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# Half-Cauchy Inverse Gompertz Distribution: Theory and Applications

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**Abstract**
In this paper, we have introduced a new distribution named half Cauchy inverse Gompertz distribution using the family of half-Cauchy distribution as base line distribution. Some crucial statistical properties of the new distribution are deliberated and analyzed. To evaluate the model parameters of the new model, we have applied the three commonly used estimation methods namely LSE, CVM and MLE. For the evaluation of flexibility and adaptability of the new model we have consider a real dataset and compared the goodness-of-fit attained by proposed distribution with some competing distribution. It is found that the suggested model is good fitted a real data set and more flexible as compared to a few other models.

**Keywords:** Half-Cauchy distribution, hazard function, inverse Gompertz distribution, reliability function, skewness

## Introduction

The Gompertz model is one of the extensively used probability model having survival function based on laws of mortality. The life-time data on human mortality can be modeled using this model, and actuarial tables can be investigated. Gompertz (1824) [11] created the Gompertz distribution, which has since been applied as a growth model and to fit tumor growth. The Gompertz model's function allows a big collection of life table data to be reduced to a single function. The number of people who will live to a specific age as a function of age is calculated using the Gompertz function. The cumulative distribution function (CDF) and probability density function (PDF) of Gompertz distribution are

$$G(x) = 1 - exp\left\{\frac{\lambda}{\alpha}(1 - e^{\alpha x})\right\};\ x > 0, \alpha\lambda > 0 \text{ And} \tag{1}$$

$$g(x) = \lambda e^{\alpha x}\, exp\left\{\frac{\lambda}{\alpha}(1 - e^{\alpha x})\right\};\ x > 0, \alpha\lambda > 0 \text{ respectively.} \tag{2}$$

By extending the inverse Gompertz distribution that has previously described by (Eliwa *et al*., 2019) [9], we have created a new distribution in this study. The CDF and PDF of inverse Gompertz distribution are

$$G(x) = exp\left\{\frac{\lambda}{\alpha}\left(1 - e^{\alpha/x}\right)\right\}; x > 0, (\alpha, \lambda) > 0 \text{ and} \tag{3}$$

$$g(x) = \lambda x^{-2} e^{\alpha/x}\, exp\left\{\frac{\lambda}{\alpha}\left(1 - e^{\alpha/x}\right)\right\}; x > 0, (\alpha, \lambda) > 0 \text{ respectively.} \tag{4}$$

El-Morshedy *et al*., (2020) [8] has defined the extension of inverse Gompertz distribution called Kumaraswamy inverse Gompertz distribution. As we know there is no any study made to extend the inverse Gompertz distribution using half-Cauchy family of distribution. Consider a positive random variable X which follows the half-Cauchy model with parameter θ and its CDF is

$$G(x;\theta) = \frac{2}{\pi} tan^{-1}\left(\frac{x}{\theta}\right), x > 0, \theta > 0.$$

and the PDF corresponding to (5) is,

$$g(x;\theta) = \frac{2}{\pi}\left(\frac{\theta}{\theta^2+x^2}\right), x > 0, \theta > 0. \tag{6}$$

The CDF of the extending family of distribution has been created by (Zografos & Balakrishnan, 2009) [23], and it is

$$F(x) = \int_0^{-ln[1-G(x)]} r(t)\ dt, \tag{7}$$

Here,$G(x)$ is the CDF of any baseline distribution and $r(t)$ is the PDF of any distribution. The half-Cauchy of family distributions whose CDF may be constructed by employing $r(t)$ as PDF of half-Cauchy distributions specified in (6) and written as

$$F(x) = \int_0^{-ln[1-G(x)]} \frac{2}{\pi}\frac{\theta}{\theta^2+t^2}\ dt\ = \frac{2}{\pi} arctan\left(-\frac{1}{\theta} ln[1-G(x)]\right);\ x > 0, \theta > 0 \tag{8}$$

The PDF corresponding to (6) can be stated as

$$f(x) = \frac{2}{\pi\theta}\frac{g(x)}{1-G(x)}\left[1+\left\{-\frac{1}{\theta} log[1-G(x)]\right\}^2\right]^{-1};x > 0, \theta > 0 \tag{9}$$

Last some decades many researchers have been used the half-Cauchy distribution as a parent distribution. The modification of the half-Cauchy distribution was introduced by (Cordeiro & Lemonte, 2011) [5] called the beta-half-Cauchy distribution, Jacob and Jayakumar (2012) [13] has presented the modification of half-Cauchy distribution applying Marshall-Olkin transformation and studied the autoregressive process of first order and The half-Cauchy distribution has been utilized by (Polson &Scott, 2012) [17] as a prior for a Bayesian analysis's universal scale parameter. The Kumaraswamy-half-Cauchy distribution is an addition to the half-Cauchy distribution that was first proposed by (Ghosh, 2014) [10]. The gamma half-Cauchy distribution has introduced by (Alzaatreh *et al*., 2016) [2]. Cordeiro *et al*. (2017) [6] has developed the family of distribution using half-Cauchy distribution as generalized odd half-Cauchy family of distribution. By utilizing the family of half-Cauchy distributions as the baseline model, the half Cauchy modified exponential distribution has suggested by (Chaudhary & Kumar, 2022) [3].

