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Reza Arabi Belaghi
Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Mehri Noori Asl
Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Hossein Bevrani
Department of Statistics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Özlem Gürünlü Alma
Department of Statistics, Faculty of Sciences, Muğla Sıtkı Koçman University, Muğla, Turkey

Corresponding Author:
Özlem Gürünlü Alma
Department of Statistics, Faculty of Sciences, Muğla Sıtkı Koçman University, Muğla, Turkey

Estimation of stress-strength reliability for poisson-exponential distribution under progressive type II censoring

Reza Arabi Belaghi, Mehri Noori Asl, Hossein Bevrani and Özlem Gürünlü Alma

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Abstract
In this study, we are aiming to estimate the stress-strength reliability, R=P(Y<X) based on progressive type-II censored samples when X and Y are independent random variables from a two parameter Poisson-Exponential distribution with the same scale but different shape parameters. The maximum likelihood and Bayesian approaches are used for estimation of R when the scale parameter is unknown. Next, in order to compare the performance of the proposed estimators, we carry out a Monte Carlo simulation study and one real data set is conducted to observe the performance of the proposed estimators.

Keywords: Bayes estimator, poisson-exponential distribution, maximum likelihood estimator, progressively type-II censoring

Introduction

In the field of reliability and life testing experiments, observing failure times of units put on tests is non-conventional. This may be due to varied restrictions on data collection like cost effectiveness, total time of test, simplicity of experimental set-up and many more. These constraints are manoeuvre by an experimenter which results in censoring of data arising in an experiment. Among various censoring schemes, the type-II progressive censoring scheme has become very popular. It can be explained as follows. Let n items be put in a life time study and $m(< n)$ items be completely observed; At the time of first failure, r_1 surviving units are removed from the $n-1$ remaining items; At the time of the next failure, r_2 items are randomly withdrawn from the $n - r_1 - 2$ remaining items; when the nth failure occurs the remaining $n - m - r_1 - \ldots - r_{m-1}$ items are removed. See Balakrishnan (2007) [2] for more details. There has been continuous interest in the problem of estimating the probability that one random variable exceeds another, that is, $R=P(X > Y)$, where X and Y are independent random variables. The parameter R is referred to as the reliability parameter. This problem arises in the classical stress-strength reliability where one is interested in assessing the proportion of the times the random strength X of a component exceeds the random stress Y to which the component is subjected. If $X \leq Y$, then either the component fails or the system that uses the component may malfunction. This problem also arises in situations where X and Y represent lifetimes of two devices and one wants to estimate the probability that one fails before the other. Estimation of the stress-strength parameter has received considerable attention in the statistical literature. Various authors have studied the estimation of R based on complete samples. A comprehensive account of this topic is given by Kots *et al.* (2003) [7]. Recently, some authors have studied the inferential procedures of R for some lifetime distributions based on progressive type-II censored samples. See, for example, Asgharzadeh *et al.* (2011) [1], Rezaei *et al.* (2015) [10] and Saraçoğu *et al.* (2015). The aim of this paper is to estimate $R = P(Y < X)$ under progressive type-II censored data on both variables X and Y, when X and Y are independent Poisson-Exponential random variables.

A random variable X is said to have a Poisson-Exponential (PE) distribution if its probability density function (pdf) and cumulative distribution (cdf) is given by

$$f_X(x,\theta,\lambda)=\frac{\theta\lambda e^{-\lambda x-\theta e^{-\lambda x}}}{1-e^{-\theta}},\quad x>0,\theta>0,\lambda>0, \tag{1.1}$$

$$F_X(x,\theta,\lambda)=1-\frac{1-e^{-\theta e^{-\lambda x}}}{1-e^{-\theta}},\quad x>0,\theta>0,\lambda>0, \tag{1.2}$$

Respectively. Here, the parameter θ is a shape parameter and λ is a scale parameter. From now on, PED with the shape parameter θ and scale parameter λ will be denoted by PE(θ, λ).

