>
<td>11.2646</td>
</tr>
<tr>
<td>(S^2_N)</td>
<td>54.360</td>
<td>2.277</td>
<td>38.99</td>
<td>8345.718</td>
<td>336.9757</td>
</tr>
<tr>
<td>(S^2_2)</td>
<td>381735</td>
<td>5772.67</td>
<td>124.1286</td>
<td>372300.5</td>
<td>70.6634</td>
</tr>
<tr>
<td>(\rho^{S2}_{22})</td>
<td>0.8277</td>
<td>0.2971</td>
<td>0.8899</td>
<td>0.657</td>
<td>0.7941</td>
</tr>
<tr>
<td>(\rho^{S2}_{2}(x))</td>
<td>2.03</td>
<td>2.773</td>
<td>2.4032</td>
<td>37.1279</td>
<td>2.2667</td>
</tr>
<tr>
<td>(\rho^{S2}_{2}(y))</td>
<td>2.696</td>
<td>2.341</td>
<td>2.993</td>
<td>17.8738</td>
<td>2.8664</td>
</tr>
<tr>
<td>(\delta^{S2}_{2})</td>
<td>2.094</td>
<td>1.458</td>
<td>2.4882</td>
<td>17.222</td>
<td>2.2209</td>
</tr>
<tr>
<td>(\lambda)</td>
<td>0.1565</td>
<td>0.037</td>
<td>0.0458</td>
<td>0.0153</td>
<td>0.0375</td>
</tr>
</tbody>
</table>

Data Set 1: \(y\) is the food cost of family employment and \(x\) is the weekly income of families.

Data Set 2: \(y\) is the number of literate persons in the village and \(x\) is the number of workers in the village.

Data Set 3: \(y\) is the leaf area for the newly developed strain of wheat and \(x\) is the weight of leaves.

Data Set 4: \(y\) is the total amount (tons) of recyclable-waste collection in Italy in 2003 and \(x\) is the number of inhabitants living in Italy in 2003.

Data Set 5: \(y\) is the output for 80 factories in a region and \(x\) is the fixed capital.

The numerical values that satisfied the condition for efficiency of the proposed estimator based on set of data described earlier, is presented in Table 2 and it is found that the proposed estimator satisfied all efficiency conditions. The MSEs and PREs results based on the data sets considered, are presented in Table 3. The expression given in (5.1) is used to obtain the percentage relative efficiency (PREs) of the proposed and the existing estimators

\[
PRE = \frac{\text{Var}(S^2_{\gamma})}{\text{minMSE}(\gamma)} \times 100, * = S^2_{\gamma}, S^2_{B}, S^2_{FR}, ..., S^2_{NEW}
\]
In Table 3, it is observed that the proposed estimator $S_{ZEW2}^2$ have the smallest MSE as compared to existing estimators for all data sets considered. Thus, the proposed estimator outperformed all the existing estimators. Note: the minimum $MSE(S_{Y}^2)$, $MSE(S_{W}^2)$ $MSE(S_{YK}^2)$, and given in Table 3 for $(\gamma, a, b, (k, \delta) and (\alpha_1, \beta_1) = (1, 1, 0), (1,1)$ and $(\frac{S_{Y}^2}{\tau}, 0)$.