In order to improve the inverse Gompertz distribution's fit to the real data, we suggest the half-Cauchy inverse Gompertz distribution, a more flexible distribution that only requires one extra parameter. We study the properties of the half -Cauchy inverse Gompertz distribution and illustrate its applicability and potentiality.

As for the remaining sections, they are handled as follows. In section 2, the novel half-Cauchy inverse Gompertz model is presented, and numerous statistical features are addressed. Three extensively used estimation approaches namely least-square (LSE), Cramer-Von-Mises (CVM) and maximum likelihood estimators (MLE) methods are used to estimate the parameters which are introduced in section 3. In section 4, utilizing a real life time dataset, we demonstrate how the proposed model is appropriate and applicable. In this section, we compute the statistics like AIC, AICC, HQIC, and BIC to assess the goodness-of-fit attain by the half Cauchy inverse Gompertz distribution. Finally, some conclusions are presented in section 5.

## 2. The Half Cauchy Inverse Gompertz (HCIG) distribution

In this section, we have introduced the new distribution called half Cauchy inverse Gompertz distribution which has been derived by combing the inverse Gompertz distribution with the family of half -Cauchy distribution. Substituting (3) and (4) in (8) and (9) we get the CDF and PDF of HCIG distribution. Let X be a random variable of non-negative values that follows the $HCIG(\alpha,\lambda,\theta)$ and if its CDF takes the form,

$$F(x) = 1 - \frac{2}{\pi} arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x}\right)\right\};\ x > 0, \alpha, \lambda, \theta > 0 \tag{10}$$

and its PDF can be stated as,

$$f(x) = \frac{2}{\pi}\frac{\lambda x^{-2}e^{\alpha/x}}{\theta}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x}\right)\right\}^2\right]^{-1};\ x > 0, \alpha, \lambda, \theta > 0 \tag{11}$$

**Reliability/survival function:**

The survival function of $HCIG(\alpha,\lambda,\theta)$ model is defined as

$$R(x) = 1 - F(x)$$

$$= \frac{2}{\pi} arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x}\right)\right\};\ x > 0, \alpha, \lambda, \theta > 0 \tag{12}$$

**Hazard rate function (hrf)**

The hrf of $HCIG(\alpha,\lambda,\theta)$ distribution can be defined as,

$$h(x) = \frac{f(x)}{R(x)}$$

$$= \frac{\lambda x^{-2} e^{\alpha/x}}{\theta}\left[arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x}\right)\right\}\right]^{-1}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x}\right)\right\}^{2}\right]^{-1} \qquad (13)$$

**Quantile function**

The quantile function of $HCIG(\alpha, \lambda, \theta)$ can be stated as,

$$Q(u) = \alpha\left[ln\left\{1+\frac{\alpha\theta}{\lambda} tan\left(\frac{\pi(1-u)}{2}\right)\right\}\right]^{-1}; 0 < u < 1. \qquad (14)$$

**The Random Deviate Generation**

The random numbers can be drawn from $HCIG(\alpha, \lambda, \theta)$ by

$$x = \alpha\left[ln\left\{1+\frac{\alpha\theta}{\lambda} tan\left(\frac{\pi(1-v)}{2}\right)\right\}\right]^{-1}; 0 < v < 1 \qquad (15)$$

Median of $HCIG(\alpha, \lambda, \theta)$ can be computed as

$$median = \alpha\left[ln\left\{1+\frac{\alpha\theta}{\lambda}\right\}\right]^{-1}$$

Plots of PDF and hrf of $HCIG(\alpha, \lambda, \theta)$ are displayed in Figure 1.