Maximum Likelihood Estimation of R

Let $X \sim PE(\theta_1, \lambda)$ and $Y \sim PE(\theta_2, \lambda)$ be independent random variables with unknown shape parameters θ_1 and θ_2 and common scale parameter λ. The stress-strength parameter, R is

$$R=P(Y<X)=\int_0^\infty\int_0^x \frac{\theta_1\theta_2\lambda^2 e^{-\lambda(x+y)-\theta_1 e^{-\lambda x}-\theta_2 e^{-\lambda y}}}{(1-e^{-\theta_1})(1-e^{-\theta_2})}dydx$$

$$=1-\frac{1}{(1-e^{-\theta_2})}\left[1-\frac{\theta_1}{\theta_1+\theta_2}\frac{1-e^{-(\theta_1+\theta_2)}}{1-e^{-\theta_1}}\right] \tag{2.1}$$

Our interest is in estimating R based on progressive censored samples on both variables. Suppose $\boldsymbol{X} = (X_{1:m1:n1}, \ldots, X_{m1:m1:n1})$ is a progressively type-II censored sample from PE(θ_1, λ) with censored scheme $\boldsymbol{r}_1$ and $Y = (Y_{1:m2:n2}, \ldots, Y_{m2:m2:n2})$ is a progressively type-II censored sample from PE(θ_2, λ) with censored scheme $\boldsymbol{r}_2$, where $\boldsymbol{r}_i = (r_{i1}, \ldots, r_{im})$ and $\sum_{j=1}^{m_i} r_{ij} = n_i$, for i = 1,2. For the sake of simplicity, we will write $(X_1, \ldots, X_{m1})$ instead of $(X_{1:m1:n1}, \ldots, X_{m1:m1:n1})$ and $(Y_1, \ldots, Y_{m2})$ instead of $(Y_{1:m2:n2}, \ldots, Y_{m2:m2:n2})$. The likelihood and log-likelihood function obtained as follows (see Balakrishnan and Aggarwala (2000)) [3].

$$L(\theta_1,\theta_2,\lambda\mid x,y)\propto\prod_{j=1}^{m_1} f_X(x_j)[1-F_X(x_j)]^{r_{1j}}\times\prod_{j=1}^{m_2} f_Y(y_j)[1-F_Y(y_j)]^{r_{2j}}, \tag{2.2}$$

$$\ln L(\theta_1,\theta_2,\lambda\mid x,y)\propto m_1\ln\theta_1+m_2\ln\theta_2+(m_1+m_2)\ln\lambda-m_1\ln(1-e^{-\theta_1})-m_2\ln(1-e^{-\theta_2})$$
$$-\sum_{j=1}^{m_1}\left(\lambda x_j+\theta_1 e^{-\lambda x_j}\right)+\sum_{j=1}^{m_1} r_{1j}\ln\left(\frac{1-e^{-\theta_1 e^{-\lambda x_j}}}{1-e^{-\theta_1}}\right)$$
$$-\sum_{j=1}^{m_2}\left(\lambda y_j+\theta_2 e^{-\lambda y_j}\right)+\sum_{j=1}^{m_2} r_{2j}\ln\left(\frac{1-e^{-\theta_2 e^{-\lambda y_j}}}{1-e^{-\theta_2}}\right).$$

The MLEs of parameters can be obtained as the simultaneous solutions of

$$\frac{\partial\ln L}{\partial\theta_1}=\frac{m_1}{\theta_1}-(n_1+m_1)\left(\frac{e^{-\theta_1}}{1-e^{-\theta_1}}\right)-\sum_{j=1}^{m_1}e^{-\lambda x_j}-\sum_{j=1}^{m_1}r_{1j}\left[\frac{e^{-\lambda x_j-\theta_1 e^{-\lambda x_j}}}{1-e^{-\theta_1 e^{-\lambda x_j}}}-\frac{e^{-\theta_1}}{1-e^{-\theta_1}}\right]=0,$$