Table 2: Numerical Values Conditions for Estimator $S_{ZEW2}^2$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Data 1</th>
<th>Data 2</th>
<th>Data 3</th>
<th>Data 4</th>
<th>Data 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{Y}^2$</td>
<td>170.82</td>
<td>0.1017</td>
<td>61.34</td>
<td>32516752</td>
<td>4633.71</td>
</tr>
<tr>
<td>$S_{W}^2$</td>
<td>88.64</td>
<td>0.1836</td>
<td>16.62</td>
<td>15932852</td>
<td>2183.53</td>
</tr>
<tr>
<td>$S_{YK}^2$</td>
<td>52.95</td>
<td>0.0716</td>
<td>10.80</td>
<td>15905962</td>
<td>1232.89</td>
</tr>
<tr>
<td>$S_{Z}^2$</td>
<td>51.98</td>
<td>0.0541</td>
<td>10.69</td>
<td>10679282</td>
<td>1198.51</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>58.91</td>
<td>0.0783</td>
<td>16.32</td>
<td>19730572</td>
<td>1421.73</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>49.03</td>
<td>0.0498</td>
<td>10.16</td>
<td>9384592</td>
<td>1130.07</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>51.87</td>
<td>0.0533</td>
<td>10.68</td>
<td>9401712</td>
<td>1195.97</td>
</tr>
<tr>
<td>$S_{YK}^2$</td>
<td>50.08</td>
<td>0.0495</td>
<td>10.39</td>
<td>8452032</td>
<td>1151.50</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>37.65</td>
<td>0.0339</td>
<td>8.26</td>
<td>7840882</td>
<td>863.56</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>36.71</td>
<td>0.0332</td>
<td>7.79</td>
<td>5896072</td>
<td>840.59</td>
</tr>
<tr>
<td>$S_{YK}^2$</td>
<td>35.36</td>
<td>0.0332</td>
<td>6.83</td>
<td>2005900</td>
<td>323.32</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>31.81</td>
<td>0.0283</td>
<td>6.80</td>
<td>4941162</td>
<td>728.73</td>
</tr>
</tbody>
</table>

Table 3: MSE and PRE values of the estimators with respect to $S_{Y}^2$

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Data 1</th>
<th>Data 2</th>
<th>Data 3</th>
<th>Data 4</th>
<th>Data 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_{Y}^2$</td>
<td>172.03</td>
<td>100.00</td>
<td>0.3401</td>
<td>100.00</td>
<td>38482180</td>
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<tr>
<td>$S_{W}^2$</td>
<td>89.86</td>
<td>191.45</td>
<td>0.4220</td>
<td>80.59</td>
<td>21898280</td>
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<tr>
<td>$S_{YK}^2$</td>
<td>54.17</td>
<td>317.59</td>
<td>0.3101</td>
<td>109.68</td>
<td>21871390</td>
</tr>
<tr>
<td>$S_{Z}^2$</td>
<td>53.19</td>
<td>323.41</td>
<td>0.2926</td>
<td>116.24</td>
<td>16644710</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>60.13</td>
<td>286.11</td>
<td>0.3168</td>
<td>107.36</td>
<td>25696000</td>
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<tr>
<td>$S_{Ye}^2$</td>
<td>50.25</td>
<td>342.34</td>
<td>0.2882</td>
<td>118.01</td>
<td>15350020</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>53.09</td>
<td>324.03</td>
<td>0.2917</td>
<td>116.58</td>
<td>15367140</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>51.30</td>
<td>335.35</td>
<td>0.2880</td>
<td>118.09</td>
<td>14471460</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>38.87</td>
<td>442.64</td>
<td>0.2724</td>
<td>124.88</td>
<td>13806310</td>
</tr>
<tr>
<td>$S_{YK}^2$</td>
<td>37.93</td>
<td>453.54</td>
<td>0.2717</td>
<td>125.18</td>
<td>11861500</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>15.29</td>
<td>1125.06</td>
<td>0.2504</td>
<td>135.81</td>
<td>7970518</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>33.03</td>
<td>520.80</td>
<td>0.2667</td>
<td>127.51</td>
<td>10906590</td>
</tr>
<tr>
<td>$S_{Ye}^2$</td>
<td>1.22</td>
<td>14132.86</td>
<td>0.2385</td>
<td>142.63</td>
<td>5965428</td>
</tr>
</tbody>
</table>

5.2 Simulation Study
A simulation study is conducted to verify the theoretical results discussed in section 3. We generated three artificial population of size N=1000 from a multivariate normal distribution of the same means $\mu = [\bar{Y}, \bar{X}] = [2, 2]$ and different covariance matrices are as follows.