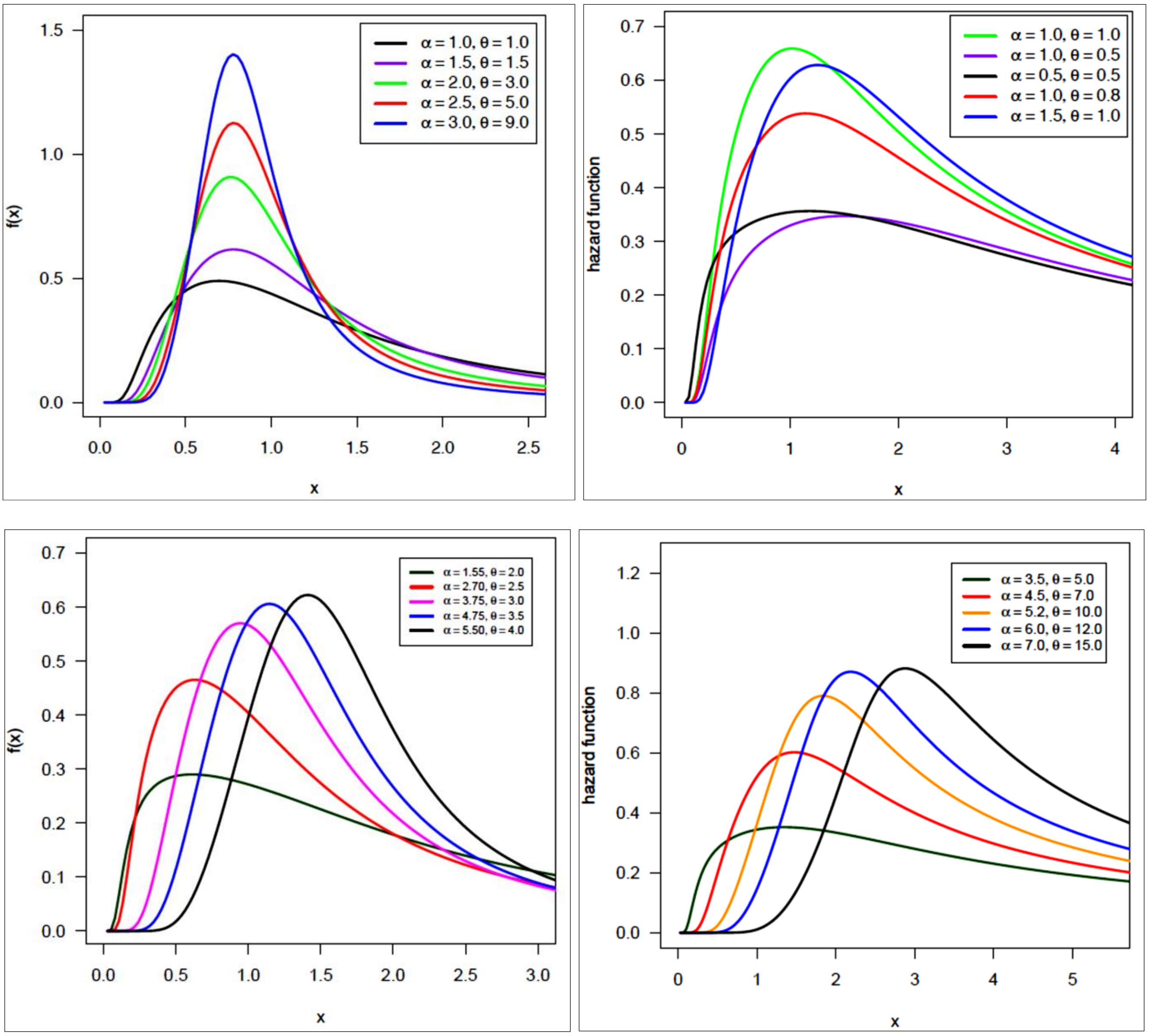


**Fig 1:** Density function's plot (left panel) and hazard function's plot (right panel) for various values of α and θ and fixed λ=1.

**Skewness and Kurtosis of HCIG distribution:** The skewness of the $HCIG(\alpha, \lambda, \theta)$ distribution based on quartiles is

$$Sk(B) = \frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}, \quad (16)$$

and the kurtosis of the $HCIG(\alpha,\lambda,\theta)$ distribution based on octiles (Moors, 1988) [15] is

$$K(moors) = \frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}, \quad (17)$$

**3. Methods of Parameter estimation**

Three widely used estimation techniques to estimate the unknown parameters of the HCIG distribution are as follows:

**3.1. Maximum Likelihood Estimation (MLE) method**

We have used the MLE approach to estimate the suggested model's parameters. Suppose a random sample $\underline{x} = (x_1, \dots, x_n)$ of size 'n' be taken from $HCIG(\alpha,\lambda,\theta)$, then the log likelihood function $l(\alpha, \lambda, \theta/\underline{x})$ is obtained as,

$$\ell(\alpha,\lambda,\theta|\underline{x}) = n\,ln\left(\frac{2}{\pi}\right) + n\,ln\,\lambda + n\,ln\,\theta - 2\sum_{i=1}^{n} ln\,x_i + \sum_{i=1}^{n}\frac{\alpha}{x_i} - \sum_{i=1}^{n} ln\left[\theta^2 + \left\{-\frac{\lambda}{\alpha}\left(1-e^{\alpha/x_i}\right)\right\}^2\right] \quad (18)$$

After differentiating (18) w. r. to $\alpha$, $\lambda$, *and* $\theta$, we have

$$\frac{\partial\ell}{\partial\alpha} = \sum_{i=1}^{n}\frac{1}{x_i} - \frac{2\lambda^2}{\alpha}\sum_{i=1}^{n}\left(1-e^{\alpha/x_i}\right)\left\{-\alpha^2\left(1-e^{\alpha/x_i}\right) + x^{-2}e^{\alpha/x_i}\right\}\left[\theta^2 + \left\{-\frac{\lambda}{\alpha}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial\ell}{\partial\lambda} = \frac{n}{\lambda} - \frac{2\lambda}{\alpha^2}\sum_{i=1}^{n}\left(1-e^{\alpha/x_i}\right)^2\left[\theta^2 + \left\{-\frac{\lambda}{\alpha}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial\ell}{\partial\theta} = \frac{n}{\theta} - 2\theta\sum_{i=1}^{n}\left[\theta^2 + \left\{-\frac{\lambda}{\alpha}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