$$\frac{\partial\ln L}{\partial\theta_2}=\frac{m_2}{\theta_2}-(n_2+m_2)\left(\frac{e^{-\theta_2}}{1-e^{-\theta_2}}\right)-\sum_{j=1}^{m_2}e^{-\lambda y_j}-\sum_{j=1}^{m_2}r_{2j}\left[\frac{e^{-\lambda y_j-\theta_2 e^{-\lambda y_j}}}{1-e^{-\theta_2 e^{-\lambda y_j}}}-\frac{e^{-\theta_2}}{1-e^{-\theta_2}}\right]=0,$$

and

$$\frac{\partial\ln L}{\partial\lambda}=\frac{m_1+m_2}{\lambda}-\sum_{j=1}^{m_1}x_j+\sum_{j=1}^{m_1}r_{1j}\frac{\theta_1 x_j e^{-\lambda x_j-\theta_1 e^{-\lambda x_j}}}{1-e^{-\theta_1 e^{-\lambda x_j}}}-\sum_{j=1}^{m_2}y_j+\sum_{j=1}^{m_2}r_{2j}\frac{\theta_2 y_j e^{-\lambda y_j-\theta_2 e^{-\lambda y_j}}}{1-e^{-\theta_2 e^{-\lambda y_j}}}=0.$$

To compute the maximum likelihood estimator (MLE) of R, we need to compute the MLEs of θ_1 and θ_2, say $\hat{\theta}_1$ and $\hat{\theta}_2$, respectively. The MLE $\hat{R}$ of R can then be obtained by substituting $\hat{\theta}_i$ in place of θ_i in (2.1), for i = 1, 2. Since, there exists no closed form for above likelihood equations, then we use the Newton-Raphson algorithm to obtain the MLEs of the unknown parameters.

Bayes Estimation of R

In this section, we obtain the Bayes estimation of *R* under assumption that all of the parameters θ_1, θ_2 and λ are unknown. We assume that θ_1 and θ_2 have the gamma prior, i.e, $\pi(\theta_1) \sim \Gamma(a_1, b_1)$ and $\pi(\theta_2) \sim \Gamma(a_2, b_2)$, respectively. Also, we consider a prior Gamma (a_3, b_3) for the scale parameter λ. Moreover, it is assumed that θ_1, θ_2 and λ are independent. Therefore the joint posterior density of θ_1, θ_2 and λ is given by

$$\pi(\theta_1,\theta_2,\lambda \mid \mathbf{x},\mathbf{y}) = \frac{L(\theta_1,\theta_2,\lambda \mid \mathbf{x},\mathbf{y})\pi(\theta_1)\pi(\theta_2)\pi(\lambda)}{\int_0^\infty\int_0^\infty\int_0^\infty L(\theta_1,\theta_2,\lambda \mid \mathbf{x},\mathbf{y})\pi(\theta_1)\pi(\theta_2)\pi(\lambda)\,d\theta_1\,d\theta_2\,d\lambda} \tag{3.1}$$

Clearly, the form of the posterior density does not lead to explicit Bayes estimators of the model parameters. For this, we need a simulation technique to compute the Bayes estimator of *R*. We adopt the importance sampling method to generate samples from the posterior distributions and then compute the Bayes estimators of *R* under Squared Error (SEL) and LINEX loss function. The posterior distributions of θ_1, θ_2 and λ can be obtained as follows:

$$\pi(\theta_1,\theta_2,\lambda \mid \mathbf{x},\mathbf{y}) = G_{\theta_1|\lambda}\left(a_1+m_1, b_1+\sum_{j=1}^{m_1} e^{-\lambda x_j}\right) G_{\theta_2|\lambda}\left(a_2+m_2, b_2+\sum_{j=1}^{m_2} e^{-\lambda y_j}\right) \tag{3.2}$$

$$\times G_{\lambda}\left(a_3+m_1+m_2, b_3+\sum_{j=1}^{m_1} x_j+\sum_{j=1}^{m_2} y_j\right) H(\theta_1,\theta_2,\lambda)$$