Table 4: Simulation Results for MSEs and PREs of different Estimators with respect to $S_{Y}^2$

<table>
<thead>
<tr>
<th>n</th>
<th>Estimator</th>
<th>Population 1</th>
<th>Population 2</th>
<th>Population 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>$S_{Y}^2$</td>
<td>3.0662</td>
<td>100.00</td>
<td>3.7855</td>
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<tr>
<td></td>
<td>$S_{YK}^2$</td>
<td>5.5261</td>
<td>55.49</td>
<td>4.1712</td>
</tr>
<tr>
<td></td>
<td>$S_{Z}^2$</td>
<td>3.0305</td>
<td>101.18</td>
<td>3.0121</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.9208</td>
<td>104.98</td>
<td>2.9237</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>3.5280</td>
<td>86.91</td>
<td>3.0305</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.8861</td>
<td>106.24</td>
<td>2.8873</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.9167</td>
<td>105.13</td>
<td>2.9203</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.8875</td>
<td>106.19</td>
<td>2.8905</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.8267</td>
<td>108.47</td>
<td>2.7800</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.7488</td>
<td>111.55</td>
<td>2.7406</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.5634</td>
<td>119.62</td>
<td>2.5328</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.7063</td>
<td>113.30</td>
<td>2.6936</td>
</tr>
<tr>
<td></td>
<td>$S_{Ye}^2$</td>
<td>2.4560</td>
<td>124.85</td>
<td>2.4101</td>
</tr>
<tr>
<td>100</td>
<td>$S_{Y}^2$</td>
<td>1.4524</td>
<td>100.00</td>
<td>1.7931</td>
</tr>
<tr>
<td></td>
<td>$S_{YK}^2$</td>
<td>2.6176</td>
<td>55.49</td>
<td>1.9758</td>
</tr>
<tr>
<td></td>
<td>$S_{Z}^2$</td>
<td>1.4355</td>
<td>101.18</td>
<td>1.4268</td>
</tr>
</tbody>
</table>
estimator were carried out and compared to the existing estimators. The findings shown that the proposed estimator
approximation. Using real data sets and Monte Carlo simulation study, the theoretical and numerical perfor-
mance of the proposed estimator random sampling has been proposed. Bias and MSE expression of the proposed

7. Conclusion

Population 1: $\Sigma_{yx} = \begin{bmatrix} 9 & 1.9 \\ 1.9 & 4 \end{bmatrix}, \rho_{yx} = 0.3209,$

Population 2: $\Sigma_{yx} = \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}, \rho_{yx} = 0.6746$ and

Population 3: $\Sigma_{yx} = \begin{bmatrix} 6 & 3 \\ 3 & 2 \end{bmatrix}, \rho_{yx} = 0.3209$

We considered different sample sizes (50, 100, 200 and 300) for each population and the values of MSE’s for all estimators are computed as presented in Table 4, using 5000 Monte Carlo samples of different sizes selected from each population. The PREs for all estimators are computed by using expression in (5.1). Observing the results in table 4, the PRE of the estimator $S^2_{NEW2}$ is always having the highest value across the three-population considered and performed better than other existing estimators. Besides, it can be seen that estimator $S^2_{tr}, S^2_{SW}$ and $S^2_{FK}$ performed equally.

7. Conclusion

In this study, an improved class of exponential estimator for finite population variance $S^2$ utilizing auxiliary information in simple random sampling has been proposed. Bias and MSE expression of the proposed estimator are derived up to first order of approximation. Using real data sets and Monte Carlo simulation study, the theoretical and numerical performance of the proposed estimator were carried out and compared to the existing estimators. The findings shown that the proposed estimator $S^2_{NEW2}$ outperformed the unit variance estimator and other existing estimators in term of MSE and PRE. Also, it is observed that increase in sample size and correlation coefficient led to gain in efficiencies of all estimators considered for the three populations (see Table 4). Estimators $S^2_{tr}, S^2_{SW}$ and $S^2_{FK}$ are found to perform equally in all data sets. Based on these findings, the proposed estimator $S^2_{NEW2}$ is recommended for efficient estimation of population variance under simple random sampling.

References