Setting $\frac{\partial\ell}{\partial\alpha} = \frac{\partial\ell}{\partial\lambda} = \frac{\partial\ell}{\partial\theta} = 0$ and solving these three non-linear equations simultaneously for α, $\lambda$, *and* $\theta$, the ML estimators of the $HCIG(\alpha,\lambda,\theta)$ model can be determined. It is not easy to resolve them manually, we may use suitable computer software like R, Matlab, Mathematica, etc. Consider $\underline{\pi} = (\alpha,\lambda,\theta)$ and $\hat{\underline{\pi}} = (\hat{\alpha},\hat{\lambda},\hat{\theta})$ be the vector of parameters and the corresponding MLE vector respectively, so an asymptotic normality results in, $(\hat{\underline{\pi}} - \underline{\pi}) \rightarrow N_3[0,\{I(\underline{\pi})\}^{-1}]$ where $I(\underline{\pi})$ stands for the Fisher's information matrix which is obtained as,

$$I = \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

Where

$$I_{11} = \frac{\partial^2 l}{\partial\alpha^2}, I_{12} = \frac{\partial^2 l}{\partial\alpha\partial\lambda}, I_{13} = \frac{\partial^2 l}{\partial\alpha\partial\theta}$$

$$I_{21} = \frac{\partial^2 l}{\partial\lambda\partial\alpha}, I_{22} = \frac{\partial^2 l}{\partial\lambda^2}, I_{23} = \frac{\partial^2 l}{\partial\theta\partial\lambda}$$

$$I_{31} = \frac{\partial^2 l}{\partial\theta\partial\alpha}, I_{32} = \frac{\partial^2 l}{\partial\theta\partial\lambda}, I_{33} = \frac{\partial^2 l}{\partial\theta^2}$$

The MLE's asymptotic variance $\{I(\underline{\pi})\}^{-1}$ serves no purpose because we lack $\underline{\pi}$ in practice. As a result, we substitute in the estimated parameter values, where $I(\underline{\pi})$ is the Fisher's information matrix, to approximate the asymptotic variance. The variance-covariance matrix can be constructed as the observed information matrix is produced via the Newton-Raphson algorithm, which maximizes the likelihood function and is therefore constructed as

$$\{I(\underline{\pi})\}^{-1} = \begin{pmatrix} var(\hat{\alpha}) & cov(\hat{\alpha},\hat{\lambda}) & cov(\hat{\alpha},\hat{\theta}) \\ cov(\hat{\alpha},\hat{\lambda}) & var(\hat{\lambda}) & cov(\hat{\lambda},\hat{\theta}) \\ cov(\hat{\alpha},\hat{\theta}) & cov(\hat{\lambda},\hat{\theta}) & var(\hat{\theta}) \end{pmatrix} \quad (19)$$

Therefore, the asymptotic normality of MLEs allows for the following derivation of the approximated 100(1-γ)% confidence intervals for α, β and λ:

$\hat{\alpha} \pm Z_{\gamma/2}SE(\hat{\alpha})$, $\hat{\lambda} \pm Z_{\gamma/2}SE(\hat{\lambda})$ and $\hat{\theta} \pm Z_{\gamma/2}SE(\hat{\theta})$.

### 3.2. Method of Least-Square Estimation (LSE)

By minimizing (20) with regard to parameters α, λ and θ, we can derive the least-square estimators of the $HCIG(\alpha,\lambda,\theta)$ distribution's unknown parameters α, λ and θ.

$$B(x;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\left[F(x_i)-\frac{i}{n+1}\right]^2 \quad (20)$$

Suppose if a random sample $\{X_1, X_2, \dots, X_n\}$ of size n is drawn from a distribution function F(.),where the CDF of the ordered random variables $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ is symbolized by$F(x_i)$, then $\hat{\alpha}, \hat{\lambda}, and\ \hat{\theta}$ are the least-square estimators of α, λ and θ respectively, which may be found through minimizing (21) with respect to α, λ and θ.

$$B(x;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{i}{n+1}\right]^2;\ x>0,(\alpha,\lambda,\theta)>0 \quad (21)$$

Differentiating (21) with respect to α, λ and θ, we get,

$$\frac{\partial B}{\partial\alpha}=-\frac{4}{\pi}\frac{\lambda}{\alpha^2\theta}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{i}{n+1}\right]\left\{1+e^{\alpha/x_i}\left(\frac{\alpha}{x_i}-1\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial B}{\partial\lambda}=\frac{4}{\pi\alpha\theta}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{i}{n+1}\right]\left\{\left(1-e^{\alpha/x_i}\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial B}{\partial\theta}=-\frac{4}{\pi\alpha\theta^2}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{i}{n+1}\right]\left\{\left(1-e^{\alpha/x_i}\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

By minimizing the following function with respect to α, λ and θ, it can be also derived the weighted least square estimators.