where

$$H(\theta_1,\theta_2,\lambda) = \frac{\prod_{j=1}^{m_1}\left(\frac{1-e^{-\theta_1 e^{-\lambda x_j}}}{1-e^{-\theta_1}}\right)^{r_{1j}} \prod_{j=1}^{m_2}\left(\frac{1-e^{-\theta_2 e^{-\lambda y_j}}}{1-e^{-\theta_2}}\right)^{r_{2j}}}{\left(1-e^{-\theta_1}\right)^{m_1}\left(1-e^{-\theta_2}\right)^{m_2}}.$$

Now consider the following steps to draw samples from the above posterior density.

Step 1: Generate $\lambda^{(k)}$ from $\text{Gamma}(a_3+m_1+m_2, b_3+\sum_{j=1}^{m_1} x_j+\sum_{j=1}^{m_2} y_j)$.

Step 2: Given $\lambda^{(k)}$ generate $\theta_1^{(k)}$ from $\text{Gamma}(a_1+m_1, b_1+\sum_{j=1}^{m_1} e^{-\lambda x_j})$.

Step 3: Given $\lambda^{(k)}$ generate $\theta_2^{(k)}$ from $\text{Gamma}(a_2+m_2, b_2+\sum_{j=1}^{m_2} e^{-\lambda y_j})$.

Step 4: Compute $R^{(k)}$ from by substituting $\theta^{(k)}$ and $\theta^{(k)}$ in place of θ_1 and θ_1 in (2.1).

Step 5: Repeat steps 2-4, *M* times.

Now the Bayes estimator of *R* under SEL and LINEX loss functions becomes:

$$\hat{R}_{SEL} = \frac{\sum_{k=1}^{M} R^{(k)} H\left(\theta_1^{(k)},\theta_2^{(k)},\lambda\right)}{\sum_{k=1}^{M} H\left(\theta_1^{(k)},\theta_2^{(k)},\lambda\right)}, \qquad \hat{R}_{LINEX} = -\frac{1}{v}\ln\left[\frac{\sum_{k=1}^{M} e^{-vR^{(k)}} H\left(\theta_1^{(k)},\theta_2^{(k)},\lambda\right)}{\sum_{k=1}^{M} H\left(\theta_1^{(k)},\theta_2^{(k)},\lambda\right)}\right].$$

Simulation study

In this section, a Monte Carlo simulation study is conducted to investigate and compare the performance of all estimators presented in this paper. We use the three censoring schemes in Table 1. The performances of the MLEs and the Bayes estimates with respect to the squared error and LINEX loss functions mainly are compared in terms of mean squares errors (MSEs). We consider the hyper parameters as $a_1 = b_1 = a_2 = b_2 = 0.0001$. The results are reported based on 1000 replications. We obtain the estimates of *R* by MLE and by using the Bayesian procedure under squared error and LINEX loss function. The results are reported in Table 2. Based on simulation results, we observed that the MSEs of the Bayes estimators are smaller than the MSEs of the MLEs. The (r_1, r_3) has smaller MSE compared to the other schemes.

Table 1: Censoring schemes.

	(n, m)	Censoring scheme
r_1	(30,20)	$(10, 0^{*19})$
r_2	(30,20)	$(0^{*19}, 10)$
r_3	(30,20)	$(0^{*9}, 10, 0^{*10})$