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}w_i\left[F(X_{(i)})-\frac{i}{n+1}\right]$$

Where

$$Weights=w_i=\frac{1}{Var(X_{(i)})}=\frac{(n+1)^2(n+2)}{i(n-i+1)}$$

As a result, by minimizing (22) with respect α, λ and θ, we may derive the weighted least square estimators of each parameter.

$$D(X;\alpha,\lambda,\theta)=\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{i}{n+1}\right]^2 \quad (22)$$

### 3.3. Cramer-Von-Mises estimation (CVME) method

Through minimizing the function (23) with respect to α, λ and θ, we can derive the Cramer-Von-Mises estimators of each parameter.

$$C(X;\alpha,\lambda,\theta)=\frac{1}{12n}+\sum_{i=1}^{n}\left[F(x_{i:n}|\alpha,\lambda,\theta)-\frac{2i-1}{2n}\right]^2$$

$$=\frac{1}{12n}+\sum_{i=1}^{n}\left[\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha x_i}\right)\right\}-\frac{2i-1}{2n}\right]^2 \quad (23)$$

Differentiating (23) with respect to α, λ and θ, we get

$$\frac{\partial C}{\partial\alpha}=-\frac{4}{\pi}\frac{\lambda}{\alpha^2\theta}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{2i-1}{2n}\right]\left\{1+e^{\alpha/x_i}\left(\frac{\alpha}{x_i}-1\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial C}{\partial\lambda}=\frac{4}{\pi\alpha\theta}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{2i-1}{2n}\right]\left\{\left(1-e^{\alpha/x_i}\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

$$\frac{\partial C}{\partial\theta}=-\frac{4}{\pi\alpha\theta^2}\sum_{i=1}^{n}\left[1-\frac{2}{\pi}arctan\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}-\frac{2i-1}{2n}\right]\left\{\left(1-e^{\alpha/x_i}\right)\right\}\left[1+\left\{-\frac{\lambda}{\alpha\theta}\left(1-e^{\alpha/x_i}\right)\right\}^2\right]^{-1}$$

We can obtain the CVM estimators of each parameter by solving $\frac{\partial C}{\partial\alpha}=0, \frac{\partial C}{\partial\lambda}=0\ and\ \frac{\partial C}{\partial\theta}=0$ simultaneously.

## 4. Applications to Real Dataset

We have demonstrated the $HCIG(\alpha, \lambda, \theta)$ distribution's applicability in this part using a genuine dataset that previous researchers have used. The real data set reported by (Gross & Clark, 1976) [12] given below shows the lifetime's data relating to relief times (in minutes) of 20 patients getting an analgesic.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1.4 | 1.1 | 1.7 | 1.3 | 1.8 | 1.9 | 1.6 | 2.2 | 1.7 | 2.7 |
| 1.8 | 4.1 | 1.5 | 1.2 | 1.4 | 3 | 1.7 | 2.3 | 1.6 | 2.0 |

Maximizing the likelihood function (18) employing the R software's optim function (R Core Team, 2022) [19] (Schmuller, 2017) [21], it can be estimated the MLEs of the $HCIG(\alpha, \lambda, \theta)$ model. The value of Log-Likelihood that we have estimated is $l$ = -15.7179, $\hat{\alpha}$= 9.0830, $\hat{\lambda}$= 0.8369 and $\hat{\theta}$= 17.9925.

The profile log-likelihood function plots for the parameters α, λ and θ demonstrate in Figure 2 that the ML estimates may be exclusively generated.

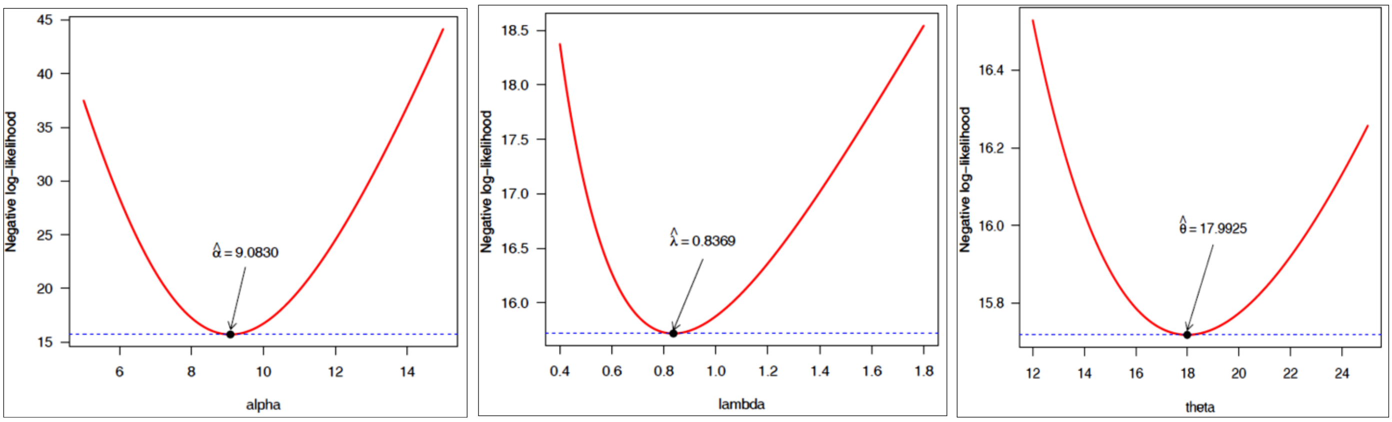


**Fig 2:** Graphs of Profile log-likelihood function for α, λ and θ.

It can be detected from the Q-Q plot and P-P plot shown in Figure 3 that the HCIG model is good fitted to the lifetime's data set.

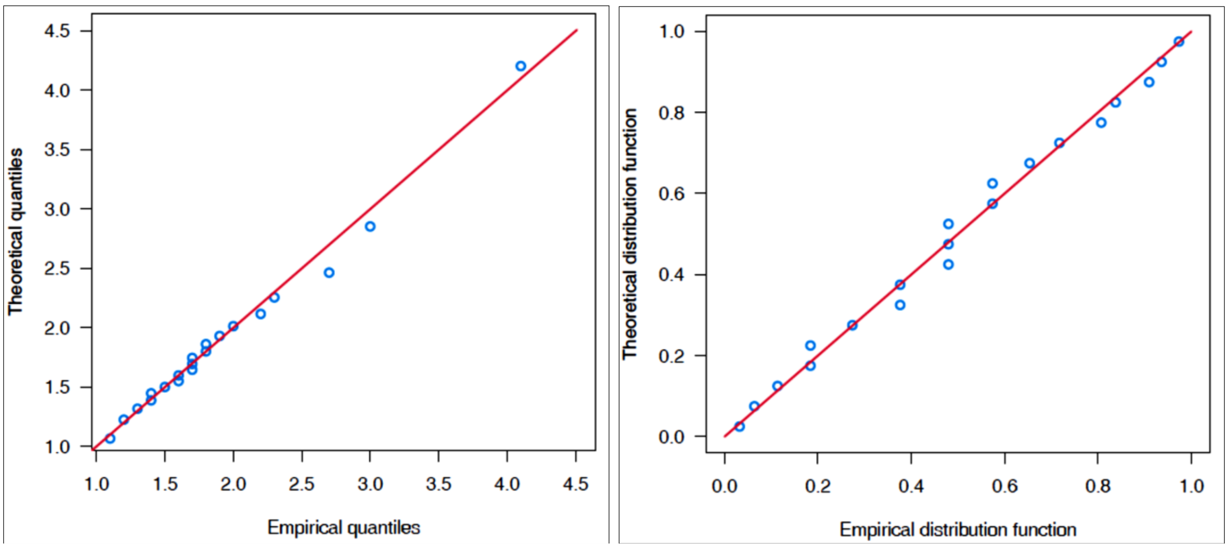


**Fig 3:** The Q-Q plot (left panel) and P-P plot (right panel) of the HCIG model.

The estimated values for the HCIG model's parameters, as determined by the MLE, LSE, and CVE methods, are shown in Table 1, together with the accompanying negative log-likelihood, AIC, and KS statistics and p-values.

**Table 1:** Values of estimated parameters, AIC and KS statistic and log-likelihood

| Estimation method | $\hat{\alpha}$ | $\hat{\lambda}$ | $\hat{\theta}$ | LL | AIC | *KS(p-value)* |
|---|---|---|---|---|---|---|
| MLE | 9.0830 | 0.8369 | 17.9925 | -15.7179 | 37.4358 | 0.0797(0.9996) |
| LSE | 8.0315 | 0.9030 | 11.8382 | -15.8761 | 37.7522 | 0.0889(0.9974) |
| CVE | 8.8151 | 0.6583 | 12.4616 | -15.7277 | 37.4553 | 0.0801(0.9995) |

Figure 4 displays the graphs of the fitted distributions' density function, histogram, and Q-Q plot from the HCIG model using the estimation methods LSE, MLE, and CVM. It turns out that the HCIG model fits the real data set fairly well.