Table 2: Average and MSEs of the estimators of R. Bayes

R	**Scheme**	**MLE**	**SEL**	**LINEX(ν=0.5)**
0.20	(r_1, r_1)	0.2459 (0.0018)	0.2133 (0.0001)	0.2004 (0.0009)
	(r_1, r_2)	0.2480 (0.0020)	0.2335 (0.0009)	0.2249 (0.0002)
	(r_1, r_3)	0.2455 (0.0018)	0.2160 (0.0001)	0.2405 (0.0001)
	(r_2, r_1)	0.2358 (0.0010)	0.2104 (0.0005)	0.1956 (0.0005)
	(r_2, r_2)	0.2417 (0.0014)	0.2453 (0.0017)	0.2394 (0.0004)
	(r_2, r_3)	0.2410 (0.0014)	0.2159 (0.0001)	0.2042 (0.0001)
	(r_3, r_1)	0.2341 (0.0009)	0.2048 (0.0003)	0.1934 (0.0007)
	(r_3, r_2)	0.2367 (0.0011)	0.2296 (0.0007)	0.2207 (0.0001)
	(r_3, r_3)	0.2419 (0.0015)	0.2107 (0.0005)	0.1986 (0.0008)
0.36	(r_1, r_1)	0.3576 (0.00005)	0.3657 (0.00002)	0.3677 (0.00002)
	(r_1, r_2)	0.3579 (0.00005)	0.3571 (0.00006)	0.3581 (0.00005)
	(r_1, r_3)	0.3559 (0.00008)	0.3615 (0.00001)	0.3588 (0.00004)
	(r_2, r_1)	0.3576 (0.00005)	0.3554 (0.00009)	0.3532 (0.00001)
	(r_2, r_2)	0.3589 (0.00003)	0.3683 (0.00009)	0.3685 (0.00001)
	(r_2, r_3)	0.3559 (0.00008)	0.3555 (0.00009)	0.3573 (0.00001)
	(r_3, r_1)	0.3529 (0.00015)	0.3642 (0.00009)	0.3748 (0.00001)
	(r_3, r_2)	0.3543 (0.00011)	0.3587 (0.00004)	0.3603 (0.00007)
	(r_3, r_3)	0.3545 (0.00011)	0.3682 (0.00008)	0.3689 (0.00002)

Numerical example

Here, we consider a data analysis for the data set reported by Pepi (1994) [9]. The data set describes the all-glass airplane window design that measure polished window strength. The data sets are as follows:

25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89,

34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

We used R package 'fitdistrplus' to compare the fitness criteria of PE model with other model such as Weibull, Generalized exponential and Burr XII distribution. Based on the minimum AIC, BIC and the p-value of Kolmogorov-Smirnov test, we find that the PE model is an appropriate model for this data set. Next we generate two independent sample form the above data set as follows;

Sample 1: 35.910, 39.580, 33.730, 33.760, 26.770, 45.381, 44.045, 27.670, 27.050, 25.800,

Sample 2: 26.770, 36.980, 35.750, 33.890, 27.670, 39.580, 45.290, 31.110, 37.090, 37.080, 33.200, 44.045.

Next we obtain the values of unknown parameters to compute the the values of R. To compute the Bayes estimate, since we do not have any prior information, we assumed that $a_1 = b_1 = a_2 = b_2 = 0.0001$. The MLE and Bayes estimators of R under SEL and LINEX loss functions become 0.1986 and 0.07075, 0.0414, respectively.

Table 3: Censoring schemes

	(*n*, *m*)	**Censoring scheme**
r_1	(10,5)	$(5, 0^{*4})$
r_2	(12,5)	$(7, 0^{*4})$

Table 4: Average of the estimators of parameters for real data set. Bayes

Parameter	**MLE**	**SEL**	**LINEX (ν = 0.5)**
θ_1	2.1403	1.5265	1.3085
θ_2	1.9863	1.6029	1.2796
λ	0.0179	0.0185	0.0167

Conclusion

In this paper, we consider the estimation of the stress-strength parameter of Poisson-Exponential distribution. It is assumed that the two populations have the same scale parameters, but different shape parameters. It is observed that the maximum likelihood estimators of the unknown parameters cannot be obtained in closed form. For this, we use the iterative method as the Newton-Raphson (NR) algorithm. Further we obtain the Bayes estimators of R under squared error and LINEX loss function. Based on simulation results it is observed that Bayes estimators based on LINEX loss function has more low risk than SEL function.

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