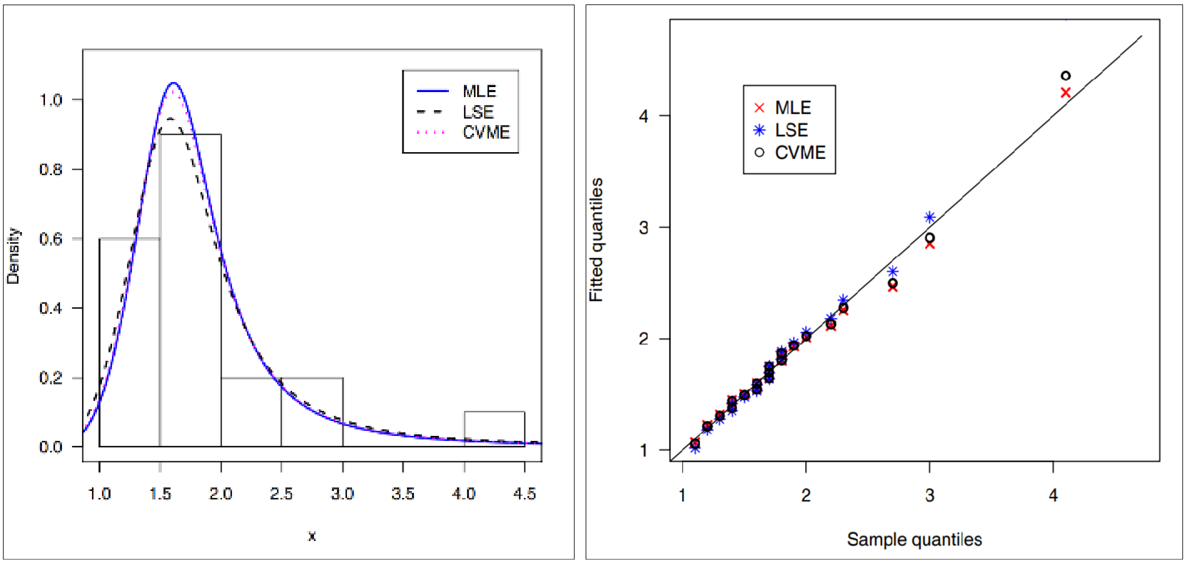

**Fig 4:** The fitted distributions' histogram and the density function (left panel) and Q-Q plot (right panel) of CVM, MLE and LSE estimation methods.

This section uses a real dataset that was previously utilized by researchers to demonstrate the applicability of the HCIG distribution. We have taken into account the following five distributions to compare the potential of the suggested model.

### i) Generalized Gompertz distribution

El-Gohary *et al.* (2013) [7] developed Generalized Gompertz model having parameters α, $\lambda$ and θ and its the PDF is

$$f_{GGZ}(x) = \theta\lambda e^{\alpha x} e^{-\frac{\lambda}{\alpha}(e^{\alpha x}-1)} \left[1 - exp\left(-\frac{\lambda}{\alpha}(e^{\alpha x}-1)\right)\right]^{\theta-1}; \ \lambda, \theta > 0, \alpha \geq 0, x \geq 0$$

### ii) Gompertz distribution (GZ)

Murthy *et al.* (2003) [16] developed Gompertz distribution with parameters α and θ and its PDF is

$$f_{GZ}(x) = \theta \ \ e^{\alpha x} \ \ exp\left\{\frac{\theta}{\alpha}(1 - e^{\alpha x})\right\} \quad ; x \geq 0, \ \theta > 0, \ -\infty < \ \alpha \ < \infty.$$

### iii) Exponentiated Exponential Poisson (EEP):

The PDF of EEP (Ristić & Nadarajah, 2014) [20] can be stated as

$$f(x) = \frac{\alpha\beta\lambda}{(1-e^{-\lambda})} e^{-\beta x}\left(1 - e^{-\beta x}\right)^{\alpha-1} exp\{-\lambda\left(1 - e^{-\beta x}\right)^{\alpha}\} \quad ; \ x > 0, \ \alpha > 0, \lambda > 0$$

### iv) Weibull Extension Model

With three parameters$(\alpha, \beta, \lambda)$, Tang *et al.* (2003) [22] suggested Weibull extension (WE) model and its the probability density function is

$$f_{WE}(x; \alpha, \ \beta, \lambda) = \ \lambda\beta \ \left(\frac{x}{\alpha}\right)^{\beta-1} exp\left(\frac{x}{\alpha}\right)^{\beta} \ exp\left\{-\lambda\alpha \ \left(exp\left(\frac{x}{\alpha}\right)^{\beta} - 1\right)\right\} \quad ; \quad x > 0$$

$\alpha > 0, \beta > 0$ and $\lambda > 0$

### v) Exponentiated Weibull (EW)

The probability density function of exponentiated Weibull extension (EW) distribution (Mudholkar and Srivastava, 1993) [18] with three parameters $(\alpha, \beta, \lambda)$ is

$$f(x) = \alpha\beta\lambda \ x^{\beta-1} \ e^{-\alpha x^{\beta}} \left(1 - e^{-\alpha x^{\beta}}\right)^{\lambda-1}; \ x > 0, \ \alpha > 0, \beta > 0, \lambda > 0$$

For the purpose of evaluating the suggested model's fit, BIC, CAIC, HQIC) and AIC values are computed and we have provided the findings in Table 2.

**Table 2:** Log-likelihood (LL), AIC, BIC, CAIC and HQIC

| Model | LL | AIC | BIC | CAIC | HQIC |
|---|---|---|---|---|---|
| HCIG | -15.7179 | 37.4358 | 40.4230 | 38.9358 | 38.0189 |
| EEP | -15.8259 | 37.6518 | 40.6390 | 39.1518 | 38.2349 |
| EW | -15.8982 | 37.7963 | 40.7835 | 39.2963 | 38.3795 |
| GGZ | -16.4717 | 38.9434 | 41.9306 | 40.4434 | 39.5266 |
| WE | -20.6181 | 47.2363 | 50.2234 | 48.7363 | 47.8194 |
| GZ | -24.5901 | 53.1802 | 55.1717 | 53.8861 | 53.5690 |

We have displayed a variety of distributions in figure 5, including a histogram, the density function of fitted models, an empirical distribution function with an estimated distribution function of the HCIG model as well as a few other models.

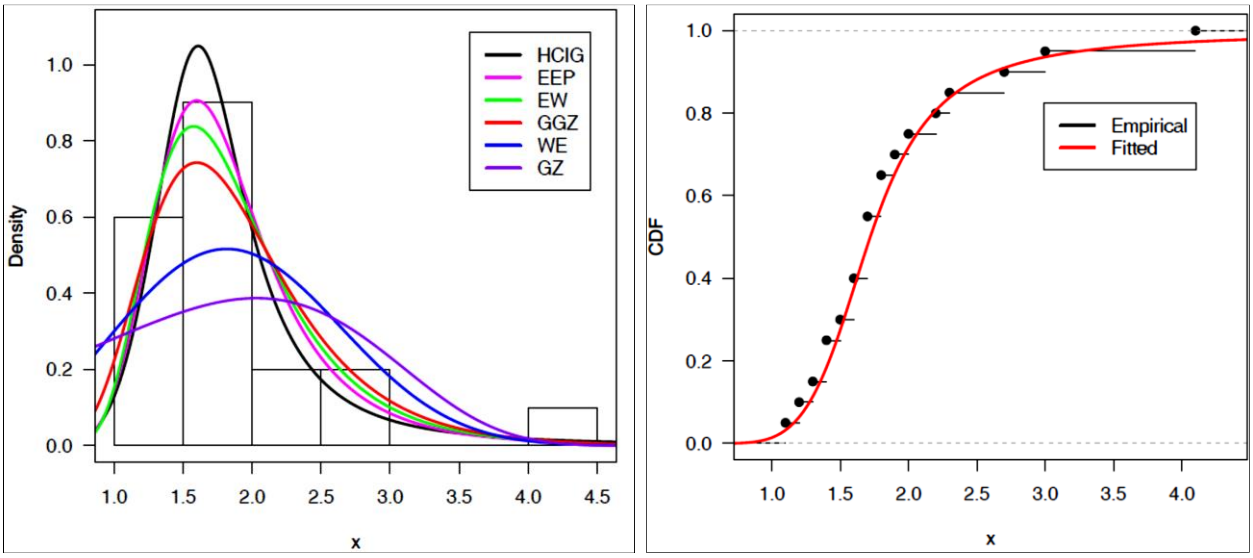


**Fig 5:** The left panel (the histogram and density function of the fitted distributions) and the right panel (the empirical and estimated distribution functions).

The results of the Anderson-Darling (AD), the Cramer-Von Mises (CVM), and Kolmogorov-Simnorov (KS) statistics are shown in Table 3 to evaluate how well the HCIG model fits with other competing models. It is clear that the HCIG model has a larger p-value and a lower value for the test statistic. Consequently, we have established that, compared to the other distributions used for comparison, the HCIG distribution exhibits a far better fit and more consistent set of data.

**Table 3:** The goodness-of-fit statistics and their corresponding p-value

| Model | KS(p-value) | AD(p-value) | CVM(p-value) |
|---|---|---|---|
| HCIG | 0.0797(0.9996) | 0.0195(0.9981) | 0.1277(0.9997) |
| EEP | 0.1086(0.9723) | 0.0299(0.9791) | 0.2157(0.9856) |
| EW | 0.1264(0.9065) | 0.0412(0.9314) | 0.2476(0.9715) |
| GGZ | 0.1446(0.7974) | 0.0628(0.8020) | 0.3564(0.8894) |
| WE | 0.1854(0.4979) | 0.1851(0.2998) | 1.0918(0.3117) |
| GZ | 0.2382(0.2065) | 0.3147(0.1225) | 1.8059(0.1183) |

## 5. Conclusion

In this article, we have developed half- Cauchy inverse Gompertz distribution using the family of half-Cauchy distribution as base line distribution .A thorough analysis of the suggested distribution's statistical and mathematical properties, including the generation of clear expressions for its skewness and kurtosis, survival function, probability density function, hazard function, cumulative distribution function, and quantile function has been examined. We estimate the parameters using three widely used estimation approaches namely maximum MLE, LSE, and CVME. We have found that MLEs outperform LSE and CVME procedures. The PDF of the suggested model shows curves that show its right-skewed, increasing-decreasing, and adjustable shape for simulating real-life data. According on model parameters values, the hazard function graph is also reverse j-shaped, constant, or monotonically increasing. Utilizing a real-life time data set, the applicability and flexibility of the suggested distribution are evaluated and the results revealed that it is noticeably more adjustable and suitable than certain other fitted distributions.

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